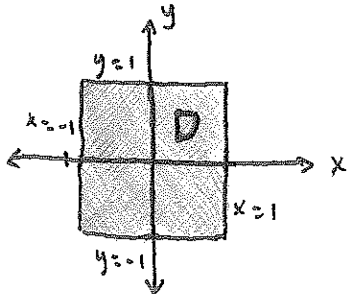
Section 10.6: 14, 18, 20, 24, 28, 32, 34

14. Find the absolute maxima and minima of $f(x,y) = 3 - x + 2y$ on $D = \{(x,y) \in \mathbb{R}^2 : x \in [-1,1], y \in [-1,1]\}$.

Answer: Since D is a closed set, you have to check for critical points both inside D and on the boundary of D. Critical points inside D occur when $\nabla f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. However, $\nabla f = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ everywhere, so there can be no critical points inside D (you should have guessed this – f is the equation of a plane). D is a square, so it has four sides: $x=1$, $y=1$, $x=-1$, and $y=-1$. Substitute each of these values into f to find the critical points on each side:



$x=1$

$f(1,y) = 3 - 1 + 2y = 2 + 2y$

$\frac{d}{dy}[f(1,y)] = 2$

$x=-1$

$f(-1,y) = 3 - (-1) + 2y = 4 + 2y$

$\frac{d}{dy}[f(-1,y)] = 2$

$y=1$

$f(x,1) = 3 - x + 2 = 5 - x$

$\frac{d}{dx}[f(x,1)] = -1$

$y=-1$

$f(x,-1) = 3 - x - 2 = 1 - x$

$\frac{d}{dx}[f(x,-1)] = -1$

So there are no critical points on the sides, either. What's wrong? Well, when we looked for critical points on the sides by finding the points where the derivative was 0, we missed the endpoints of the sides, or the vertices of the square. Just as we have to check the boundary of our two-dimensional region D for critical points, we have to check the endpoints of the one-dimensional sides for critical points.

There are four endpoints: $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$. Just plug them into f to find the values:

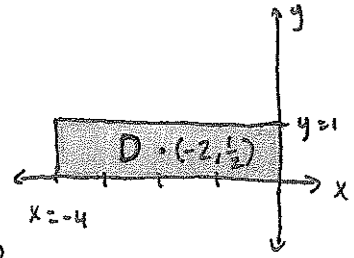
$f(1,1) = 3 - 1 + 2 = 4$; $f(1,-1) = 3 - 1 - 2 = 0$; $f(-1,-1) = 3 + 1 - 2 = 2$; $f(-1,1) = 3 + 1 + 2 = 6$.

$\boxed{\text{So the absolute maximum occurs at } (-1,1) \text{ with value } 6\text{, and the absolute minimum occurs at } (1,-1) \text{ with value } 0.}$ Note that it also makes intuitive sense that the extrema should occur at the endpoints, as f is a plane, and its restriction to the boundary of D consists of lines.

14. Find the absolute maxima and minima of $f(x,y) = x^2 - y^2 + 4x + y$ on the set $D = \{(x,y) \in \mathbb{R}^2 : x \in [-4,0], y \in [0,1]\}$.

<u>Answer</u>: Critical points inside D occur when $\nabla f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$\nabla f = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix} = \begin{bmatrix} 2x+4 \\ -2y+1 \end{bmatrix}$. So the only critical point is $(-2, \frac{1}{2})$, which conveniently lies right in the middle of D. To classify it, compute the Hessian of f:



$$\text{Hess} f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

So $\det(\text{Hess} f) = -4 < 0$, which means that $(-2, \frac{1}{2})$ is a saddle point.

So there are no <u>extrema</u> within D; now check the boundaries:

<u>$x=0$</u>:
$f(0,y) = -y^2 + y$
$\frac{d}{dy}[f(0,y)] = -2y + 1$
Local maximum at $y = \frac{1}{2}$, or $(0, \frac{1}{2})$. Value is $f(0, \frac{1}{2}) = \frac{1}{4}$.

<u>$x = -4$</u>
$f(-4,y) = 16 - y^2 - 16 + y$
$= -y^2 + y$
As with $x=0$, local maximum at $y = \frac{1}{2}$, or $(-4, \frac{1}{2})$.
Value is $f(-4, \frac{1}{2}) = \frac{1}{4}$.

<u>$y=0$</u>
$f(x,0) = x^2 + 4x$
$\frac{d}{dx}[f(x,0)] = 2x + 4$
Local minimum at $x = -2$ or $(-2,0)$. Value is $f(-2,0) = -4$.

<u>$y=1$</u>
$f(x,1) = x^2 + 4x$
As with $y=0$, local minimum at $x = -2$, or $(-2,1)$
Value is $f(-2,1) = -4$.

Finally, check the vertices: $(0,0)$, $(-4,0)$, $(-4,1)$, and $(0,1)$:

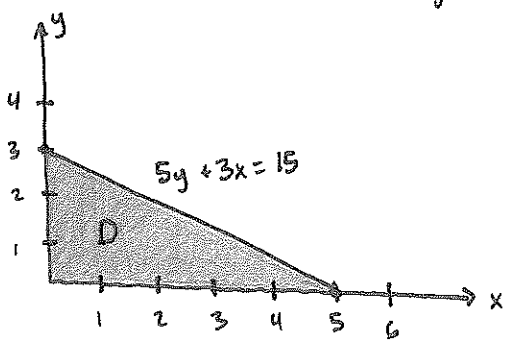
$f(0,0) = 0$; $f(-4,0) = (-4)^2 + 0 + 4(-4) + 0 = 0$; $f(-4,1) = (-4)^2 - 1 + 4(-4) + 1 = 0$

$f(0,1) = 0 - 1 + 4(0) + 1 = 0$. So the vertices all conveniently have value 0.

$\boxed{\text{The absolute maxima are at } (0,\tfrac{1}{2}) \text{ and } (-4,\tfrac{1}{2}) \text{ with value } \tfrac{1}{4}, \text{ and the absolute minima are at } (-2,0) \text{ and } (-2,1) \text{ with value } -4.}$

20. Maximize the function $f(x,y) = xy(15 - 5y - 3x)$ on the triangle bounded by the line $5y + 3x = 15$, the x-axis, and the y-axis.

<u>Answer</u>: First, graph the region. Note that, on each of the sides of the boundary, $f = 0$. This is obvious when $x = 0$ or $y = 0$; on the line $5y + 3x = 15$, $15 - 5y - 3x = 0$, so $f = 0$.



Now find the critical points inside D:

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \nabla f = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix}$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[xy(15 - 5y - 3x)]$$
$$= \frac{\partial}{\partial x}[15xy - 5xy^2 - 3x^2 y] = 15y - 5y^2 - 6xy$$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[15xy - 5xy^2 - 3x^2y\right] = 15x - 10xy - 3x^2$. So we have the system

$$\begin{cases} 15y - 5y^2 - 6xy = 0 & (1) \\ 15x - 10xy - 3x^2 = 0 & (2) \end{cases}$$

First look at (1): We can factor out a y: $y(15 - 5y - 6x) = 0$. So either $y = 0$ or $15 - 5y - 6x = 0$. If $y = 0$, then equation (2) gives $15x - 10x(0) - 3x^2 = 0$, or $15x - 3x^2 = 0 \Rightarrow x(5 - x) = 0$. So $x = 0$ or $x = 5$. So we found two critical points: $(0,0)$ and $(5,0)$. Both of these lie on the boundary and so have $f = 0$.

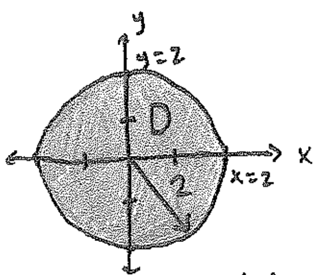
Going back to (1), we see that we have another option: $15 - 5y - 6x = 0$, or $5y = 15 - 6x$. We can substitute this into (2):

$$15x - 10xy - 3x^2 = 0$$
$$15x - 2x(5y) - 3x^2 = 0$$
$$15x - 2x(15 - 6x) - 3x^2 = 0$$
$$12x^2 + 15x - 30x - 3x^2 = 0$$
$$9x^2 - 15x = 0$$
$$x(3x - 5) = 0$$
$$x = 0, \tfrac{5}{3}$$

When $x = 0$, we get the solution $5y = 15$, or $y = 3$.
When $x = \frac{5}{3}$, $5y = 15 - 6\left(\frac{5}{3}\right) = 5$, so $y = 1$.
We get two more critical points: $(0,3)$ and $\left(\frac{5}{3}, 1\right)$.
$(0,3)$ lies on the boundary and so has value 0.
We have to check whether $\left(\frac{5}{3}, 1\right)$ lies in D.
We know that it lies in the first quadrant, but we also need that $5y + 3x \le 15$.
Since $5(1) + 3\left(\frac{5}{3}\right) = 10 < 15$, $\left(\frac{5}{3}, 1\right)$ lies within our domain. Finally, we don't have to compute the Hessian; only one of our critical points is inside the boundary. The rest are on the boundary and have value 0. $\boxed{\text{So } \left(\tfrac{5}{3}, 1\right) \text{ is the absolute maximum with value } 25/3.}$

24. Find the absolute maxima and minima of $f(x,y) = x^2 + y^2 + x + 2y$ on the disk $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 4\}$.

Answer: First, graph D. You should know what D looks like. Now find the critical points of f inside D:



(Pretend this is a circle.)

$$\nabla f = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix} = \begin{bmatrix} 2x + 1 \\ 2y + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So the only critical point is $\left(-\frac{1}{2}, -1\right)$, which does indeed lie within D. You should be able to tell that the absolute minimum of f occurs here, but I'll go ahead and compute the Hessian:

$$\text{Hess } f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \quad \begin{matrix} tr(\text{Hess } f) = 4 > 0 \\ \det(\text{Hess } f) = 4 > 0 \end{matrix}.$$

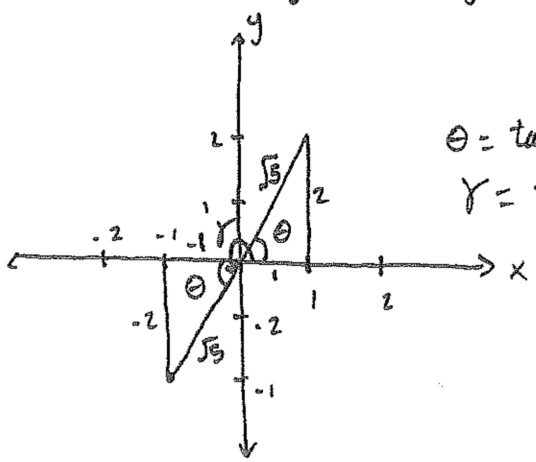
So by the test on p. 553, $\left(-\frac{1}{2}, -1\right)$ is a local minimum with value $-5/4$.

Now we have to find critical points on the boundary, where $x^2 + y^2 = 4$. Parametrize the boundary as follows: $\vec{r}(t) = (2\cos t, 2\sin t)$, $t \in [0, 2\pi]$. Then $f(\vec{r}(t)) = (2\cos t)^2 + (2\sin t)^2 + 2\cos t + 2(2\sin t) = 4 + 2\cos t + 4\sin t$.

Critical points occur when $f'(t)=0$:

$\frac{d}{dt}[f(\vec{r}(t))] = \frac{d}{dt}[4 + 2\cos t + 4\sin t] = -2\sin t + 4\cos t$

$0 = -2\sin t + 4\cos t$. Since $\cos t = 0$ does not give a solution, we can divide by $\cos t$ to get $\tan t = 2$. So there are two solutions: $t = \tan^{-1} 2$ and $t = \pi + \tan^{-1} 2$. If you don't understand why, hopefully this diagram will explain:



$\theta = \tan^{-1} 2$

$\gamma = \pi + \theta$

Using the triangles in the diagram, you should be able to see that

$\cos(\tan^{-1} 2) = \frac{1}{\sqrt{5}}$, $\sin(\tan^{-1} 2) = \frac{2}{\sqrt{5}}$

$\cos(\tan^{-1} 2 + \pi) = -\frac{1}{\sqrt{5}}$, $\sin(\tan^{-1} 2 + \pi) = -\frac{2}{\sqrt{5}}$.

So $f(\vec{r}(\tan^{-1} 2)) = 4 + 2\cos(\tan^{-1} 2) + 4\sin(\tan^{-1} 2)$

$= 4 + \frac{2}{\sqrt{5}} + \frac{8}{\sqrt{5}} = 4 + \frac{10}{\sqrt{5}}$

$= 4 + 2\sqrt{5}$,

and $f(\vec{r}(\tan^{-1} 2 + \pi)) = 4 + 2\cos(\tan^{-1} 2 + \pi) + 4\sin(\tan^{-1} 2 + \pi) = 4 - \frac{2}{\sqrt{5}} - \frac{8}{\sqrt{5}} = 4 - 2\sqrt{5}$.

So we know that the absolute maximum on D occurs at $t = \tan^{-1} 2$, or $(x,y) = \left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$, with value $4 + 2\sqrt{5}$.

But what about the absolute minimum? Well, the second derivative test tells us that $(-\frac{1}{2}, -1)$ is the only critical point of f over all of $\mathbb{R}^2$, so that should be the absolute minimum on D. But let's make sure that $-\frac{5}{4} < 4 - 2\sqrt{5}$:

$-\frac{5}{4} < 4 - 2\sqrt{5} \Rightarrow 2\sqrt{5} < 4 + \frac{5}{4} \Rightarrow 2\sqrt{5} < \frac{21}{4} \Rightarrow 8\sqrt{5} < 21$

$\Rightarrow 64 \cdot 5 < 21^2 \Rightarrow 320 < 441$, which is true. Thus $-\frac{5}{4} < 4 - 2\sqrt{5}$, and

the absolute minimum of f on D occurs at $(-\frac{1}{2}, -1)$ with value $-\frac{5}{4}$.

28. Choose three numbers x, y, z so that their sum is equal to 60 and their product is maximal.

<u>Answer</u>: We have that $x+y+z=60$, and we want to maximize $f(x,y,z)=xyz$.
First, solve for z in terms of x and y: $z=60-x-y$. Then f is a function of two variables: $f(x,y)=xy(60-x-y)$. Now find the critical points of f:

$$\begin{bmatrix}0\\0\end{bmatrix}=Df=\begin{bmatrix}60y-2xy-y^2\\60x-x^2-2xy\end{bmatrix}.\quad \text{So we get the system } \begin{cases}60y-2xy-y^2=0 & (1)\\ 60x-x^2-2xy=0 & (2)\end{cases}.$$

First manipulate (1). We can factor out a y to get $y(60-2x-y)=0$, so either $y=0$ or $(60-2x-y)=0$. If $y=0$, equation (2) gives $60x-x^2=0$, or $x(60-x)=0$ so we have two critical points: $(0,0)$ and $(60,0)$.

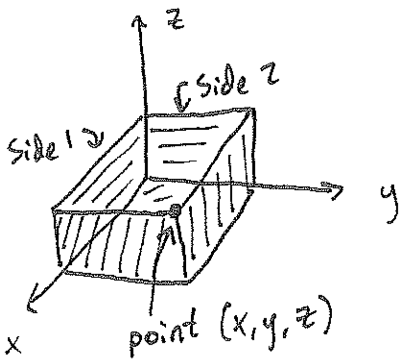
The other option from equation (1) was that $60-2x-y=0$, or $y=60-2x$. Substitute that into equation (2):

$$\begin{aligned}&60x-x^2-2x(60-2x)=0\\ \Rightarrow\ &60x-x^2-120x+4x^2=0\\ \Rightarrow\ &3x^2-60x=0\\ \Rightarrow\ &x(x-20)=0.\end{aligned}$$

So we get two more critical points: $(0,60)$ and $(20,20)$. You can evaluate the types of the points if you want, or you could see that three of them give $xyz=0$ and one ($(20,20)$) gives $xyz=20^3=8000$.
Thus the correct choice is $\boxed{x=20,\ y=20,\ z=20.}$

32. Find the minimum surface area of a rectangular open (bottom and four sides, no top) box with volume $256\,m^3$.

<u>Answer</u>: First, ignore the $256\,m^3$; let the volume be V. Let $V(x,y,z)=xyz$; what's the surface area? Well, the box has four rectangular sides and a bottom; just add their areas:



$$A(x,y,z)=\underset{\text{(bottom)}}{xy}+\underset{(2\cdot\text{side }1)}{2xz}+\underset{(2\cdot\text{side }2)}{2yz}.$$

But we fixed $xyz=V$, so we can let $z=\dfrac{V}{xy}$. (We can do this because $xy\neq 0$. Why?)

Now substitute $z=\dfrac{V}{xy}$ into our area function:

$A(x,y) = xy + 2x\left(\frac{V}{xy}\right) + 2y\left(\frac{V}{xy}\right) = xy + 2V\left(\frac{1}{x} + \frac{1}{y}\right).$

Now find the critical points of A:

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \nabla A = \begin{bmatrix} y - \frac{2V}{x^2} \\ x - \frac{2V}{y^2} \end{bmatrix}$. We get the equations $\begin{cases} y - \frac{2V}{x^2} = 0 & (1) \\ x - \frac{2V}{y^2} = 0 & (2) \end{cases}$.

So, from (1), $y = \frac{2V}{x^2}$. Substituting that into (2) gives $x = \frac{2V}{\left(\frac{2V}{x^2}\right)^2}$,

or $x = \frac{x^4}{2V} \Rightarrow x^3 = 2V$. So $x = (2V)^{1/3}$, and

$y = \frac{2V}{(2V)^{2/3}} = (2V)^{1/3}$. Now we need to check that this is a minimum:

$$\text{Hess } A = \begin{bmatrix} \frac{\partial^2 A}{\partial x^2} & \frac{\partial^2 A}{\partial x \partial y} \\ \frac{\partial^2 A}{\partial x \partial y} & \frac{\partial^2 A}{\partial y^2} \end{bmatrix} = \begin{bmatrix} +\frac{4V}{x^3} & 1 \\ 1 & \frac{4V}{y^3} \end{bmatrix}.$$

So $\det(\text{Hess } A) = \frac{16V^2}{x^3 y^3} - 1$, and $\text{tr}(\text{Hess } A) = 4V\left(\frac{1}{x^3} + \frac{1}{y^3}\right)$.

At $((2V)^{1/3}, (2V)^{1/3})$, $\det(\text{Hess } A) = \frac{16V^2}{4V^2} - 1 = 4 - 1 = 3 > 0$.

Also, $\text{tr}(\text{Hess } A) = 4V\left(\frac{1}{2V} + \frac{1}{2V}\right) = 4 > 0$. Thus $((2V)^{1/3}, (2V)^{1/3})$

is indeed a minimum. The value of A at this point is

$A((2V)^{1/3}, (2V)^{1/3}) = (2V)^{2/3} + 2V\left(\frac{2}{(2V)^{1/3}}\right) = 3(2V)^{2/3}$.

In our case $V = 256\,m^3$, so $A = 3(2 \cdot 256\,m^3)^{2/3} = 3(2^9 m^3)^{2/3} = 3 \cdot 2^6 m^2$

$= \boxed{192\,m^2.}$

34. Given the symmetric matrix $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$, where $a, b, c \in \mathbb{R}$, show that the eigenvalues of A are real. (Hint: compute the eigenvalues.)

Answer: Do what the hint says: eigenvalues are the solutions λ to the equation $\det(A - \lambda I_2) = 0$:

$$0 = \begin{vmatrix} a-\lambda & c \\ c & b-\lambda \end{vmatrix} = (a-\lambda)(b-\lambda) - c^2 = ab - (a+b)\lambda + \lambda^2 - c^2 = \lambda^2 - (a+b)\lambda + (ab - c^2)$$

$$\lambda = \frac{a+b \pm \sqrt{(a+b)^2 - 4(ab-c^2)}}{2} = \frac{a+b \pm \sqrt{a^2 + 2ab + b^2 - 4ab + 4c^2}}{2}$$

$$= \frac{a+b \pm \sqrt{a^2 - 2ab + b^2 + 4c^2}}{2} = \frac{a+b \pm \sqrt{(a-b)^2 + 4c^2}}{2}.$$

Since $(a-b)^2 + 4c^2 \geq 0$ for all real numbers, λ must be a real number. Thus the eigenvalues of A are real.

Section 11.1: 2, 4, 6, 10, 14, 16, 20, 24

2. Write the system $\frac{dx_1}{dt} = x_1 + x_2$, $\frac{dx_2}{dt} = -2x_2$ in matrix form.

Answer: $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, or $\vec{x}'(t) = A\vec{x}(t)$, where $A = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$.

4. Write the system $\frac{dx_1}{dt} = 2x_2 - 3x_1 - x_3$, $\frac{dx_2}{dt} = -x_1 + x_2$, $\frac{dx_3}{dt} = 5x_1 + x_3$ in matrix form.

Answer: $\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 & -1 \\ -1 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$, or $\vec{x}'(t) = A\vec{x}(t)$, where $A = \begin{bmatrix} -3 & 2 & -1 \\ -1 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$.

6. Consider $\frac{dx_1}{dt} = 2x_1 - x_2$, $\frac{dx_2}{dt} = -x_2$. Determine the direction vectors associated with the following points in the x_1x_2-plane, and graph the direction vectors in the x_1x_2-plane: $(2,0)$, $(1.5,1)$, $(1,0)$, $(0,-1)$, $(1,1)$, $(0,0)$, and $(-2,-2)$.

Answer: First, write the system in matrix form: $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

Then it will be easier to compute the 7 direction vectors:

$(2,0)$: $\begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

$(1.5,1)$: $\begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$(1,0)$: $\begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

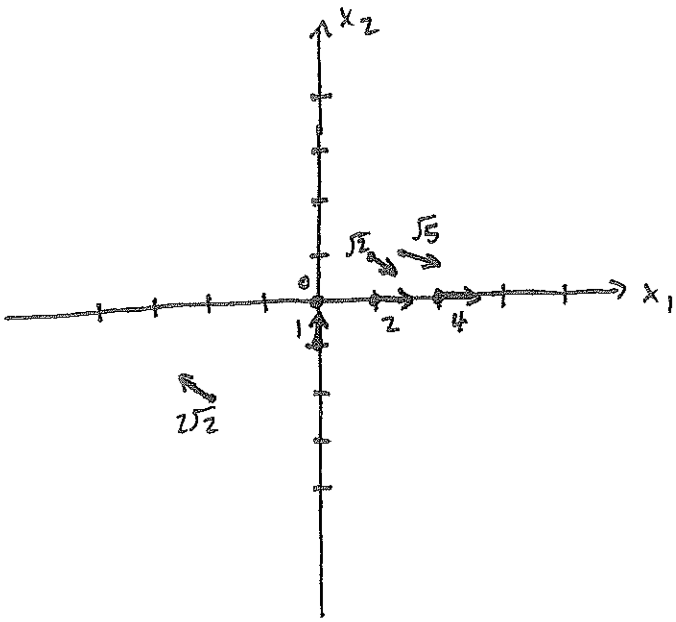
$(0,-1)$: $\begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$(1,1)$: $\begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$(0,0)$: $\begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

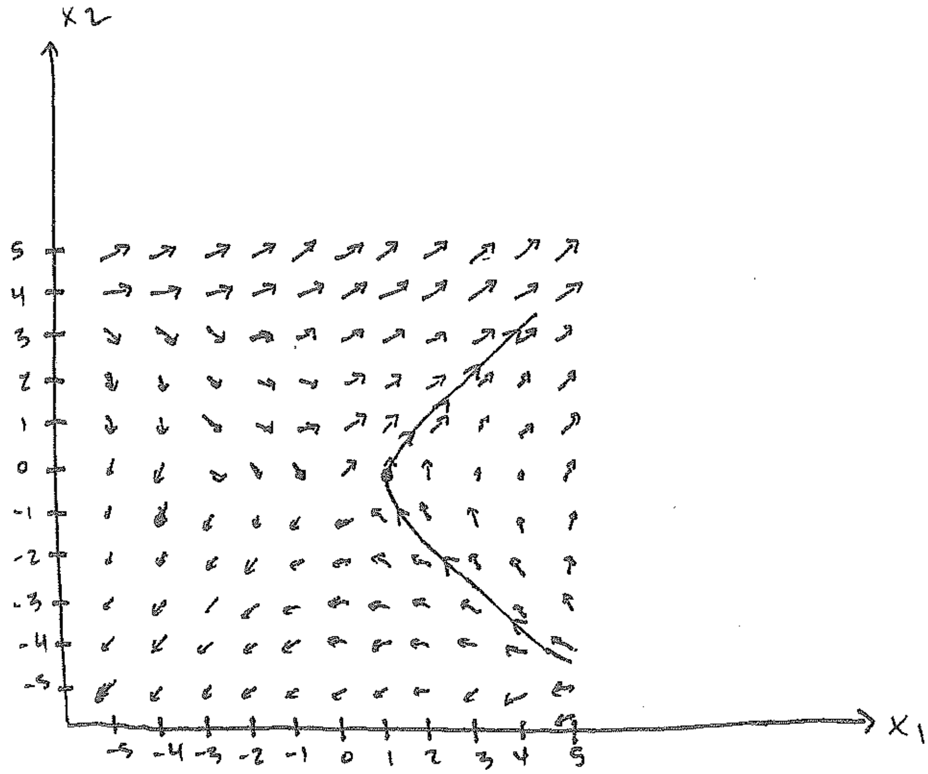
$(-2,-2)$: $\begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} -4+2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

I am going to draw all the vectors on the same graph, but I will not draw them to scale; I'll put the length next to the vector.



10. The direction field of $\frac{dx_1}{dt} = x_1 + 3x_2$, $\frac{dx_2}{dt} = 2x_1 + 3x_2$ is given in figure 11.22. Sketch the solution curve that goes through the point $(1,0)$.

Answer: I'm going to try to redraw the direction field here:



You can see that the origin is a saddle point. To draw the solution curve through $(1,0)$, just follow the direction of the arrows with your pencil. What you get will be approximately correct.

14. Find the general solution to $\begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. Sketch the lines in the direction of the eigenvectors and indicate the direction a solution on that line would move.

Answer: First, find the eigenvalues:

$$0 = \begin{vmatrix} 2-\lambda & 1 \\ 4 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) - 4 = -2 - 2\lambda + \lambda^2 - 4 = \lambda^2 - \lambda - 6 = (\lambda+2)(\lambda-3)$$

So eigenvalues are $\lambda = -2, 3$. Now find the eigenvectors (use the augment matrix)

$\lambda = -2$: $\left[\begin{array}{cc|c} 2-(-2) & 1 & 0 \\ 4 & -1-(-2) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 4 & 1 & 0 \\ 4 & 1 & 0 \end{array}\right] \xrightarrow{-R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 4 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to 4x_1 + x_2 = 0$, so $x_2 = -4x_1$.

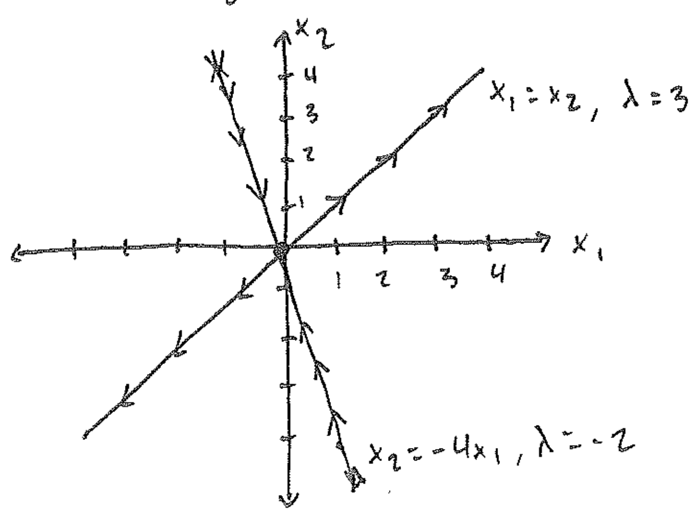
If $x_1 = 1$, $x_2 = -4$, so an eigenvector is $\vec{v}_{-2} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$.

$\lambda = 3$: $\left[\begin{array}{cc|c} 2-3 & 1 & 0 \\ 4 & -1-3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 4 & -4 & 0 \end{array}\right] \xrightarrow{4R_1+R_2 \to R_2} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to x_1 = x_2$,

so an eigenvector is $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The general solution is therefore $\vec{x}(t) = c_{-2}\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$.

If I start along $\vec{v}_{-2}$ ($c_3 = 0$), then as $t \to \infty$, $e^{-2t} \to 0$, so the solution approaches zero. If I start along $\vec{v}_3$ ($c_{-2} = 0$), then $e^{3t} \to \infty$ as $t \to \infty$, so the solution diverges. The graph looks like this.



16. Find the general solution of $\begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ and sketch the lines in the direction of the eigenvectors. Indicate on the lines the direction in which the solution would move if it starts on that line.

Answer: First, find the eigenvalues:

$$0 = \begin{vmatrix} -5-\lambda & 3 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 + 5\lambda + 6 = (\lambda+2)(\lambda+3).$$ So the eigenvalues are $\lambda = -2, -3$.

Now compute the eigenvectors:

$\lambda = -2$: $\left[\begin{array}{cc|c} -5+2 & 3 & 0 \\ -2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -3 & 3 & 0 \\ -2 & 2 & 0 \end{array}\right] \xrightarrow{-\frac{2}{3}R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x_1 = x_2,$

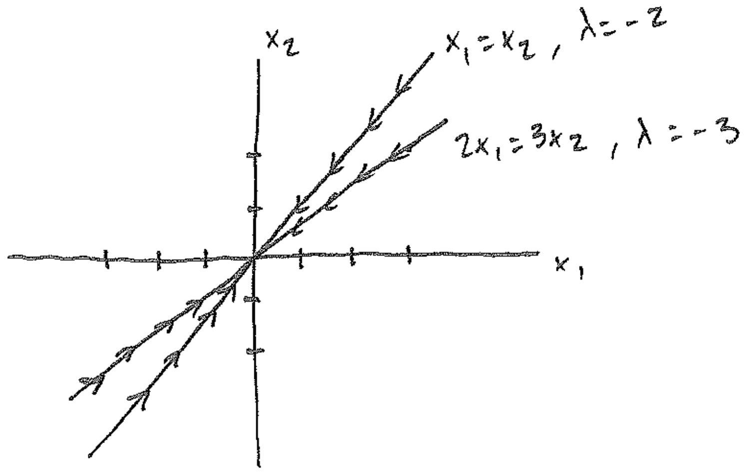
so an eigenvector is $\vec{v}_{-2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\lambda = -3$: $\left[\begin{array}{cc|c} -5+3 & 3 & 0 \\ -2 & 3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2 & 3 & 0 \\ -2 & 3 & 0 \end{array}\right] \xrightarrow{-R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow 2x_1 = 3x_2$

so an eigenvector is $\vec{v}_{-3} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

The general solution is therefore $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-3t}$.

As $t \to \infty$, both e^{-2t} and $e^{-3t} \to 0$, so all solutions along the direction ~~both~~ of the eigenvectors tend to 0:



16. Solve the initial value problem $\vec{x}'(t) = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \vec{x}(t)$, when $\vec{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Answer: First find the eigenvalues:

$$0 = \begin{vmatrix} 1-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) \rightarrow \lambda = 1, 2.$$

Now find the eigenvectors:

$\lambda = 1$: $\left[\begin{array}{cc|c} 1-1 & 3 & 0 \\ 0 & 2-1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & 1 & 0 \end{array}\right] \rightarrow x_2 = 0$, so set $x_1 = 1$;

the eigenvector is $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\lambda = 2$: $\left[\begin{array}{cc|c} 1-2 & 3 & 0 \\ 0 & 2-2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x_1 = 3x_2$, so an eigenvector is

$\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

The general solution is

$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t}$. When $t = 0$,

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{0} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{0} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

So $c_2 = -1$ and $c_1 = 5$, and the solution is

$$\boxed{\vec{x}(t) = 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t}.}$$

24. Solve the initial value problem $\vec{x}'(t) = \begin{bmatrix} -3 & 4 \\ -1 & 2 \end{bmatrix} \vec{x}(t)$, when $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Answer: First, find the eigenvalues:

$$0 = \begin{vmatrix} -3-\lambda & 4 \\ -1 & 2-\lambda \end{vmatrix} = -6 + 3\lambda - 2\lambda + \lambda^2 + 4 = \lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1).$$

So $\lambda = -2, 1$.

Now find the eigenvectors:

$\lambda = -2$: $\left[\begin{array}{cc|c} -3+2 & 4 & 0 \\ -1 & +2+2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 4 & 0 \\ -1 & 4 & 0 \end{array}\right] \xrightarrow{-R_1+R_2 \to R_2} \left[\begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \to x_1 = 4x_2$

So an eigenvector is $\vec{v}_{-2} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

$\lambda = 1$: $\left[\begin{array}{cc|c} -3-1 & 4 & 0 \\ -1 & 2-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -4 & 4 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow{-4R_2+R_1 \to R_1} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 1 & 0 \end{array}\right] \to x_1 = x_2$.

So an eigenvector is $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The general solution is

$\vec{x}(t) = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$. When $t = 0$,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

$$\left[\begin{array}{cc|c} 4 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{-4R_2+R_1 \to R_1} \left[\begin{array}{cc|c} 0 & -3 & -7 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{-\frac{1}{3}R_1 \to R_1} \left[\begin{array}{cc|c} 0 & 1 & 7/3 \\ 1 & 1 & 2 \end{array}\right]$$

$\xrightarrow{-R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 0 & 1 & 7/3 \\ 1 & 0 & -1/3 \end{array}\right]$. So $c_1 = -\frac{1}{3}$ and $c_2 = \frac{7}{3}$, and

the solution is $\boxed{\vec{x}(t) = -\frac{1}{3}\begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{-2t} + \frac{7}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}.}$