3.4.2. Let $f(x) = x\cos\frac{1}{x}$, $x \neq 0$.

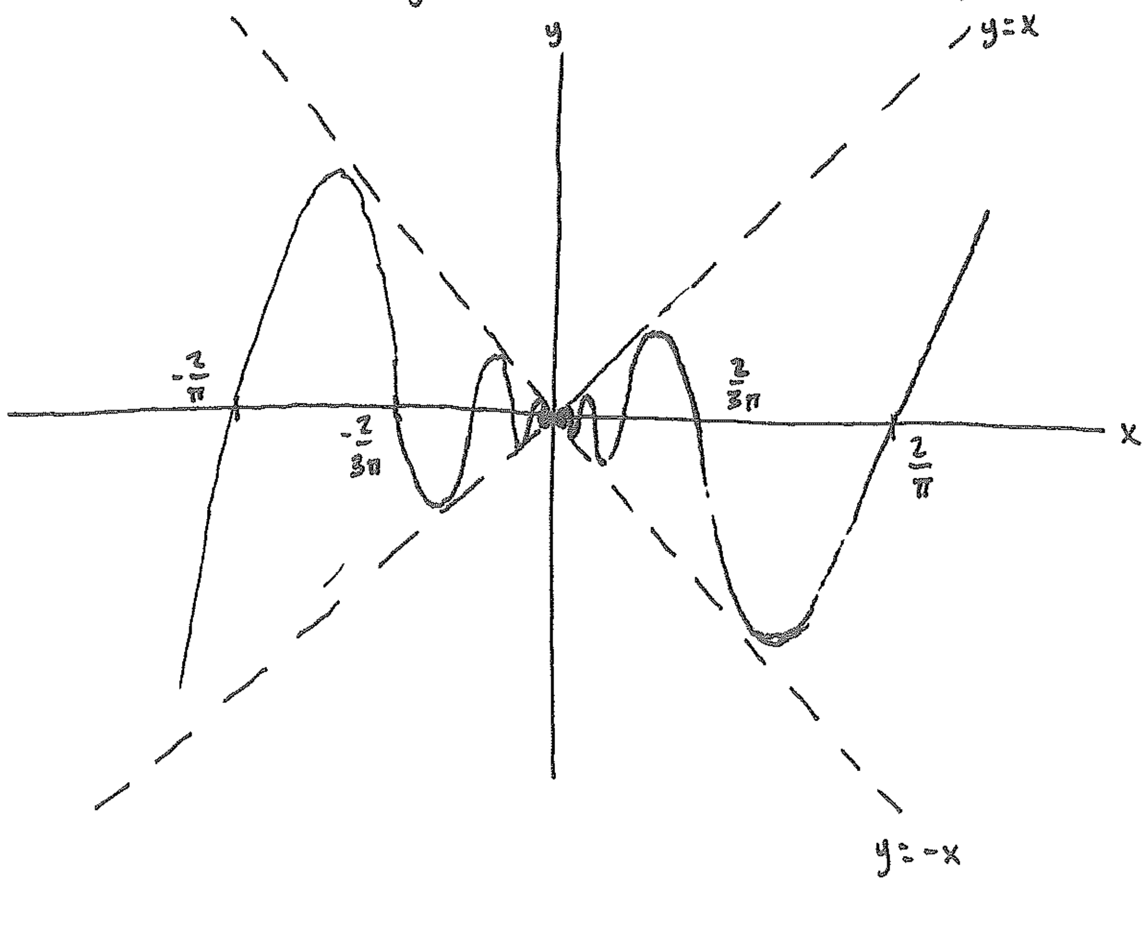
(a) Use a graphing calculator to sketch the graph of $y = f(x)$.

Answer: To draw this, the first thing you should note is that $f(x)$ is an odd function, i.e. $f(x) = -f(-x)$. This is because $\cos\frac{1}{x}$ is even and x is odd, and the product of an even function and an odd function is odd.

The second thing you should note is that $f(x) = 0$ whenever $\frac{1}{x} = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$, because $\cos(\frac{\pi}{2} + n\pi) = 0$. So $f\left(\frac{1}{\frac{\pi}{2} + n\pi}\right) = 0$ for all $n \in \mathbb{Z}$.

Also, since $\cos(n\pi) = (-1)^n$, $f\left(\frac{1}{n\pi}\right) = \frac{1}{n\pi}(-1)^n$, $n \in \mathbb{Z}$, $n \neq 0$.

Finally, since $|\cos\frac{1}{x}| \leq 1$, $|f(x)| = |x\cos\frac{1}{x}| \leq |x|$ for all $x \in \mathbb{R}$, $x \neq 0$, and we already saw that $|f(x)| = |x|$ if and only if $x = \frac{1}{n\pi}$, $n \in \mathbb{Z}$, $n \neq 0$. This is enough information to draw the graph:



(b) Use the sandwich theorem to show that $\lim_{x \to 0} x \cos \frac{1}{x} = 0$.

Answer: The sandwich theorem is stated in section 3.4, on page 114:

Sandwich Theorem: If $f(x) \le g(x) \le h(x)$ for all x in an open interval that contains c (except possibly at c) and

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$$

then

$$\lim_{x \to c} g(x) = L$$

We already stated that $|x \cos \frac{1}{x}| \le |x|$ for all $x \in \mathbb{R}, x \neq 0$; rewriting it gives

$$-|x| \le x \cos \frac{1}{x} \le |x|.$$

Since $\lim_{x \to 0} |x| = \lim_{x \to 0} -|x| = 0$, $\lim_{x \to 0} x \cos \frac{1}{x} = 0$ by the sandwich theorem.

3.4.6. Calculate the limit $\lim_{x \to 0} \frac{\sin(2x)}{3x}$.

Answer: Make the substitution $z = 2x$; this works because $z = 2x \to 0$ as $x \to 0$. The limit becomes

$$\lim_{x \to 0} \frac{\sin(2x)}{3x} = \lim_{z \to 0} \frac{\sin z}{3z/2} = \frac{2}{3} \lim_{z \to 0} \frac{\sin z}{z} = \frac{2}{3}$$

since $\lim_{z \to 0} \frac{\sin z}{z} = 1$. So $\boxed{\lim_{x \to 0} \frac{\sin(2x)}{3x} = \frac{2}{3}}$.

7.1.6. Evaluate the indefinite integral $\int 5\sin(1-2x)\,dx$

by making the substitution $u = 1-2x$.

Answer: Setting $u = 1-2x$ gives $\frac{du}{dx} = -2$, or $du = -2dx$.
Substituting into the integral gives

$$\int 5\sin(1-2x)\,dx = \int 5\sin u\left(-\frac{1}{2}\,du\right) = \frac{5}{2}\int(-\sin u)\,du$$

$$= \frac{5}{2}\cos u + C = \boxed{\frac{5}{2}\cos(1-2x) + C}.$$

7.1.22. Use substitution to evaluate the indefinite integral

$$\int \frac{x^2-1}{x^3-3x+1}\,dx.$$

Answer: Let $u = x^3-3x+1$; then $\frac{du}{dx} = 3x^2-3$, or $du = (3x^2-3)dx$.
Dividing by 3 gives $\frac{1}{3}du = (x^2-1)\,dx$, and now we can substitute everything in:

$$\int \frac{x^2-1}{x^3-3x+1}\,dx = \int \frac{\frac{1}{3}du}{u} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C$$

$$= \boxed{\frac{1}{3}\ln|x^3-3x+1| + C}.$$

7.2.3. Use integration by parts to evaluate $\int 2x\cos(3x-1)\,dx$.

Answer: The integration by parts formula is on page 335, in section 7.2:

> Integration by Parts If $u(x)$ and $v(x)$ are differentiable functions, then
>
> $$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$
>
> or in short form,
>
> $$\int u\,dv = uv - \int v\,du.$$

I will make the following choices for u and v:

$u = 2x$ $\qquad$ $dv = \cos(3x-1)\,dx$

$du = 2\,dx$ $\qquad$ $v = \int dv = \int \cos(3x-1)\,dx$. Let $w = 3x-1$; then $dw = 3dx$, or $\frac{1}{3}dw = dx$. The integral becomes

$$\int \cos(3x-1)\,dx = \int \cos w\left(\tfrac{1}{3}dw\right) = \frac{1}{3}\int \cos w\,dw$$
$$= \frac{1}{3}\sin w = \frac{1}{3}\sin(3x-1)$$

That choice will make the integral $\int v\,du$ simpler than $\int u\,dv$, as we differentiated away a power of x. (Note that I left out the constant when I finished integrating for v. I will explain why in a moment.)

Substituting in gives

$$\int 2x\cos(3x-1)\,dx = (2x)\left[\frac{1}{3}\sin(3x-1)\right] - \int\left[\frac{1}{3}\sin(3x-1)\right](2\,dx)$$

$$= \frac{2}{3}x\sin(3x-1) + \frac{2}{3}\int[-\sin(3x-1)]\,dx$$

Let $z = 3x-1$
$dz = 3dx \Rightarrow \frac{1}{3}dz = dx$

$$= \frac{2}{3}x\sin(3x-1) + \frac{2}{3}\int(-\sin z)\left(\frac{1}{3}dz\right)$$

$$= \frac{2}{3}x\sin(3x-1) + \frac{2}{9}\int(-\sin z)\,dz$$

$$= \frac{2}{3}x\sin(3x-1) + \frac{2}{9}\cos z + C$$

$$= \boxed{\frac{2}{3}x\sin(3x-1) + \frac{2}{9}\cos(3x-1) + C}.$$

So why did I leave out the constant when I finished integrating for v? Well, say we go back to $u(x)$ and $v'(x)$; we have the integral $\int u(x)v'(x)\,dx$. The Fundamental Theorem of Calculus says (something like) $\int v'(x) = v(x) + C$. So say we use $v(x)+C$ instead of $v(x)$ in our formula:

$$\int u(x)v'(x)\,dx = u(x)[v(x)+C] - \int u'(x)[v(x)+C]\,dx$$

$$= u(x)v(x) + Cu(x) - C\int u'(x)\,dx - \int u'(x)v(x)\,dx$$

$$= u(x)v(x) + Cu(x) - C[u(x)+C_1] - \int u'(x)v(x)\,dx$$

$$= u(x)v(x) + Cu(x) - Cu(x) - CC_1 - \int u'(x)v(x)\,dx$$

$$= u(x)v(x) - \int u'(x)v(x)\,dx.$$

So the term with $Cu(x)$ cancels out, and CC_1 can be absorbed into the integral $\int u'(x)v(x)$. So in the end, the constant C didn't affect our answer; that's why we let $C=0$ when we solve for v.

7.2.12. Use integration by parts to evaluate the integral $\int x^2 \ln x \, dx$.

Answer: Let

$$u = \ln x \qquad dv = x^2 dx$$
$$du = \frac{dx}{x} \qquad v = \int x^2 dx = \frac{x^3}{3}$$;

the integral becomes

$$\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3\left(\frac{dx}{x}\right) = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$$
$$= \boxed{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}.$$

7.3.14. Use partial-fraction decomposition to evaluate $\int \frac{1}{x(2x+1)}dx$.

Answer: Set up the decomposition as follows:

$$\frac{1}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}$$
$$= \frac{A(2x+1) + B(x)}{x(2x+1)} = \frac{2Ax + A + Bx}{x(2x+1)}$$
$$= \frac{(2A+B)x + A}{x(2x+1)}$$

We want $(2A+B)x + A = 1$; write out a system of equations by equating the coefficients of like powers of x:

$$\begin{cases} A = 1 \\ 2A + B = 0 \implies 2(1) + B = 0 \implies B = -2. \end{cases}$$

So our integral becomes

$$\int \frac{1}{x(2x+1)}\,dx = \int\left(\frac{1}{x} - \frac{2}{2x+1}\right)dx = \int\frac{dx}{x} - \int\frac{2}{2x+1}\,dx$$

$$= \ln|x| - \int \frac{2}{2x+1}\,dx \quad \left|\ \begin{array}{l} \text{Let } u = 2x+1 \\ du = 2\,dx \end{array}\right.$$

$$= \ln|x| - \int\frac{du}{u} = \ln|x| - \ln|u| + C$$

$$= \ln|x| - \ln|2x+1| + C = \boxed{\ln\left|\frac{x}{2x+1}\right| + C}.$$

42. Evaluate the definite integral $\int_2^3 \frac{1}{1-x^2}\,dx$.

<u>Answer</u>: If you try to use a trigonometric substitution for this integral, funny things start happening. (They have to do with the fact that $\sin x$ (or $\cos x$) <u>cannot</u> be 3 if x is a real number.) So we should use partial-fraction decomposition:

$$\frac{1}{1-x^2} = \frac{1}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x} = \frac{A(1-x) + B(1+x)}{(1+x)(1-x)}$$

$$= \frac{A - Ax + B + Bx}{1-x^2} = \frac{A + B + (B-A)x}{1-x^2}$$

So we have $\begin{cases} A + B = 1 \\ B - A = 0 \implies B = A \end{cases}$ $\quad A + A = 1 \implies A = \frac{1}{2},\ B = \frac{1}{2}$

The integral becomes

$$\int_2^3 \frac{1}{1-x^2}\,dx = \int_2^3 \left(\frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x} \right) dx = \frac{1}{2}\int_2^3 \frac{dx}{1+x} + \frac{1}{2}\int_2^3 \frac{dx}{1-x}$$

I will do these integrals one at a time:

Let $u = 1+x$; then $du = dx$. When $x=2$, $u = 1+2 = 3$; when $x=3$, $u = 1+3 = 4$. The first integral becomes

$$\frac{1}{2}\int_2^3 \frac{dx}{1+x} = \frac{1}{2}\int_3^4 \frac{du}{u} = \frac{1}{2}\Big[\ln|u|\Big]_3^4 = \frac{1}{2}(\ln 4 - \ln 3)$$

$$= \frac{1}{2}\ln\left(\frac{4}{3}\right).$$

For the second integral, let $u = 1-x$; then $du = -dx$. When $x = 2$, $u = 1-2 = -1$, and when $x = 3$, $u = 1-3 = -2$.

The second integral becomes

$$\frac{1}{2}\int_2^3 \frac{dx}{1-x} = \frac{1}{2}\int_{-1}^{-2} \frac{(-du)}{u} = -\frac{1}{2}\Big[\ln|u|\Big]_{-1}^{-2}$$

$$= -\frac{1}{2}\big[\ln|-2| - \ln|-1|\big] = -\frac{1}{2}\ln 2.$$

So the whole integral equals

$$\int_2^3 \frac{1}{1-x^2}\,dx = \frac{1}{2}\int_2^3 \frac{dx}{1+x} + \frac{1}{2}\int_2^3 \frac{dx}{1-x} = \frac{1}{2}\ln\left(\frac{4}{3}\right) - \frac{1}{2}\ln 2$$

$$= \frac{1}{2}\left[\ln\left(\frac{4}{3}\right) - \ln 2\right] = \boxed{\frac{1}{2}\ln\left(\frac{2}{3}\right)}.$$

EP1: For the function $f(x) = x^x$, calculate $f'(x)$ and $f''(x)$. $(x \in (0, \infty))$

Answer: By the properties of the natural log, we can write $f(x) = x^x = e^{\ln(x^x)} = e^{x \ln x}$. Differentiation is now simple:

$$f'(x) = \frac{d}{dx}[e^{x \ln x}] = e^{x \ln x} \frac{d}{dx}[x \ln x] = e^{x \ln x}\left(x \cdot \frac{1}{x} + 1 \cdot \ln x\right)$$

$$= e^{x \ln x}(1 + \ln x) = x^x(1 + \ln x).$$

So $\boxed{f'(x) = x^x(1 + \ln x)}$.

We can just use the product rule to find $f''(x)$:

$$f''(x) = \frac{d}{dx}[f'(x)] = \frac{d}{dx}[x^x(1+\ln x)] = x^x \frac{d}{dx}[1 + \ln x] + (1 + \ln x)\frac{d}{dx}[x^x]$$

$$= x^x \cdot \frac{1}{x} + (1 + \ln x)[x^x(1 + \ln x)]$$

$$= x^{x-1} + x^x(1 + \ln x)^2.$$

So $\boxed{f''(x) = x^{x-1} + x^x(1 + \ln x)^2}$.

EP2: Find an antiderivative of $g(x) = \frac{\ln x}{x}$ using integration by parts.

Answer: First, note that the domain of $g(x)$ cannot include $(-\infty, 0]$, because $\ln x$ is not defined in that region. So $x > 0$.

Now perform integration by parts:

Let

$$u = \ln x \qquad dv = \frac{1}{x} dx$$

$$du = \frac{dx}{x} \qquad v = \ln|x| = \ln x \quad (\text{since } x > 0)$$

So the integral becomes

$$\int g(x)\,dx = \int \frac{\ln x}{x}\,dx = (\ln x)^2 - \int \frac{\ln x}{x}\,dx$$

$$\Rightarrow\ 2\int \frac{\ln x}{x}\,dx = (\ln x)^2 + C$$

$$\Rightarrow\ \int \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2 + C$$

So $\boxed{\text{any function } G(x) = \frac{1}{2}(\ln x)^2 + C \text{ is an antiderivative of } g(x) = \frac{\ln x}{x}.}$

Section 7.4

These two integrals are improper, and they converge. Explain why they are improper, and evaluate each integral.

2. $\displaystyle\int_0^\infty x e^{-x}\,dx$.

Answer: The integral is improper because the upper bound extends to infinity. Use integration by parts to evaluate the integral: Let $u = x$, $du = dx$, $dv = e^{-x}\,dx$, $v = -e^{-x}$. The integral becomes

$$\int xe^{-x}\,dx = -xe^{-x} - \int (-e^{-x})\,dx = -xe^{-x} - e^{-x} + C.$$

So the improper integral is

$$\int_0^\infty xe^{-x}\,dx = \lim_{z\to\infty}\int_0^z xe^{-x}\,dx = \lim_{z\to\infty}\left[-xe^{-x} - e^{-x}\right]_0^z$$

$$= \lim_{z\to\infty}\left[-ze^{-z}-e^{-z}-(0\cdot e^{-0}-e^{-0})\right] = -\lim_{z\to\infty} ze^{-z} - \underbrace{\lim_{z\to\infty} e^{-z}}_{\to 0} + 1$$

$$= 1 - \lim_{z\to\infty} ze^{-z}.$$

To calculate the limit, use L'Hopital's rule:

$$\lim_{z\to\infty} ze^{-z} = \lim_{z\to\infty}\frac{z}{e^z} = \lim_{z\to\infty}\frac{\frac{d}{dz}[z]}{\frac{d}{dz}[e^z]} = \lim_{z\to\infty}\frac{1}{e^z} = 0.$$

So $\boxed{\int_0^\infty xe^{-x}\,dx = 1}$.

7. $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}\,dx.$

<u>Answer</u>: This integral is improper because both bounds extend to infinity. To evaluate the integral, note that

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}, \quad \text{so} \quad e^{-|x|} = \begin{cases} e^{-x} & x \geq 0 \\ e^{x} & x < 0 \end{cases}.$$

Now we can evaluate the integral:

$$\int_{-\infty}^{\infty} e^{-|x|}\,dx = \lim_{z\to-\infty}\int_z^0 e^{-|x|}\,dx + \lim_{z\to\infty}\int_0^z e^{-|x|}\,dx$$

$$= \lim_{z\to-\infty}\int_z^0 e^{x}\,dx + \lim_{z\to\infty}\int_0^z e^{-x}\,dx$$

$$= \lim_{z\to-\infty}\left[e^x\right]_z^0 + \lim_{z\to\infty}\left[-e^{-x}\right]_0^z$$

$$= \lim_{z\to -\infty} (e^0 - e^z) + \lim_{z\to\infty} \left[-e^{-z} - (-e^{-0})\right]$$

$$= 2 + \underbrace{\lim_{z\to-\infty} e^z}_{\to 0} + \underbrace{\lim_{z\to\infty}(-e^{-z})}_{\to 0} = 2.$$

So $\boxed{\int_{-\infty}^{\infty} e^{-|x|}\,dx = 2}$.

Determine whether the integral is convergent. If it is, compute its value.

18. $\int_1^\infty \frac{1}{x^{1/3}}\,dx$.

Answer: $\int_1^\infty \frac{1}{x^{1/3}}\,dx = \lim_{z\to\infty}\int_1^z \frac{1}{x^{1/3}}\,dx = \lim_{z\to\infty}\left[\frac{3}{2}x^{2/3}\right]_1^z$

$$= \lim_{z\to\infty}\left(\frac{3}{2}z^{2/3} - \frac{3}{2}\right) = \infty.$$

Thus the integral diverges.

25. $\int_e^\infty \frac{dx}{x\ln x}$.

Answer: $\int_e^\infty \frac{dx}{x\ln x} = \lim_{z\to\infty}\int_e^z \frac{dx}{x\ln x}$ | Let $u = \ln x$, $du = \frac{dx}{x}$ $u(e) = \ln e = 1$, $u(z) = \ln z$

$$= \lim_{z\to\infty}\int_1^{\ln z}\frac{du}{u} = \lim_{z\to\infty}\left[\ln|u|\right]_1^{\ln z}$$

$$= \lim_{z\to\infty} \ln(\ln z) - 0 = \infty.$$

Thus the integral diverges.

32. Determine the constant c so that $\int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx = 1$.

Answer: $$\int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx = c\left[\lim_{z\to\infty}\int_0^z \frac{dx}{1+x^2} + \lim_{z\to-\infty}\int_z^0 \frac{dx}{1+x^2}\right]$$

$$= c\left[\lim_{z\to\infty}\left[\tan^{-1}x\right]_0^z + \lim_{z\to-\infty}\left[\tan^{-1}x\right]_z^0\right]$$

$$= c\left[\lim_{z\to\infty}\tan^{-1}z - \tan^{-1}0 + \lim_{z\to-\infty}\left[-\tan^{-1}z\right] + \tan^{-1}0\right]$$

$$= c\left[\frac{\pi}{2} + \left(-\left(-\frac{\pi}{2}\right)\right)\right] = c\pi$$

$$\boxed{\text{For the integral to equal } 1,\ c \text{ must equal } \frac{1}{\pi}.}$$

33. In this problem, we investigate the integral

$$\int_1^{\infty} \frac{1}{x^p}\,dx, \quad p \in (0,\infty).$$

(a) For $z > 1$, set $A(z) = \int_1^z \frac{1}{x^p}\,dx$ and show that

$$A(z) = \begin{cases} \frac{1}{1-p}\left(z^{-p+1} - 1\right) & p \neq 1 \\ \ln z & p = 1 \end{cases}$$

Answer: When $p \neq 1$, we have

$$A(z) = \int_1^z \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_1^z = \frac{1}{1-p}\left(z^{1-p} - 1^{1-p}\right) = \frac{1}{1-p}\left(z^{1-p} - 1\right).$$

When $p = 1$,

$$A(z) = \int_1^z \frac{dx}{x} = \left[\ln|x|\right]_1^z = \ln|z| - \ln 1 = \ln z.$$

(b) Use your results in (a) to show that, for $0 < p \leq 1$,

$$\lim_{z \to \infty} A(z) = \infty.$$

Answer: When $0 < p < 1$,

$$\lim_{z \to \infty} A(z) = \lim_{z \to \infty}\left[\frac{1}{1-p}\left(z^{1-p} - 1\right)\right] = \lim_{z \to \infty} \frac{z^{1-p}}{1-p} - \frac{1}{1-p}.$$

Since $p < 1$, $1-p > 0$. I will take as given that

$$\lim_{z \to \infty} z^k = \begin{cases} \infty & k \geq 1 \\ 0 & k < 1 \end{cases}.$$ Since $1-p > 0$, $\lim_{z \to \infty} \frac{z^{1-p}}{1-p} = \infty$,

and $\lim_{z \to \infty} A(z) = \infty$.

When $p = 1$, $\lim_{z \to \infty} A(z) = \lim_{z \to \infty} \ln z = \infty$.

(c) Use your results in (a) to show that, for $p > 1$,

$$\lim_{z \to \infty} A(z) = \frac{1}{p-1} .$$

<u>Answer</u>: When $p > 1$, $1 - p < 0$, so $\lim_{z \to \infty} z^{1-p} = 0$.

Thus $$\lim_{z \to \infty} A(z) = \lim_{z \to \infty} \frac{1}{1-p} \left(z^{1-p} - 1 \right)$$

$$= \frac{1}{1-p} \underbrace{\lim_{z \to \infty} z^{1-p}}_{\to 0} - \frac{1}{1-p} = - \frac{1}{1-p} = \frac{1}{p-1} .$$