Example of the use of differential forms

Example 0.1. Let $M = \left\{(x, y, z \in \mathbb{R}^3 \mid z = \sqrt{x^2 + y^2}\right\}$ be the northern hemisphere of the unit sphere in $\mathbb{R}^3$, and $\omega = z^2\, dx \wedge dy$ be a 2-form on $\mathbb{R}^3$. Calculate $\displaystyle\int_M \omega$.

Solution: Via the parameterization $\varphi(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{1-r^2})$, φ takes the rectangle

$$D = \left\{(r, \theta) \in \mathbb{R}^2 \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\right\}$$

in the rectilinear (r, θ)-space onto M. We can calculate directly using forms:

$$\begin{aligned}\int_M \omega &= \int_D \omega_{\varphi(r,\theta)}\left(\frac{\partial\varphi}{\partial r}(r,\theta), \frac{\partial\varphi}{\partial\theta}(r,\theta)\right) dr \wedge d\theta \\ &= \int_D \omega_{\varphi(r,\theta)}\left(\left\langle \cos\theta, \sin\theta, \frac{-r}{\sqrt{1-r^2}}\right\rangle, \langle -r\sin\theta, r\cos\theta, 0\rangle\right) dr \wedge d\theta.\end{aligned}$$

Note that the tangent vectors are vectors in $\mathbb{R}^3$ even though we are integrating ω across a surface. You should think of the tangent vectors as begin in a sort of cylindrical coordinates on $\mathbb{R}^3$. Note also that the coefficient of ω along the surface can also be written in (r, θ), as $z^2 = 1 - r^2$. Now

$$\int_M \omega = \int_D (1-r^2) \begin{vmatrix} dr\langle \cos\theta, \sin\theta, \frac{-r}{\sqrt{1-r^2}}\rangle & dr\langle -r\sin\theta, r\cos\theta, 0\rangle \\ d\theta\langle \cos\theta, \sin\theta, \frac{-r}{\sqrt{1-r^2}}\rangle & d\theta\langle -r\sin\theta, r\cos\theta, 0\rangle \end{vmatrix} dr \wedge d\theta \tag{0.1}$$

$$= \int_D (1-r^2) \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr \wedge d\theta \tag{0.2}$$

$$= \int_D (1-r^2)\, r\, dr \wedge d\theta \tag{0.3}$$

$$= \int_0^{2\pi}\int_0^1 (r - r^3)\, dr\, d\theta = \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^1 d\theta = \frac{1}{4}\,\theta\Bigg|_0^{2\pi} = \frac{\pi}{2}. \tag{0.4}$$

Note, one can associate to any 2-form on $\mathbb{R}^3$ a vector field by using the function coefficients. Under this association, the vector field $\vec{F} = (F_1, F_2, F_3) = \left(0, 0, z^2\right)$ corresponds to ω. As an exercise, perform the same calculation by evaluating the vector surface integral

$$\iint_M \vec{F} \cdot d\vec{S}$$

in the more classical way, by using the same parameterization. You will get the same answer. Indeed,

$$\int_M \omega = \iint_D \vec{F}\left(\varphi(r,\theta)\right) \cdot \vec{N}(r,\theta)\, dr\, d\theta,$$

where

$$\vec{N}(r,\theta) = \frac{\partial(y,z)}{\partial(r,\theta)}\vec{\mathbf{i}} + \frac{\partial(z,x)}{\partial(r,\theta)}\vec{\mathbf{j}} + \frac{\partial(x,y)}{\partial(r,\theta)}\vec{\mathbf{k}}.$$

Hence

$$\begin{aligned}\int_M \omega &= \iint_D \vec{F}\left(\varphi(r,\theta)\right) \cdot N(r,\theta)\, dr\, d\theta \\ &= \iint_D \left(F_1\left(\varphi(r,\theta)\right)\frac{\partial(y,z)}{\partial(r,\theta)} + F_2\left(\varphi(r,\theta)\right)\frac{\partial(z,x)}{\partial(r,\theta)} + F_3\left(\varphi(r,\theta)\right)\frac{\partial(x,y)}{\partial(r,\theta)}\right) dr\, d\theta \\ &= \iint_D z^2\left(\varphi(r,\theta)\right)\frac{\partial(x,y)}{\partial(r,\theta)}\, dr\, d\theta \\ &= \iint_D (1-r^2)\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\, d\theta\end{aligned}$$

which is the same as Equation 0.2, and the rest follows as above.

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