110.107 - Homework #12 Solutions

Jake Mokris
Section 1

Section 12.2: 4-8, 16, 18, 20, 24, 26, 28, 40, 46

4. The sample space is the number of combinations of five of the balls, which is $\binom{6}{5} = \frac{6!}{5!\,1!} = 6$. Note that this is the same as choosing the ball you leave out. $\square$

We have that $\Omega = \{1,2,3,4,5,6\}$, $A = \{1,3,5\}$ and $B = \{1,2,3\}$.

5. Find $A \cup B$ and $A \cap B$.

Answer: $A \cup B = \{1,3,5\} \cup \{1,2,3\} = \{1,2,3,5\}$

$A \cap B = \{1,3,5\} \cap \{1,2,3\} = \{1,3\}$.

6. Find A^c and show that $(A^c)^c = A$.

Answer: $A^c = \Omega - A = \{1,2,3,4,5,6\} - \{1,3,5\} = \{2,4,6\}$.

$(A^c)^c = \Omega - A^c = \{1,2,3,4,5,6\} - \{2,4,6\} = \{1,3,5\} = A$.

7. Find $(A \cup B)^c$.

Answer: $(A \cup B)^c = \Omega - (A \cup B) = \{1,2,3,4,5,6\} - \{1,2,3,5\} = \{4,6\}$.

8. Are A and B disjoint?

Answer: Since $A \cap B = \{1,3\} \neq \emptyset$, A and B are not disjoint.

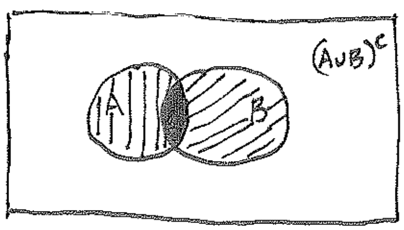
16. Assume that $P(A \cap B^c) = 0.1$, $P(B \cap A^c) = 0.5$, and $P((A \cup B)^c) = 0.2$. Find $P(A \cap B)$.

Answer: $P(A \cap B) = 1 - P(A \cap B^c) - P(B \cap A^c) - P((A \cup B)^c)$

$= 1 - 0.1 - 0.5 - 0.2 = 0.2$.

This is because $A \cap B = [(A \cap B^c) \cup (B \cap A^c) \cup (A \cup B)^c]^c$.

Think about the set diagram:



$|||| = A \cap B^c$

$= B \cap A^c$

■ $= A \cap B$

□ $= (A \cup B)^c$

18. Assume $P(A) = 0.4$, $P(B) = 0.4$, and $P(A \cup B) = 0.7$. Find $P(A \cap B)$ and $P(A^c \cap B^c)$.

<u>Answer</u>: $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.1$.

Since $A^c \cap B^c = (A \cup B)^c$, and $P((A \cup B)^c) = 1 - P(A \cup B)$,

$P(A^c \cap B^c) = 1 - 0.7 = 0.3$.

20. If $A \subset B$, we can define the difference between the two sets A and B, denoted $B - A$, $B - A = B \cap A^c$.

(a) Clearly $A \cap (B \cap A^c) = 0$ (if $x \in B \cap A^c$, $x \notin A$).

Suppose $x \in B$; if $x \notin A$, then $x \in A^c$. If $x \in A$, then $x \notin A^c$ so $x \notin B \cap A^c$. So $B \subset A \cup (B \cap A^c)$.

Conversely, if $x \in A \cup (B \cap A^c)$, $x \in A \subset B$ or $x \in B \cap A^c \subset B$.

Thus $A \cup (B \cap A^c) \subset B$, so $A \cup (B \cap A^c) = B$.

(b) We know that $P(A \cup (B - A)) = P(A) + P(B - A) + P(A \cap (B - A))$

But $A \cup (B - A) = B$ and $A \cap (B - A) = \emptyset$. Thus

$$P(B) = P(A) + P(B - A).$$

(c) Since $P(B - A) \geq 0$, $P(B) = P(A) + P(B - A) \geq P(A)$.

24. Number of possible outcomes: $2^4 = 16$

Number with three or more heads: 5.

{HHHT, HHTH, HTHH, THHH, HHHH}

So the probability is $\frac{5}{16}$.

26. Number of possible rolls: 36
Number of rolls in which both dice come up odd: $3 \cdot 3 = 9$
Number of rolls in which both dice come up even: $3 \cdot 3 = 9$

Probability is $\frac{9+9}{36} = \frac{1}{2}$.

28. Number of rolls in which both dice come up greater than 4: $2 \cdot 2 = 4$.

Probability is $\frac{4}{36} = \frac{1}{9}$.

40. First, the size of our sample space is $8 \cdot 7 \cdot 6$. The number of events in which at least three balls are green is equal to the number of events in which there are no blue balls plus the number of events in which there is only one blue ball.

No blue balls: $3 \cdot 2 \cdot 1 = 6$ (we chose green every time)

One blue ball: $(5 \cdot 3 \cdot 2) \cdot 3 = 60$ (choose blue first, then two green) $\times 3$ (for when you choose blue second or third)

Probability is $\frac{6+60}{8 \cdot 7 \cdot 6} = \frac{16}{56} = \frac{2}{7}$.

You could also do this using combinations:

Size of sample space: $\binom{8}{3} = 56$

No blue balls: $\binom{3}{3} = 1$

One blue ball: $\frac{5 \cdot 3 \cdot 2}{2!} = 15$ ← since the two green are indistinguishable

So the probability would be $\frac{1+15}{56} = \frac{2}{7}$, same as before.

46. Sample space size (all possible 4-card draws): $\binom{52}{4} = \frac{52 \cdot 51 \cdot 50 \cdot 49}{4!}$

Number of combinations with exactly one pair:

$$52 \times 3 \times 48 \times 44 \times \frac{1}{4!}$$

3: must be same number as first choice

48: must be different number from first choice

44: must be different number from first and second choice

So the probability is $\frac{52 \cdot 3 \cdot 48 \cdot 44 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1056}{20825} \approx \frac{1}{20}$

Section 17.3: 4, 6, 12, 16, 20, 30, 34, 40

4. $P(\text{third card is a club} \mid \text{first two are clubs}) = \dfrac{P(\text{all three are clubs})}{P(\text{first two are clubs})}$

$$= \frac{\frac{13}{52}\cdot\frac{12}{51}\cdot\frac{11}{50}}{\frac{13}{52}\cdot\frac{12}{51}} = \frac{11}{50}$$

6. $P(\text{third ball is green} \mid \text{first two are red}) = \dfrac{P(\text{third ball is green and first two are red})}{P(\text{first two are red})}$

$$= \frac{\frac{4}{15}\cdot\frac{3}{14}\cdot\frac{5}{13}}{\frac{4}{15}\cdot\frac{3}{14}} = \frac{5}{13}$$

12. $P(\text{two heads at least} \mid \text{second toss heads}) = \dfrac{P(\text{second toss heads and at least two heads})}{P(\text{second toss heads})}$

$$= \frac{\frac{3}{8}}{\frac{1}{2}} = \frac{3}{4}$$

16. $P(\text{test positive}) = P(\text{test positive} \mid \text{person infected})\,P(\text{person infected})$
$\qquad + P(\text{test positive} \mid \text{person not infected})\,P(\text{person not infected})$

$$= \frac{92}{100}\cdot\frac{1}{600} + \frac{7}{100}\cdot\frac{599}{600} = \frac{857}{12000} \approx 0.07$$

20. $P(\text{blue}) = \displaystyle\sum_{i=1}^{6} P(\text{blue} \mid \text{bag } i)\,P(\text{bag } i)$

$$= \sum_{i=1}^{6} \frac{i}{i+2}\cdot\frac{1}{6} = \frac{1}{6}\left(\frac{1}{3}+\frac{2}{4}+\frac{3}{5}+\frac{4}{6}+\frac{5}{7}+\frac{6}{8}\right)$$

$$= \frac{499}{840}$$

30. $P(A \cap B) = \frac{5}{11} \cdot \frac{5}{11} = P(A)P(B)$, so the events are independent.
This is because we replaced the balls after we drew them.

34. Since A and B are disjoint, $A \cap B = \emptyset$, so $P(A \cap B) = 0$.
Since $P(A)$ and $P(B)$ are greater than 0,
$(P(A))(P(B)) > 0 = P(A \cap B)$. Thus $P(A)P(B) \neq P(A \cap B)$ which means A and B are not independent

40. $P(\text{first card is a spade} \mid \text{second card is a spade})$

$$= \frac{P(\text{both are spades})}{P(\text{second card is a spade})}$$

$$= \frac{\frac{13}{52} \cdot \frac{12}{51}}{\underbrace{\frac{13}{52}}_{\text{first is spade}} \cdot \underbrace{\frac{12}{51}}_{\text{second is spade}} + \underbrace{\frac{39}{52}}_{\text{first isn't spade}} \cdot \underbrace{\frac{13}{51}}_{\text{second is spade}}} = \frac{12}{12+39} = \boxed{\frac{12}{51}}$$

Of course, that should be obvious; the probability that the first card is a spade is the same as the probability that the second card is a spade.

Section 12.4: 4, 6, 12, 20, 24, 34, 62

4.

x	Number of rolls w/ X = x
1	1
2	3
3	5
4	7
5	9
6	11

So the probability mass function is

x	P(X = x)
1	1/36
2	1/12
3	5/36
4	7/36
5	1/4
6	11/36

6.

x	Number of draws w/ X = x
0	$7 \cdot 6 \cdot 5$
1	$7 \cdot 6 \cdot 3 \cdot 3$
2	$7 \cdot 3 \cdot 2 \cdot 3$
3	$3 \cdot 2 \cdot 1$

because you can draw them in any order.

So the probability mass function is

x	P(X = x)
0	7/24
1	21/40
2	7/40
3	1/120

12.

x	P(X = x)
0	0.05
1.3	0.25
1.7	0.55
1.9	0.05
2	0.1

20.

x	$P(X=x)$
0	0.3
1	0.3
2	0.1
3	0.1
4	0.2

(a) $E(X) = \sum_{i=0}^{4} i\,P(i) = 0\cdot 0.3 + 1\cdot 0.3 + 2\cdot 0.1 + 3\cdot 0.1 + 4\cdot 0.2$

$= \boxed{1.6}$

(b) $E(X^2) = \sum_{x=0}^{4} x^2 P(x) = 0^2\cdot 0.3 + 1^2\cdot 0.3 + 2^2\cdot 0.1 + 3^2\cdot 0.1 + 4^2\cdot 0.2$

$= 0.3 + 0.4 + 0.9 + 3.2 = \boxed{4.8}$

(c) $E(2X-1) = 2E(X) - 1 = 2(1.6) - 1 = \boxed{2.2}$

24. (a) $E(X) = \sum_{x=1}^{n} x\,P(X=x) = \sum_{x=1}^{n} \frac{x}{n} = \frac{1}{n}\sum_{x=1}^{n} x = \frac{n(n+1)}{2n} = \frac{n+1}{2}$.

(b) $E(X^2) = \sum_{x=1}^{n} x^2 P(X=x) = \frac{1}{n}\sum_{x=1}^{n} x^2 = \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right]$

So $Var(X) = E(X^2) - [E(X)]^2$

$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$

$= \frac{n+1}{2}\left[\frac{2(2n+1) - 3(n+1)}{6}\right] = \frac{n+1}{2}\left(\frac{n-1}{6}\right)$

$= \boxed{\frac{n^2-1}{12}}$

34. (a) $P(X=2) = \binom{5}{3}(0.3)^3(0.7)^2 = 0.1323$

(b) $P(X \geq 1) = 1 - P(X=0) = 1 - \binom{5}{5}(0.3)^5(0.7)^0$

$= 0.99757$

62. This is a geometric distribution:

$P(X=5) = (1-0.3)^{5-1} \cdot 0.3 = \boxed{0.07203}$

Section 12.5: 2, 4, 6, 12

2. Show that $f(x) = \begin{cases} \frac{1}{2} & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$ is a density function.

Find the corresponding distribution function.

Answer: If f is nonnegative and $\int_{-\infty}^{\infty} f(x)dx = 1$, then f is a density function. Clearly f is nonnegative, and

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^2 \frac{1}{2}\, dx = \frac{1}{2}[x]_0^2 = \frac{1}{2}[2-0] = 1,$$

so f is a density function. The corresponding distribution function is

$$F(x) = \int_{-\infty}^{x} f(u)du = \begin{cases} 0 & x < 0 \\ x/2 & 0 \le x < 2 \\ 1 & x \ge 2 \end{cases}.$$

4. Determine c such that

$$f(x) = \begin{cases} \frac{c}{x^2} & x > 1 \\ 0 & x \le 1 \end{cases}$$

is a density function.

Answer: We want

$$1 = \int_{-\infty}^{\infty} f(x)dx = c\int_1^{\infty} \frac{1}{x^2}dx = c\left[-\frac{1}{x}\right]_1^{\infty}$$

$$= -c \lim_{z\to\infty}\left[\frac{1}{z} - 1\right] = c.$$

So $c = 1$.

6. Let X be a continuous random variable with density function $f(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$. Find $E(X)$ and $\mathrm{var}(X)$.

<u>Answer</u>:

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} x e^{-|x|}$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} x e^{x}\,dx + \int_{0}^{\infty} x e^{-x}\,dx\right].$$

Now $$\int_0^{\infty} x e^{-x}\,dx = \lim_{z\to\infty}\int_0^{z} x e^{-x}\,dx = \lim_{z\to\infty}\left[-xe^{-x}\Big|_0^z + \int_0^z e^{-x}\,dx\right]$$

$$= \lim_{z\to\infty}\left[-ze^{-z} - e^{-z} + 1\right] = 1, \text{ so}$$

$$\int_{-\infty}^{0} xe^{x}\,dx = -\int_0^{-\infty} xe^{x}\,dx = -\int_0^{\infty}(-u)e^{-u}(-du) = -\int_0^{\infty} ue^{-u}\,du$$

$$= -1; \text{ so}$$

$$E(x) = \frac{1}{2}[-1+1] = \boxed{0.}$$

Also, $$\int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|}\,dx = \frac{1}{2}\left[\int_{-\infty}^{0} x^2 e^{x}\,dx + \int_0^{\infty} x^2 e^{-x}\,dx\right]$$

$$= \frac{1}{2}[2! + 2!] = 2$$ (you should be able to do the integrals),

so $$\mathrm{var}(X) = E(X^2) - [E(X)]^2 = 2 - 0 = \boxed{2.}$$

12. Suppose that $f(x)$ is the density function of a normal distribution with mean μ and standard deviation σ. Show that

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx \quad \text{is the mean of the distribution.}$$

<u>Answer</u>: So we need to integrate that, with

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}.$$

So
$$\int_{-\infty}^{\infty} \frac{x}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{(\mu+v)}{\sigma\sqrt{2\pi}} e^{-v^2/2\sigma^2} dv$$

$$= \mu \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-v^2/2\sigma^2} dv + \int_{-\infty}^{\infty} \frac{v}{\sigma\sqrt{2\pi}} e^{-v^2/2\sigma^2} dv$$

$$= \mu \int_{-\infty}^{\infty} f(v)\,dv$$

$$+ \int_{-\infty}^{\infty} \frac{\sigma\sqrt{2}}{\sqrt{\pi}} v e^{-v^2} dv$$

$$= \mu \cdot 1 + \frac{\sigma\sqrt{2}}{2\sqrt{\pi}} \left[-e^{-v^2}\right]_{-\infty}^{\infty}$$

because f is a density function, which means $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$= \mu + \frac{\sigma\sqrt{2}}{2\sqrt{\pi}} \left[\lim_{z\to\infty} \cancelto{0}{\left(-e^{-z^2}\right)} - \lim_{z\to-\infty} \cancelto{0}{\left(-e^{-z^2}\right)}\right]$$

$$= \mu. \quad \square$$