110.107 - Homework #11 Solutions

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Section #1

Section 11.1: 32, 38, 42, 46, 50, 56, 58, 66, 68

In these problems, classify the equilibrium at $(0,0)$ for each matrix.

32. $A = \begin{bmatrix} -5 & -2 \\ 6 & 3 \end{bmatrix}$.

Answer: All you have to do for these problems is find the eigenvalues of the matrix; then use the test at the bottom of page 605 to classify the equilibrium.

$$0 = \det(A - \lambda I_2) = \begin{vmatrix} -5-\lambda & -2 \\ 6 & 3-\lambda \end{vmatrix} = (-5-\lambda)(3-\lambda) - (6)(-2)$$

$$= -15 + 5\lambda - 3\lambda + \lambda^2 + 12 = \lambda^2 + 2\lambda - 3 = (\lambda+3)(\lambda-1).$$

So the eigenvalues are -3 and 1; thus the origin is a saddle point. □

38. $A = \begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix}$.

Answer: $0 = \det(A - \lambda I_2) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (-3-\lambda)(-2-\lambda) - (1)(1)$

$$= \lambda^2 + 5\lambda + 6 - 1 = \lambda^2 + 5\lambda + 5.$$

Eigenvalues are $\lambda = \frac{-5 \pm \sqrt{25-20}}{2} = -\frac{5}{2} \pm \frac{\sqrt{5}}{2}$. Since $\sqrt{5} < 5$, both eigenvalues are negative, which means the origin is a sink. □

42. $A = \begin{bmatrix} 4 & -1 \\ 5 & -1 \end{bmatrix}$.

Answer: $0 = \det(A - \lambda I_2) = \begin{vmatrix} 4-\lambda & -1 \\ 5 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) - (5)(-1)$

$$= -4 - 4\lambda + \lambda + \lambda^2 + 5 = \lambda^2 - 3\lambda + 1.$$

Eigenvalues are $\lambda = \frac{3 \pm \sqrt{5}}{2}$. Since $\sqrt{5} < 3$, both eigenvalues are positive, which means the origin is a source. □

46. $A = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$.

<u>Answer</u>: $0 = \det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 3 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (-2)(3)$

$= \lambda^2 - 2\lambda + 1 + 6 = \lambda^2 - 2\lambda + 7$.

Eigenvalues are $\lambda = \dfrac{2 \pm \sqrt{4-28}}{2} = 1 \pm \sqrt{6}\, i$; thus the origin is an unstable spiral. □

50. $A = \begin{bmatrix} 2 & 2 \\ -6 & -4 \end{bmatrix}$.

<u>Answer</u>: $0 = \det(A - \lambda I_2) = \begin{vmatrix} 2-\lambda & 2 \\ -6 & -4-\lambda \end{vmatrix} = (2-\lambda)(-4-\lambda) - (-6)(2)$

$= -8 - 2\lambda + 4\lambda + \lambda^2 + 12 = \lambda^2 + 2\lambda + 4$.

Eigenvalues are $\lambda = \dfrac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{3}\, i$; thus the origin is a stable spiral. □

56. $A = \begin{bmatrix} 2 & -3 \\ 3 & -2 \end{bmatrix}$.

<u>Answer</u>: $0 = \det(A - \lambda I_2) = \begin{vmatrix} 2-\lambda & -3 \\ 3 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) - (3)(-3)$

$= -4 - 2\lambda + 2\lambda + \lambda^2 + 9 = \lambda^2 + 5 = (\lambda + \sqrt{5}\, i)(\lambda - \sqrt{5}\, i)$.

So the eigenvalues are $\pm\sqrt{5}\, i$, which means the origin is a center. □

58. $A = \begin{bmatrix} -2 & 2 \\ -4 & 3 \end{bmatrix}$.

<u>Answer</u>: $0 = \det(A - \lambda I_2) = \begin{vmatrix} -2-\lambda & 2 \\ -4 & 3-\lambda \end{vmatrix} = (-2-\lambda)(3-\lambda) - (-4)(2)$

$= -6 + 2\lambda - 3\lambda + \lambda^2 + 8 = \lambda^2 - \lambda + 2$.

Eigenvalues are $\lambda = \dfrac{1 \pm \sqrt{7}\, i}{2}$; thus the origin is an unstable spiral. □

66. $A = \begin{bmatrix} 3 & 6 \\ -1 & -4 \end{bmatrix}$.

<u>Answer</u>: $0 = \det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & 6 \\ -1 & -4-\lambda \end{vmatrix} = (3-\lambda)(-4-\lambda) - (-1)(6)$

$= -12 - 3\lambda + 4\lambda + \lambda^2 + 6 = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$.

So the eigenvalues are -3 and 2, which means the origin is a saddle point.

68. The following system has two distinct real eigenvalues, but one eigenvalue is equal to 0:

$$\vec{x}'(t) = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \vec{x}(t).$$

(a) Find both eigenvalues and the associated eigenvectors.
(b) Use the general solution (11.26) to find $x_1(t)$ and $x_2(t)$.
(c) The direction field is shown in Figure 11.32. Sketch the lines corresponding to the eigenvectors. Compute dx_2/dx_1, and conclude that all direction vectors are parallel to the line in the direction of the eigenvector corresponding to the nonzero eigenvalue. Describe in words how solutions starting at different points behave.

<u>Answer</u>: (a) First find the eigenvalues:

$$0 = \det(A - \lambda I_2) = \begin{vmatrix} 2-\lambda & 4 \\ 3 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda) - (3)(4) = 12 - 8\lambda + \lambda^2 - 12$$

$$= \lambda^2 - 8\lambda = \lambda(\lambda - 8).$$

So the eigenvalues are $\boxed{\lambda = 0 \text{ and } \lambda = 8.}$ Now find the eigenvectors:

<u>$\lambda = 0$</u>:

$$\left[\begin{array}{cc|c} 2-0 & 4 & 0 \\ 3 & 6-0 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 3 & 6 & 0 \end{array}\right] \xrightarrow[\frac{1}{2}R_1 \to R_1]{-\frac{3}{2}R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} 1x_1 + 2x_2 = 0 \\ x_1 = -2x_2. \end{array}$$

So an eigenvector for $\lambda = 0$ is $\boxed{\bar{u}_0 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.}$

<u>$\lambda = 8$</u>:

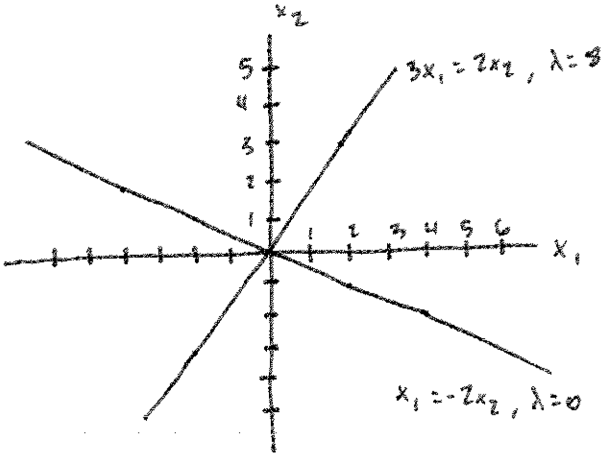
$$\left[\begin{array}{cc|c} 2-8 & 4 & 0 \\ 3 & 6-8 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -6 & 4 & 0 \\ 3 & -2 & 0 \end{array}\right] \xrightarrow[-\frac{1}{6}R_1 \to R_1]{\frac{1}{2}R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} x_1 - \frac{2}{3}x_2 = 0 \\ 3x_1 = 2x_2 \end{array}$$

So an eigenvector for $\lambda = 8$ is $\boxed{\bar{u}_8 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.}$

(b) The general solution is

$$\vec{x}(t) = c_0 \vec{v}_0 e^{0t} + c_8 \vec{v}_8 e^{8t} = c_0 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_8 e^{8t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

(c)



To find dx_2/dx_1, first find $\vec{x}'(t)$:

$$\begin{aligned} \vec{x}'(t) &= A\vec{x}(t) \\ &= c_0 A\vec{v}_0 + c_8 e^{8t} A\vec{v}_8 \\ &= c_0 \cdot (0\vec{v}_0) + c_8 e^{8t}(8\vec{v}_8) \\ &= 8c_8 e^{8t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \end{aligned}$$

Then $$\frac{dx_2}{dx_1} = \frac{dx_2/dt}{dx_1/dt} = \frac{8c_8 e^{8t} \cdot 3}{8c_8 e^{8t} \cdot 2} = \frac{3}{2}.$$

Since $\vec{x}'(t)$ is parallel to $\vec{v}_8$, all the direction vectors are parallel to $\vec{v}_8$. Now suppose $\vec{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix}$. If $\vec{x}(0)$ is parallel to $\vec{v}_0$, then

$\begin{bmatrix} a \\ b \end{bmatrix} = c\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, so we can solve for c_0 and c_8:

$$c\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} c_0 \\ c_8 \end{bmatrix} \Longrightarrow \left[\begin{array}{cc|c} -2 & 2 & -2c \\ 1 & 3 & c \end{array}\right] \xrightarrow{2R_2 + R_1 \to R_1} \left[\begin{array}{cc|c} 0 & 8 & 0 \\ 1 & 3 & c \end{array}\right]$$

$$\xrightarrow{-\frac{3}{8}R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 0 & 8 & 0 \\ 1 & 0 & c \end{array}\right] \longrightarrow c_8 = 0,\ c_0 = c.$$ So the solution is

$\vec{x}(t) = c\begin{bmatrix} -2 \\ 1 \end{bmatrix} + 0e^{8t}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, which means that the solution does not move. So if the initial value of $\vec{x}$ lies on the line $x_1 = -2x_2$, $\vec{x}$ remains constant.

Now suppose $a \neq -2b$, i.e. the initial condition does not lie on the line $x_1 = -2x_2$. We can still solve for c_0 and c_8:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} c_0 \\ c_8 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|c} -2 & 2 & a \\ 1 & 3 & b \end{array}\right] \xrightarrow{+2R_2+R_1 \to R_1} \left[\begin{array}{cc|c} 0 & 8 & a+2b \\ 1 & 3 & b \end{array}\right]$$

$$\xrightarrow{-\frac{3}{8}R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 0 & 8 & a+2b \\ 1 & 0 & b-\frac{3}{8}(a+2b) \end{array}\right] \xrightarrow[\frac{1}{8}R_2 \to R_2]{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{8}(2b-3a) \\ 0 & 1 & \frac{1}{8}(a+2b) \end{array}\right]$$

So the solution is

$$\vec{x}(t) = \frac{1}{8}(2b-3a)\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \frac{1}{8}(a+2b)e^{8t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Since $a \neq -2b$, $a+2b \neq 0$, which means that $|\vec{x}(t)| \to \infty$ as $t \to \infty$, and the solution is a line parallel to $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

<u>Section 12.1:</u> 2, 6, 8, 10, 16, 18, 20, 22, 24, 32, 34, 36, 38

2. (3 levels of leaf damage) $\times$ (4 watering protocols) $\times$ (3 of each combination) $= \boxed{36}$

6. (3 choices of food) $\times$ (12 for each choice) $\times$ (2 genders) $= \boxed{72}$

8. If I want a strand of DNA with n nucleotides, the number of possibilities is 4^n since there are four kinds of nucleotides (as long as you care which is the first nucleotide and which is the last one. If you don't, this problem becomes a bit more difficult.) If one nucleotide has mass $m = 5.6 \cdot 10^{-22}$ g, then the mass of a strand of length n is nm. So if you want to order all possible DNA strands of length n, the mass of your order is $4^n nm$.

The problem wants us to solve for n when $4^n nm = M$ for some mass M. But you can't; the n in the exponent combined with the other n makes this a transcendental equation, so all you can do is stick in numbers. (6)

(a) Here $M = 100kg$, so we want

$$n 4^n \geq \frac{100kg \cdot 10^{22} g^{-1}}{5.6} \approx 1.79 \cdot 10^{26}.$$

Here's a table:

n	$n \cdot 4^n$
1	4
2	32
3	192
⋮	⋮
10	10485760
⋮	⋮
40	$4.83 \cdot 10^{25}$
41	$1.98 \cdot 10^{26}$

So the order exceeds 100kg when $n \geq 41$.

symbol for earth

(b) Here, $M = M_\oplus = 5.97 \cdot 10^{24} kg$.

So we want $n 4^n \geq \dfrac{5.97 \cdot 10^{27} g \cdot 10^{22} g^{-1}}{5.6}$

$= 1.07 \cdot 10^{49}$.

If you keep looking down a table (make one in your calculator), you'll find that the mass exceeds $M_\oplus$ when $n \geq 79$.

10. (5 possibilities for first spot) $\times$ (4 people left to choose) $\times$
$\cdots \times$ (2 people left) $\times$ (1 person left) $= 5! = 120$

16. (10 possibilities for first song) $\times$ (9 songs left) $\times$ (8) $\times$ (7)
$= 5040$ (assuming you don't listen to a song more than once). Also, who carries around a CD player anymore?

18. If the order of the nucleotides matters, and you can choose a nucleotide more than once, the answer is 4^3.
If you can't choose a nucleotide more than once, the answer is $4!$.

20. Assume you don't have to see them in any specific order; the answer is $\binom{60}{5} = \dfrac{60!}{5!\,55!}$

22. Again, the answer is $\binom{52}{5} = \dfrac{52!}{5!\,47!}$.

24. $\dfrac{12!}{5!\,7!} = \binom{12}{5}$.

32. (a) I have to choose three of each kind of bean. The number of ways I can choose three of the same kind of bean is $\dfrac{15 \cdot 14 \cdot 13}{3!} = \dfrac{15!}{3!\,12!} = \binom{15}{3}$.
Since I do this for each kind of bean, the answer is $\binom{15}{3}^3 = \left(\dfrac{15!}{3!\,12!}\right)^3$.

(b) The number of ways I can choose one kind of bean 9 times is $\binom{15}{9} = \frac{15!}{6!9!}$.

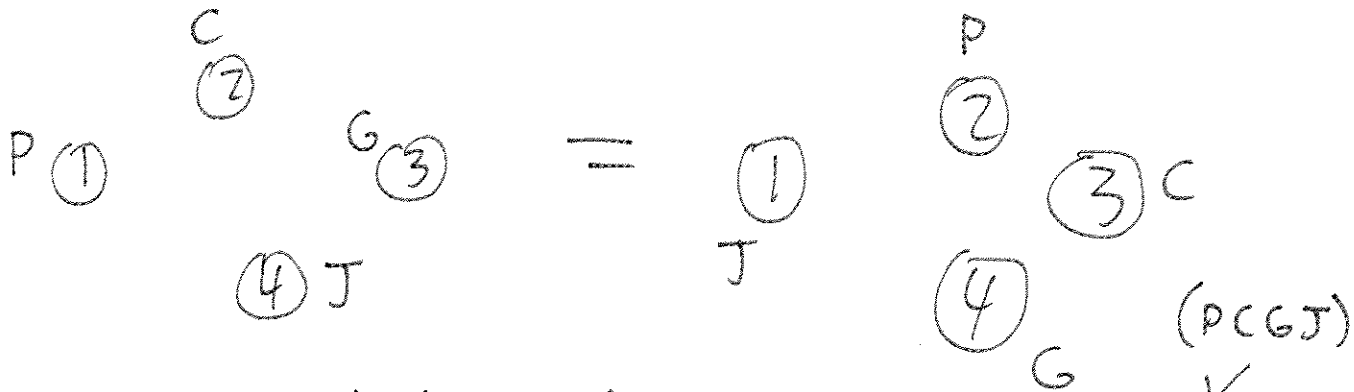
Since there are three kinds of beans, the number of ways I can choose all one kind of bean is $3 \cdot \frac{15!}{6!9!}$.

34. Think of it this way: there are n elements, and for each set A you have to choose whether each of the n elements is in the set. You have two choices per element: yes or no, and there are n elements - so you make n choices. So the number of possible ways you can choose elements (by labeling each one yes or no) is

$$\underbrace{2 \cdot 2 \cdot \cdots \cdot 2 \cdot 2 \cdot 2}_{n \text{ times}} = 2^n.$$

This is why sets are the same as <u>choice functions</u>.

36. The point of this problem is the seating order - it doesn't matter <u>which</u> of the four chairs each person is seated in. We're choosing the seating order. This means that, if you label the chairs, the two choices below



are equivalent. So for each seat placement (PCGJ), there are three others with the same order (JPCG, GJPC, and CGJP). So we have to divide the number of permutations by 4.

Once we choose Paula, though, we've chosen Cindy's spot too, because Cindy must sit next to Paula for some reason. So now we can answer the question:

$$\frac{(4 \text{ choices for Cindy}) \times (1 \text{ for Paula}) \times (2 \text{ for Gloria}) \times (1 \text{ for Jenny})}{4}$$

$= 2$ (poor Jenny; of course, it doesn't matter whether Gloria or Jenny picks first; the number of arrangements is the same.)

38 (a)

$$\frac{(9 \text{ choices for first committee}) \times (8) \times (7) \times (6 \text{ choices for second committee}) \times 5 \times 4}{(3! \text{ ways to order the people in each committee})^{2}}$$

$$= \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{3!\,3!} = \boxed{\frac{9!}{3!\,3!\,3!}}$$

(b)

$$\frac{\underbrace{(9 \cdot 8 \cdot 7)}_{\text{committee \#1}} \cdot \underbrace{(9 \cdot 8 \cdot 7)}_{\text{committee \#2}}}{(3!) \cdot (3!)} = \boxed{\binom{9}{3}\binom{9}{3} = \frac{9!\,9!}{3!\,6!\,3!\,6!}}$$