SPRING 2008: 110.211 HONORS MULTIVARIABLE CALCULUS

Examining Theorem 6.5: The Implicit Function Theorem for real-valued functions

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The following theorem was presented in lecture on Wednesday, February 27. I believe it is poorly worded and am writing this note to explain why I think this is the case.

Theorem 6.5 (Colley, p.163)**.** *Let $F : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be of class C^1 and let $\mathbf{a}$ be a point of the level set $S = \{\mathbf{x} \in \mathbb{R}^n | F(\mathbf{x}) = c\}$. If $F_{x_n}(\mathbf{a}) \neq 0$, then there is a neighborhood U of $(a_1, \ldots, a_{n-1})$ in $\mathbb{R}^{n-1}$, a neighborhood V of $\mathbf{a}$ in $\mathbb{R}$, and a function $f : U \subseteq \mathbb{R}^{n-1} \to V$ of class C^1 such that if $(x_1, \ldots, x_{n-1}) \in U$ and $x_n \in V$ satisfy $F(x_1, \ldots, x_{n-1}, x_n) = c$ (i.e., $(x_1, \ldots, x_{n-1}, x_n) \in S$), then $x_n = f(x_1, \ldots, x_{n-1})$.*

In the lecture discussion, a question was posed as to why does the above theorem, like the Inverse Function Theorem, guarantee that there exists an open interval V of the dependent variable x_n as the codomain of the implicit function. This stands in contrast to the general Implicit Function Theorem, given in Colley as Theorem 6.6, where there is no mention of an open interval in the implicitly defined codomain.

The question was, why is this V present in this special case, and not in the general Theorem 6.6.

To answer this question, consider the following example:

Example. Let $F(x, y, z) = x^2+y^2+z^2$. F is differentiable at the north pole of any of the c-level sets S_c of F, for $c > 0$. Let $\mathbf{a} = (a_1, a_2, a_3) = (0, 0, 1) \in S_1$. Since $F_z(\mathbf{a}) = 1 \neq 0$, it follows that there is a neighborhood U of the origin $(a_1, a_2) = (0, 0) \in \mathbb{R}^2$, a neighborhood V of $1 \in \mathbb{R}$ and a C^1 function $z = f(x, y)$ defined on U so that if $(x, y) \in U$ and $z \in V$ satisfy $x^2 + y^2 + z^2 = 1$, then $z = f(x, y)$

Well, we know the implicit function: In the northern hemisphere away from the equator, the implicit function can be made explicit: $z = \sqrt{1 - x^2 - y^2}$. Here, for any choice of δ satisfying $0 < \delta < 1$,

$$(x, y) \in U = B_\delta(0, 0) = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; ||(x, y)|| < \delta \right\},$$

we can choose V to be any open interval bigger in size than a ball about $1 \in \mathbb{R}$ of radius $\epsilon = 1 - \sqrt{1 - \delta^2}$ (verify this); Hence

$$V = B_\epsilon(1) = \left\{ z \in \mathbb{R} \;\middle|\; |z - 1| < \epsilon \right\}.$$

However, the range of the implicit function here is $f(U) = (\sqrt{1 - \delta^2}, 1]$, which is not open in $\mathbb{R}$. While stressing the existence of V open is not technically wrong, it is an odd wording, and the existence of V is really non-important. Better (like in the general case) to not bother to place any import on the image in the codomain at all.

Incidentally, there is great need for the range of the function of the Inverse Function Theorem to be open, when the domain is open. It helps (but only helps) to verify that the inverse relation as a function is continuous. This is not really important for this class, but it is worth mentioning.

As a final note, It is common when proving the Implicit Function Theorem to use the Inverse Function Theorem. Maybe this is where the odd wording of Theorem 6.5 comes in. If I have time, I will write up a proof of the Implicit Function Theorem for a real-valued function in three dimensions, without any mention of V. I am hopeful that I will have time.

Date: February 29, 2008.