ANOSOV MAPPING CLASS ACTIONS ON THE SU(2)-REPRESENTATION VARIETY OF A PUNCTURED TORUS

RICHARD J. BROWN

Abstract. Recently, Goldman [2] proved that the mapping class group of a compact surface S, $MCG(S)$, acts ergodically on each symplectic stratum of the Poisson moduli space of flat $SU(2)$-bundles over S, $X(S, SU(2))$. We show that this property does not extend to that of cyclic subgroups of $MCG(S)$, for S a punctured torus. The symplectic leaves of $X(T^2 - pt., SU(2))$ are topologically copies of the 2-sphere S^2, and we view mapping class actions as a continuous family of discrete Hamiltonian dynamical systems on S^2. These deformations limit to finite rotations on the degenerate leaf corresponding to $-Id.$ boundary holonomy. Standard KAM techniques establish that the action is not ergodic on the leaves in a neighborhood of this degenerate leaf.

(*Ergod. Th. & Dynam. Sys.*, (1998), **18**, 539-554)

Contents

Date: November 8, 2002.

1. Introduction

Let $S_{g,n}$ be an orientable surface of genus-g with n (possibly 0) boundary components, and G a compact, connected Lie group. The set $Hom(\pi_1(S_{g,n}), G)$ is an analytic subset of G^{2g+n}. G acts on G^{2g+n} by diagonal conjugation, leaving invariant $Hom(\pi_1(S_{g,n}), G)$. The resulting quotient variety

$$X(S_{g,n}, G) = Hom(\pi_1(S_{g,n}), G)/G$$

is a Poisson space. The symplectic leaves correspond to fixed conjugacy class of holonomy over $\partial S_{g,n}$, and have finite symplectic volume [4]. Since G is compact, the leaf space $[G]^n$, defined as the product of the sets of conjugacy classes of G over each element of the boundary of $S_{g,n}$, is a standard measure space. The mapping class group $MCG(S_{g,n}) = \pi_0 Diff(S_{g,n})$ acts on $X(S_{g,n}, G)$, preserving the Poisson structure.

Recently Goldman proved that, for compact groups locally isomorphic to the product of copies of $U(1)$ and $SU(2)$, this action is ergodic with respect to symplectic measure on every leaf [2]. Can a single element $\sigma \in MCG(S_{g,n})$ act ergodically? This paper is the first step in a general study of the action of individual mapping classes on moduli spaces of this type. Specifically, the main result of this paper is the following:

Theorem 1.1. *For any $\sigma \in MCG(S_{1,1})$, there exists a positive measure set of leaves such that σ does not act ergodically on $X(S_{1,1}, SU(2))$.*

This paper is organized as follows: In Section 2, we review and reinterpret Goldman's result in an appropriate context. Abelian representations ($G = U(1)$) provide the proper starting point for the analysis of representations into the non-abelian group $SU(2)$, given in Section 3. In the case of a closed surface $S_g = S_{g,0}$, $X(S_g, U(1))$ is a torus. Here individual mapping classes can act ergodically, and we classify these types. In Section 4, we restrict our attention to $SU(2)$-representations of the fundamental groups of a torus T^2 and a torus with one boundary component $S_{1,1}$. In the latter case, we embed $X(S_{1,1}, SU(2))$ into $\mathbb{R}^3$ as the $SU(2)$-character variety of $S_{1,1}$ and analyze mapping class actions there. The symplectic leaves in $X(S_{1,1}, SU(2))$ are concentric two spheres, and the action of a mapping class may be viewed as a continuous deformation of area preserving sphere maps. This deformation is a deformation of the action on the orbifold $X(T^2, SU(2))$ – the symplectic leaf of $X(S_{1,1}, SU(2))$ corresponding to trivial boundary holonomy – and limits to a finite rotation on the sphere of directions at the degenerate leaf corresponding to $-Id$ boundary holonomy. In a neighborhood of this degenerate leaf the action is not ergodic. Section 4.3 deduces this result from the existence of elliptic fixed points (Theorem 4.4). Theorem 1.1 follows as a corollary. Section 5 develops some notation used to prove Theorem 4.4, which is proved in Section 6. The Appendix gives an example of a nonhyperbolic (non-Anosov) linear symplectic automorphism of the torus $X(S_{8,0}, U(1))$ which is ergodic.

The author would like to thank Bill Goldman for many helpful conversations during the course of this work, and for proofreading a late version.

2. Preliminaries

2.1. **Moduli spaces of surfaces.** Let S_g be a closed, genus-g, orientable surface (i.e., $n = 0$.), and G a compact, connected Lie group. The set $Hom(\pi_1(S_g), G)$ is an algebraic subset of G^{2g} formed by evaluation on a set of generators of $\pi_1(S_g)$. This set is independent of the generating set, however. The relation given by a presentation of $\pi_1(S_g)$ defines an analytic function on G^{2g} such that $Hom(\pi_1(S_g), G)$ is an analytic variety. G acts on G^{2g} by diagonal conjugation, leaving invariant $Hom(\pi_1(S_g), G)$. The quotient variety

$$X(S_g, G) = Hom(\pi_1(S_g), G)/G$$

is a stratified symplectic space in the sense of [11]. $X(S_g, G)$ has finite symplectic volume (Huebschmann, [4]). Note that a symplectic space possesses a nondegenerate Lie algebra structure on smooth functions. Hence a symplectic space is a Poisson space with a single symplectic leaf. A geometric interpretation of $X(S_g, G)$ is the space of flat G-bundles over S_g.

For $n > 0$, denote $\partial S_{g,n} = c = \coprod_{i=1}^{n} c_i$. For each i, there is a map

$$f_i \colon Hom(\pi_1(S_{g,n}), G) \to G, \quad f_i(\phi) = \phi(c_i),$$

formed by restricting a representation to an element in $\pi_1(S_{g,n})$ homotopic to the boundary element c_i. This map is surjective, and descends to a map on conjugacy classes of representations, which we also denote by f_i. Denote by $[G]$ the set of conjugacy classes in G. For $r \in [G]$, the set $f_i^{-1}(r)$ corresponds to the set of representation classes with fixed conjugacy class of boundary holonomy over c_i. Consider the map

$$f \colon X(S_{g,n}, G) \to [G]^n, \quad [\phi] \mapsto ([\phi(c_1)], \ldots, [\phi(c_n)]) .$$

It can be shown (see, for instance [4]) that $X_{\overline{r}}(S_{g,n}, G) = f^{-1}(\overline{r})$ is symplectic. For a fixed complex structure on $S_{g,n}$, $X_{\overline{r}}(S_{g,n}, G)$ corresponds to the set of holomorphic G-bundles with fixed parabolic structure; the parabolic structure corresponds to fixing the conjugacy class of holonomy over each $c_i \in \partial S_{g,n}$. Evidently, $X(S_{g,n}, G)$ is Poisson, with symplectic leaves corresponding to points $\overline{r} \in [G]^n$. In the language of Poisson manifold theory, f is a Casimir map, and describes the kernel of the Lie bracket on smooth functions over $X(S_{g,n}, G)$. The image of this map is called the leaf space.

2.2. **Mapping class groups.** For a closed surface (i.e., $n = 0$), the mapping class group $MCG(S_g)$ is defined as the set of all isotopy classes of orientation preserving diffeomorphisms $\pi_0(Diff(S_g))$. By a theorem of J. Nielsen,

$$MCG(S_g) \cong Out(\pi_1(S_g)).$$

A Dehn twist about a simple closed curve α on S_g is a map on a tubular neighborhood of α formed by cutting along α, twisting one side around once, and reconnecting along α. For any g, there exists a finite generating set for $Out(\pi_1(S_g))$ given by Dehn twists [6].

For $n > 0$, define $MCG(S_{g,n})$ as the isotopy classes of diffeomorphisms which pointwise fix $\partial(S_{g,n})$. Notice that for $n = 1$, $\pi_1(S_{g,1})$ is free on $2g$ generators, such that the element homotopic to the single boundary component c is conjugate to the relation R in any presentation of $\pi_1(S_{g,0})$. Consequently, there is an identification

$$MCG(S_{g,1}) \cong MCG(S_{g,0}) = MCG(S_g).$$

2.3. **Structure preserving actions.** It is easy to see that a diffeomorphism of $S_{g,n}$ will correspond to a self-map of $X(S_{g,n},G)$, and that one which fixes $\partial S_{g,n}$ will also leave invariant each of the symplectic leaves $X_{\overline{r}}(S_{g,n},G)$. Hence $MCG(S_{g,n})$ acts on $X(S_{g,n},G)$, preserving $X_{\overline{r}}(S_{g,n},G)$, for all $\overline{r} \in [G]^n$. What is not so obvious is the following fact [2]:

Proposition 2.1. *$MCG(S_{g,n}) \subset Symp(X_{\overline{r}}(S_{g,n},G))$.*

Since $MCG(S_{g,n})$ preserves the symplectic structure on each $X_{\overline{r}}(S_{g,n},G)$, it preserves the corresponding volume form, and hence a measure on $X_{\overline{r}}(S_{g,n},G)$. The volume form and the measure induced by the symplectic form are called symplectic.

Theorem 2.2. *(Goldman, [2]) For compact G locally isomorphic to a product of copies of $SU(2)$ and $U(1)$, $MCG(S_{g,n})$ acts ergodically with respect to symplectic measure on each $X_{\overline{r}}(S_{g,n},G)$.*

For G compact, the leaf space $[G]^n$ is a measure space. Denote this measure on $[G]^n$ by ℓ. Define a measure μ on $X(S_{g,n},G)$ as follows: Denote by $\omega_{\overline{r}}$ the symplectic structure on the leaf $f^{-1}(\overline{r})$. Then $\omega_{\overline{r}}^n$ is a nondegenerate volume form on $f^{-1}(\overline{r})$, where $n = \frac{1}{2}\dim(f^{-1}(\overline{r}))$. Denote also by $\omega^n_{f^{-1}(\overline{r})}$ the symplectic measure associated to this volume form. Then for S a Borel set

$$\mu(S) = \int_{[G]^n} \omega^n_{f^{-1}(\overline{r})}(S \cap f^{-1}(\overline{r}))d\ell(\overline{r}).$$

Note that the integrand above is a function on $[G]^n$ supported on $f(S)$.

Proposition 2.3. *μ is invariant under $MCG(S_{g,n})$.*

Proof. The symplectic volume and hence the symplectic measure on each leaf is invariant by Proposition 2.1. The induced action of $MCG(S_{g,n})$ on $[G]^n$ is trivial, hence μ is invariant. □

Hence $MCG(S_{g,n})$ acts as measure preserving transformations on the Poisson space $X(S_{g,n},G)$. Since the symplectic measure is finite and $[G]^n$ is compact, μ is a finite measure.

In this context, Goldman's theorem is a statement on the ergodic decomposition of the action of $MCG(S_{g,n})$ on $X(S_{g,n},G)$: The ergodic decomposition of the action of $MCG(S_{g,n})$ on $X(S_{g,n},G)$ is equal to the symplectic stratification on $X(S_{g,n},G)$ induced by the Poisson structure. Casimir functions form the center of the set of smooth functions on a Poisson manifold with respect to the Lie bracket. This result is a statement that the Casimir functions are the only nontrivial functions invariant under $MCG(S_{g,n})$. We sum up this statement by saying that the mapping class group acts leaf-wise ergodically on the Poisson space.

Fact 2.4. *$MCG(S_{g,n})$ acts leaf-wise ergodically on $X(S_{g,n},G)$.*

We pose the following question: Can a cyclic subgroup of $MCG(S_{g,n})$ act leaf-wise ergodically on $X(S_{g,n},G)$? In general, the answer is unknown.

The strategy for this endeavor involves using the abelian $U(1)$ representations as a model to understand the nonabelian representations into $SU(2)$.

3. Abelian representations

Let G be an abelian Lie group. Then

$$X(S_g,G) = Hom(\pi_1(S_g),G)/G = Hom(\pi_1(S_g),G)$$

since the action of G on itself by conjugation is trivial. Also due to the fact that G is abelian, homomorphisms of $\pi_1(S_g)$ into G lift through the Hurewicz homomorphism

$$\rho\colon \pi_1(S_g) \to \pi_1(S_g)/[\pi_1(S_g), \pi_1(S_g)] = H_1(S_g) \cong \mathbb{Z}^{2g}$$

inducing the isomorphism $Hom(\pi_1(S_g), G) \cong Hom(H_1(S_g), G)$. It follows that

$$X(S_g, G) \cong Hom(H_1(S_g), G) \cong H^1(S_g; G) \cong G^{2g}.$$

Every connected abelian Lie group is isomorphic to a product $\mathbb{R}^n \times T^m$, where T^m is a m-torus and $\dim G = n + m$. Since any homomorphism into a product may be uniquely decomposed into a product of homomorphisms into the factors, the only 2 abelian groups to consider are $G = \mathbb{R}$ and $U(1)$.

3.1. $G = \mathbb{R}$. The automorphism group $Out(\pi_1(S_g))$ acts on $X(S_g, \mathbb{R})$ by pullback through the $Hom(\cdot, \mathbb{R})$ functor and through ρ. Specifically,

Proposition 3.1. *([7]) $\rho_*(MCG(S_g)) = Sp(\mathbb{Z}^{2g}) \subset Aut(H_1(S_g))$.*

It follows by the above discussion that mapping classes act as symplectic automorphisms of the linear symplectic space $X(S_g, \mathbb{R}) = \mathbb{R}^{2g}$. Observe that, for $g = 1$, ρ is an isomorphism, so that

$$MCG(S_1) \cong Sp(\mathbb{Z}^2) = SL(\mathbb{Z}^2),$$

where $S_1 = T^2$, the standard torus.

Note the integer points of $X(S_g, \mathbb{R})$ are precisely those representations $\phi\colon \pi_1(S_g) \to \mathbb{Z} \subset \mathbb{R}$. Elements of $Sp(\mathbb{Z}^{2g})$ leave invariant this set of integer points. Hence $MCG(S_g)$ leaves invariant the integer lattice $\mathbb{Z}^{2g} \subset \mathbb{R}^{2g}$. It follows that there is an induced action on the quotient space $\mathbb{R}^{2g}/\mathbb{Z}^{2g} \cong T^{2g}$.

3.2. $G = U(1)$. In this case,

$$Hom(\pi_1(S_g), U(1)) \cong X(S_g, U(1)) \cong T^{2g}.$$

All automorphisms of a torus T^n are induced from linear automorphisms of $\mathbb{R}^n$ which leave invariant the integer lattice $\mathbb{Z}^n$ [8]. The dynamical properties of toral homomorphisms have been extensively studied in recent years. For our purposes, we offer the following:

Theorem 3.2. *(Mañe, [8]) A continuous surjective homomorphism of T^n is ergodic if and only if none of its eigenvalues is a root of unity.*

An Anosov diffeomorphism of a closed manifold is one in which there exists a continuous splitting of the tangent bundle into invariant complementary subbundles, together with a technical condition on the growth of vectors under the transformation. In our context, the Anosov automorphisms of T^n are the linear automorphisms whose spectrum is disjoint from the unit circle; the hyperbolic automorphisms. In the language at hand, the preceding theorem implies the following.

Theorem 3.3. *$\sigma \in MCG(S_g)$ acts ergodically on $X(S_g, U(1))$ if σ is Anosov.*

For $g = 1$, the converse is also true. In general, however, the converse is false. This is because there exist unit modulus algebraic integers with algebraic conjugates off $U(1)$. These algebraic integers are necessarily not roots of unity. Hence they would satisfy Theorem 3.2 while not being hyperbolic (Anosov). In the appendix, we present an explicit example of one of these integers. This example establishes the following:

Fact 3.4. *There exist non-Anosov mapping class actions on $X(S_g, U(1))$ for certain g which are ergodic.*

4. $SU(2)$-representations

4.1. **Representations of $\pi_1(T^2)$.** The simplest case for $SU(2)$-representations is the case $g = 1$, $n = 0$. Here $S_1 = T^2$, the standard 2-torus, whose fundamental group is abelian. Consider the standard embedding of $U(1)$ into $SU(2)$ as the set

$$T = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in [0, 2\pi).$$

Representations of $\pi_1(S_1)$ are necessarily abelian. Therefore, homomorphisms of $\pi_1(S_1)$ into $SU(2)$ have image in some maximal torus, a conjugate of $T \subset SU(2)$. However, the action by conjugation in $SU(2)$ is not trivial. Indeed,

$$Hom(\pi_1(S_1), T) \hookrightarrow Hom(\pi_1(S_1), SU(2))$$

as the standard 2-torus T^2. But the map on the quotient varieties is not injective. This is due to the $\mathbb{Z}_2$ action by the Weyl group generated by the element $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (Note this is an action by conjugation). Thus $X(S_1, SU(2)) \cong T^2/w$. Topologically, $X(S_1, SU(2))$ is a 2-sphere, but there are four distinguished points, corresponding to the places where the $\mathbb{Z}_2$ action by w is not free. These are the places where the fundamental domain of T^2 in $\mathbb{R}^2$ intersects the $\frac{1}{2}$-integer lattice. The action by conjugation by w is a reflection through the origin in $\mathbb{R}^2$, and takes

$$(x, y) \mapsto (-x, -y) \in \mathbb{R}^2.$$

The automorphism group $SL(\mathbb{Z}^2)$ has image through the quotient map $T^2 \to T^2/w$ the group

$$MCG(S_1) = SL(\mathbb{Z}^2) \mapsto SL(\mathbb{Z}^2)/\pm Id. = \mathbb{P}SL(\mathbb{Z}^2).$$

Any linear automorphism of T^2 necessarily leaves invariant the $\frac{1}{2}$-integer lattice, hence descends to an automorphism of T^2/w. Hyperbolic automorphisms descend to automorphisms which are hyperbolic everywhere except the four singularities. In [5], Katok shows that hyperbolic automorphisms of T^2 induce Bernoulli automorphisms of T^2/w, which are ergodic. Thus, the action of any hyperbolic element of $\mathbb{P}SL(\mathbb{Z}^2)$ on $X(S_1, SU(2))$ is ergodic:

Theorem 4.1. *$\sigma \in MCG(S_1)$ acts ergodically on $X(S_1, SU(2))$ iff σ is Anosov.*

4.2. **Representations of $F_2 = \pi_1(S_{1,1})$.** For the rest of this paper, we will restrict our attention to the case of the torus with one boundary component. In this section, we describe the basic structure of $X(S_{1,1}, SU(2))$.

4.2.1. *The character variety.* For the surface $S_{1,1}$ (See Figure 1), $X(S_{1,1}, SU(2)) = SU(2)^2/SU(2)$ since $\pi_1(S_{1,1})$ is free. For G any complex matrix group, the ring of invariants of n $m \times m$ complex matrices $\mathbb{C}[G^n]^G$ is finitely generated by the traces of monomials in the n coordinate places $(A_1, \ldots, A_n) \in G^n$ (Procesi [10]). For $G = SU(2)$, these polynomial functions are real-valued, and

$$\mathbb{R}[SU(2)^2]^{SU(2)} = \mathbb{R}[tr(A_1), tr(A_2), tr(A_1A_2)]$$

is freely generated. Consider the presentation of $\pi_1(S_{1,1})$ given in Figure 1. Using the trace coordinates

$$x = tr(\phi(a)), \quad y = tr(\phi(b)), \quad z = tr(\phi(ab)),$$

$X(S_{1,1}, SU(2))$ embeds into $\mathbb{R}^3$. This is the $SU(2)$-character variety associated to $S_{1,1}$ (see Goldman [3]), and was called the "tetrahedral pillow" by Dan Asimov. Figure 1 is a depiction of this character variety. Although the picture is that of a tetrahedron, the variety is smooth on the edges. The vertices are the only true singularities. For G a compact group, the character variety is birationally isomorphic to the representation variety for any genus. For $g > 1$, however, the trace functions which determine the coordinates do not form an independent set. Indeed, for $g > 2$, the full set of trace relations (equations relating the trace functions) is not even known (see Whittemore [13]).

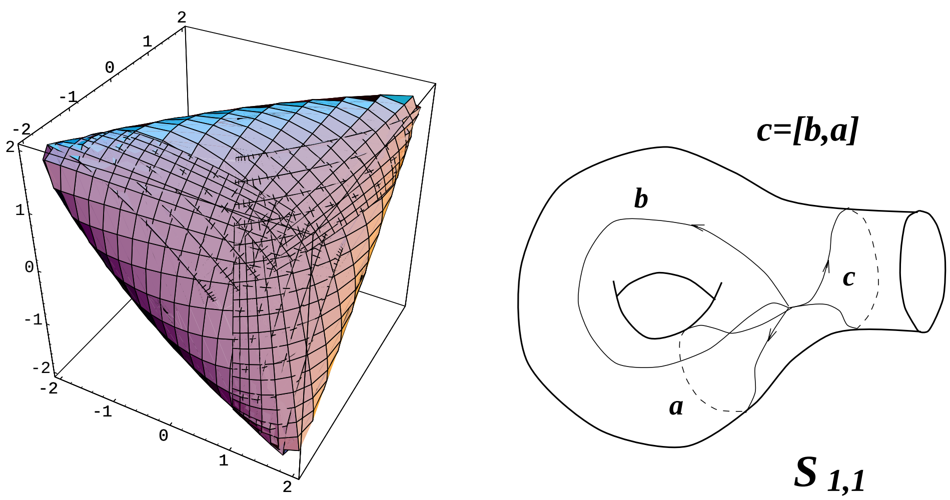


FIGURE 1. Character variety (left) for the punctured torus (right).

Recall that the level sets $X_{\overline{r}}(S_{1,1}, SU(2))$ correspond to conjugacy classes of the boundary holonomy. In $SU(2)$, the trace of the boundary holonomy $tr(\phi([a,b]))$ completely determines these sets. In $\mathbb{R}^3$, this condition determines a cubic

$$\kappa(x,y,z) = tr(\phi(aba^{-1}b^{-1})) = x^2 + y^2 + z^2 - xyz - 2.$$

Level sets are parametrized by $k \in [-2, 2] \subset \mathbb{R}$. Topologically, these level sets are concentric 2-spheres, where

$$\begin{array}{lcll} k & = & -2 & \text{is } \textit{the point } (0,0,0) \\ k & \neq & -2 & \text{is } \cong S^2 \\ k & = & 2 & X_2(S_{1,1}, SU(2)) \cong T^2/W = \partial X(S_{1,1}, SU(2)) = X(S_1, SU(2)). \end{array}$$

Recall also that, by choosing $\partial S_{1,1}$ as the origin of T^2,

$$MCG(S_{1,1}) \cong MCG(S_{1,0}).$$

Thus the action of $MCG(S_{1,1})$ on $X(S_{1,1}, SU(2))$ factors through $\mathbb{P}SL(\mathbb{Z}^2)$.

Mapping classes for $S_{1,1}$ are generated by Dehn twists about the curves a and b (refer to Figure 1. Notice that any sequence of these Dehn twists will leave invariant the boundary curve c. Through the trace coordinates defined above, the two generators act as the following transformations on $\mathbb{R}^3$ [2]:

$$\begin{aligned} A(x,y,z) &= (x, z, xz-y) \\ B(x,y,z) &= (xy-z, y, x). \end{aligned}$$

These transformations are polynomial automorphisms of $\mathbb{R}^3$ which leave invariant the cubic $\kappa(x,y,z)$. To understand the limiting behavior of these transformations near the origin, replace the origin with a copy of S^2. We shall call this shell (the $k=-2$ shell) the sphere of directions, so that σ may be considered as a map

$$\sigma\colon [-2,2]\times S^2 \to S^2,$$

where $\sigma(k,\cdot) = \sigma|_{\kappa^{-1}(k)}$ and $\kappa^{-1}(k) \cong S^2$. Thus one may view σ as a continuous deformation of area preserving (symplectic) sphere maps. The end points of this deformation are of particular interest and we will discuss them at length below.

4.2.2. *Basic results.* First, we establish the following:

Proposition 4.2. *$MCG(S_{1,1})$ does not act ergodically on $X(S_{1,1}, SU(2))$.*

Proof. The cubic $\kappa \in \mathbb{R}[X(S_{1,1}, SU(2))]$ is a nonconstant invariant function. □

Hence the cyclic subgroup generated by any single element of $MCG(S_{1,1})$ also cannot act ergodically. Also note that if σ is not Anosov, then there exists a conjugacy class of some nontrivial element $\alpha \in \pi_1(S_{1,1})$ which is invariant under σ. The trace of the holonomy around α, tr_α, is a nontrivial invariant real-valued function over $X(S_{1,1}, SU(2))$. It will restrict to a nontrivial invariant function over each level set $k = const$. Hence we have the following:

Proposition 4.3. *Let $\sigma \in MCG(S_{1,1})$ be non-Anosov. Then σ does not act ergodically on any $\kappa^{-1}(k)$, $k \in [-2,2]$.*

Much of the following discussion will be limited to cyclic subgroups generated by Anosov mapping classes. We begin with an analysis of mapping class actions on the extreme points of the set $[-2,2]$.

4.2.3. *Trivial boundary holonomy.* Let $\sigma \in MCG(S_{1,1})$ be Anosov, and consider the level set corresponding to $k=2$. Recall k is the trace of a representation class $[\phi]$ evaluated on a fundamental group element homotopic to $c = \partial S_{1,1}$. Since $trace(\phi(c)) = 2$, $\phi(c)$ is necessarily the Id. This will correspond to a representation into $SU(2)$ of the fundamental group of a closed torus $S_1 = T^2$. Hence by Theorem 4.1, σ acts ergodically on the $k=2$ shell. The action here descends from a linear action on T^2. Many of the dynamical properties of linear actions of the torus are passed through the quotient map, like topological transitivity, hyperbolicity of all nonsingular points, and the fact that the set of periodic points is dense.

In [5], Katok develops smooth models of these actions by slowly killing off the action at the singularities. He then proves many of the dynamical properties of the resulting smooth action are preserved under the deformation.

4.2.4. *Nontrivial central boundary holonomy.* At the other extreme, consider the sphere of directions at the origin. The two generators A, B of the action of $MCG(S_{1,1})$ on $\mathbb{R}^3$ degenerate to linear isometries on this sphere of directions. Indeed, to smallest order,

$$\begin{aligned} A|_{k=-2} &= (x, z, -y) \\ B|_{k=-2} &= (-z, y, x). \end{aligned}$$

Hence for any $\sigma \in MCG(S_{1,1})$,

$$\begin{aligned} \lim_{k\to-2} \sigma|_{\kappa^{-1}(k)} &\in \{\textit{group generated by } 90^o \textit{ rotations about the } x, y \textit{ axis}'\} \\ &= \{X, Y | X^4 = Y^4 = (XY)^3 = \cdots = Id.\} \\ &= \text{octahedral group.} \end{aligned}$$

Recall that the octahedral group is the set of rigid motions of a regular octahedron (equivalently, the cube), and is a finite group of order 24, with the dihedral group D_4 as a subgroup. Using this fact, we will establish the following result, which forms the basis for the main result of this section:

4.3. **Main result.** In the space of symplectomorphisms (area-preserving diffeomorphisms, for 2-dimensional manifolds) of S^2, k parametrizes a path which limits at -2 to a finite order element of $SO(3)$. This path represents a deformation of an element of $SO(3)$ induced by a mapping class $\sigma \in MCG(S_{1,1})$. In this section, we essentially prove that symplectomorphisms sufficiently close to a finite rotation of S^2 and reachable by a mapping class deformation cannot act ergodically. The following theorem will establish this fact. Recall that for a smooth map from a manifold to itself, an elliptic fixed point is a fixed point such that the Jacobian has all eigenvalues of unit modulus. Elliptic periodic points are elliptic fixed points for the nth iterate of the map. When the period is not relevent, we will call these points elliptic points.

Theorem 4.4. *Let $\sigma \in MCG(S_{1,1})$. Then there exists $\epsilon > 0$ such that $\forall k \in (-2, -2+\epsilon)$, there exists either an elliptic fixed point, or an elliptic period 2 point.*

We will prove this theorem in Section 6. We place it here to establish a connection between the Hamiltonian deformations of a finite rotation of S^2 induced by these mapping class actions and their dynamical properties. We start with a general theorem on the fixed points of surface diffeomorphisms:

Theorem 4.5. *(Kolmogorov-Arnolд́-Moser) Let f be a volume preserving analytic diffeomorphism of a surface M. If x is a non-degenerate elliptic fixed point, then for every $\epsilon > 0$, there exists an arbitrarily small neighborhood U of x and a set $U_0 \subset U$ with the following properties:*
a) U_0 is a union of f-invariant analytic simple closed curves containing x in their interior.
b) The restriction of f to each of these curves is topologically equivalent to an irrational rotation.
c) Denoting by μ the measure associated with the volume form of M, we have

$$\mu(U - U_0) \leq \epsilon\mu(U).$$

The proof of this theorem can be found in [1]. This theorem establishes the relevance of elliptic fixed points of area preserving diffeomorphisms of a surface to the dynamical properties of their action on the surface. The theorem basically says in any neighborhood of a nondegenerate elliptic fixed point of a surface diffeomorphism is a positive measure set of invariant curves. The nondegeneracy condition is that the eigenvalues of the Jacobian at the fixed points not be nth order roots of unity, for $n < 5$. In the language of Hamiltonian dynamical systems, the eigenvalues of the Jacobian at a fixed point p are called the multipliers of p. Explicitly, we establish the following:

Corollary 4.6. *(of Theorem 4.4) σ does not act ergodically on these shells.*

Proof. On the set of shells possessing nondegenerate elliptic fixed points, there exist invariant sets of intermediate measure, thus the action is not ergodic. To prove this corollary, therefore, it is sufficient to establish the nondegeneracy requirement of Theorem 4.5. Note that the set of nth roots of unity, $n < 5$ on the unit circle is a finite, discrete set (see Figure 4 in Section 6). For any σ Anosov, the fixed points on the sphere of directions at the origin (the $k = -2$ shell) have multipliers which are roots of unity of order less than 4. We will establish in the next section that fixed points vary smoothly from shell to shell, and the multipliers at these fixed points vary smoothly and in a non-constant fashion. Thus in an open neighborhood of -2 in $[-2, 2]$, the multipliers at certain fixed points will satisfy the nondegeneracy requirement posed above. Thus this corollary and hence Theorem 1.1 will be proved. □

Corollary 4.7. *The cyclic subgroup generated by any $\sigma \in MCG(S_{1,1})$ does not act leaf-wise ergodically on $X(S_{1,1}, SU(2))$.*

An alternative description of this phenomenon is the statement that the ergodic decomposition of the action of an Anosov mapping class on $X(S_{1,1}, SU(2))$ is finer than the stratification given by κ. Thus, for σ Anosov, σ induces a continuous deformation of sphere maps from an ergodic action to a nonergodic action (by a finite element of $SO(3)$). This is a continuous family of discrete Hamiltonian dynamical systems on the surface S^2. Many of the dynamical properties associated to such symplectic perturbations, including period doubling and creation/annhilation bifurcations, are explicitly embedded in the dynamical system formed by these mapping class actions on $X(S_{1,1}, SU(2))$. In the following section, we will analyze and illustrate some of this dynamical behavior via a specific example.

5. Fixed point varieties

For σ a polynomial automorphism of $\mathbb{R}^3$, the set of fixed points $Fix(\sigma\colon \mathbb{R}^3 \to \mathbb{R}^3)$ is a real algebraic set. Let $\sigma \in MCG(S_{1,1})$ determine such an automorphism, where we abuse notation and use the same symbol. Define $L_\sigma = Fix(\sigma\colon \mathbb{R}^3 \to \mathbb{R}^3) \cap \kappa^{-1}([-2, 2])$.

Proposition 5.1. $\dim_{\mathbb{R}} L_\sigma = 1$.

Proof. L_σ is the 0-set of the polynomial map

$$\widetilde{\sigma}\colon \mathbb{R}^3 \to \mathbb{R}^3 \quad \widetilde{\sigma}(p) = \sigma(p) - p.$$

The Zariski tangent space to L_σ at p is the 0-eigenspace (the kernel) of the linear map

$$T_p\widetilde{\sigma}\colon \mathbb{R}^3 \to \mathbb{R}^3.$$

Choose a point $p \in L_\sigma$ such that $\kappa(p) = k \in (-2, 2)$. Since σ is volume preserving, the determinant of the Jacobian is 1. σ also acts as area preserving diffeomorphisms on $\kappa^{-1}(k) \cong S^2$. Hence, two of the eigenvalues are reciprocals, forcing the third eigenvalue to be 1. Indeed, the eigenvalues of $T_p\sigma$ are $(1, \lambda, \lambda^{-1})$, where $\lambda \in \mathbb{R} - \{0\}$ or $\lambda \in U(1)$. Hence the eigenvalues of $T_p\widetilde{\sigma}$ are $(0, 1-\lambda, 1-\lambda^{-1})$, where λ varies continuously with p. Hence, for $\lambda \neq 1$, $\dim_{\mathbb{R}} L_\sigma = 1$. □

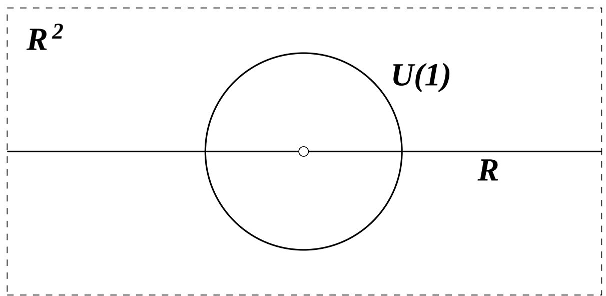


FIGURE 2. For $\lambda \in Sp(\mathbb{R}^2)$, $\lambda \in U(1) \cup \{\mathbb{R} - \{0\}\}$.

Notice that if 1 is a simple eigenvalue of the Jacobian of σ at p, then it follows that L_σ is transverse to $\kappa^{-1}(\kappa(p))$. By restriction to certain σ, we can ensure that this is almost always the case:

Proposition 5.2. *Let σ be Anosov. Then $Fix(\sigma|_{\kappa^{-1}(k)})$, $k \in (-2, 2)$ is discrete, and L_σ is transverse to κ almost everywhere.*

proof. On the shell $k = 2$, $Fix(\sigma|_{\kappa^{-1}(2)})$ is discrete. This follows from the fact that the fixed points of any linear hyperbolic automorphism of T^2 are isolated. Disregarding the four singular points, the multipliers of any point $p \in L_\sigma \cup \kappa^{-1}(2)$ lie off the unit circle. Hence at these points, L_σ is transverse to κ, and $p \in Nonsing(L_\sigma)$. This condition persists for all $p \in L_\sigma \cap \kappa^{-1}(k)$, k in a neighborhood of 2. Suppose now there exists a $k \in (-2, 2)$ such that $Fix(\sigma|_{\kappa^{-1}(k)})$ is not discrete. Then $\dim(L_\sigma \cap \kappa^{-1}(k)) > 0$. Choose p in a connected component of $L_\sigma \cap \kappa^{-1}(k)$ which has dimension greater than 0. The multipliers of p here are necesarily 1, since T_pL_σ is tangent to $\kappa^{-1}(k)$. Hence, on this component

$$\{L_\sigma \cap \kappa^{-1}(k)\} \subset Sing(L_\sigma).$$

But L_σ is an algebraic set, so

$$\dim(Sing(L_\sigma) < \dim(Nonsing(L_\sigma)) = 1.$$

By this contradiction, $Fix(\sigma|_{\kappa^{-1}(k)})$ is discrete for all k. It follows immediately that L_σ is transverse to κ almost everywhere, since this is not true only on $Sing(L_\sigma)$. □

$Sing(L_\sigma)$ is a discrete set of points. But the points in $Sing(L_\sigma)$ are precisely the places where L_σ is not transverse to κ.

Remark 5.3. There are three equivalent phenomena occurring for Anosov mapping class actions on $X(S_{1,1}, SU(2))$: The singular points of L_σ are precisely the places where L_σ is not transverse to κ. These singular points p are precisely the places where the multipliers are 1. In summary.

$$Sing(L_\sigma) = \{p \in L_\sigma | \lambda = 1\} = \{p \in L_\sigma | L_\sigma \pitchfork \kappa\}.$$

Example. Let σ correspond to the $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ Anosov map of the torus. σ induces a polynomial automorphism of $\mathbb{R}^3$, also denoted by σ, given by

$$\sigma = B^{-1}A\colon (x, y, z) = (z, zy - x, z(zy - x) - y).$$

L_σ is defined by the set of equations

$$\begin{aligned} x &= z \\ y &= zy - x \\ z &= z(yz - x) - y, \end{aligned}$$

and is topologically an embedded line.

Hence, in general, L_σ is a 1-dimensional algebraic set composed of a finite number of irreducible components. Denote an irreducible component by ℓ_σ.

Lemma 5.4. *Let σ be Anosov. Then, restricted to each ℓ_σ, λ is a nonconstant function of p.*

proof. There are four cases to consider (see Figure 3):

(1) $\ell_\sigma \cap \{k = 2\} \neq \emptyset$, $\ell_\sigma \cap \{k = -2\} \neq \emptyset$. Here

$$\lim_{k \to -2} eig(Jac_\sigma(p)) \in U(1), \quad p \in \ell_\sigma$$

while for $p \in \ell_\sigma \cap \{k = 2\}$, $eig(Jac_\sigma(p)) \notin U(1)$. Thus λ is not constant on all of ℓ_σ.

(2) $\ell_\sigma \cap \{k = 2\} \neq \emptyset$, $\ell_\sigma \cap \{k = -2\} = \emptyset$. As in Case 1, for $p \in \ell_\sigma \cap \{k = 2\}$, $eig(Jac_\sigma(p)) \notin U(1)$. Since ℓ_σ does not meet the origin (the $k = -2$ set), there must be a point where ℓ_σ is tangent to a level set. Call this level set $\{k = k_0\}$. Then at this intersection point p_0,

$$eig(Jac_\sigma(p_0)) = 1.$$

Thus, again, λ is not constant on ℓ_σ.

(3) $\ell_\sigma \cap \{k = 2\} = \emptyset$, $\ell_\sigma \cap \{k = -2\} \neq \emptyset$. Since ℓ_σ does not intersect the shell $k = 2$, ℓ_σ achieves a maximum with respect to k in $(-2, 2)$. As algebraic sets cannot be locally modeled on closed intervals, ℓ_σ must turn back toward the origin. Hence there exists a point where ℓ_σ is tangent to a level set, again denoted $\{k = k_0\}$. It may happen here, though that at both the origin and on the level set $k = k_0$, that the multipliers at fixed points are 1. But by Proposition 5.2, the set $\{p \in \ell_\sigma | eig(Jac_\sigma(p))\} \subset Sing(L_\sigma)|_{\ell_\sigma}$, and is discrete. Hence, there exists a $k_1 \in (-2, k_0)$ where $eig(Jac_\sigma(p)) \neq 1$ for $p \in \ell_\sigma \cap \{k = k_1\}$.

(4) $\ell_\sigma \cap \{k = 2\} = \ell_\sigma \cap \{k = -2\} = \emptyset$. Here there exist 2 distinct shells where ℓ_σ is tangent. The same reasoning used in case 3 can be used here to show there is a shell between these two where ℓ_σ is transverse. Thus the multipliers are not 1 here, and we are done. □

6. Proof of Theorem 4.4

Let σ be Anosov. To show the theorem, we shall need two facts:

Fact 6.1. [12] *Any real analytic variety is locally homeomorphic to the cone over a polyhedron with even Euler characteristic.*

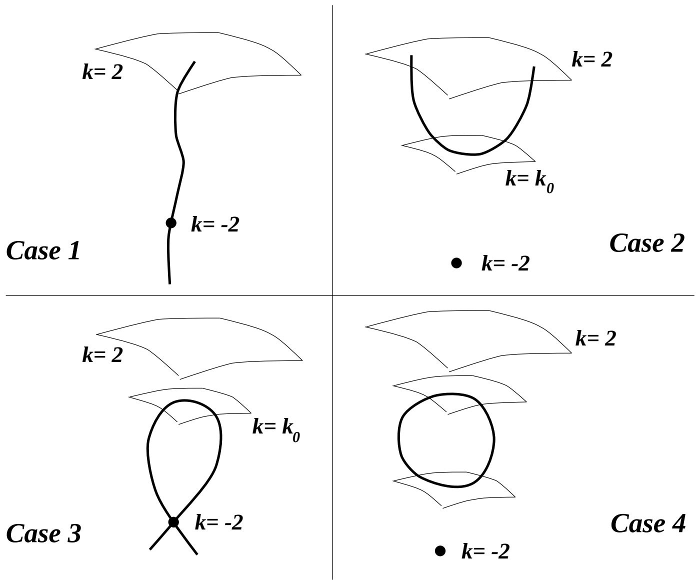
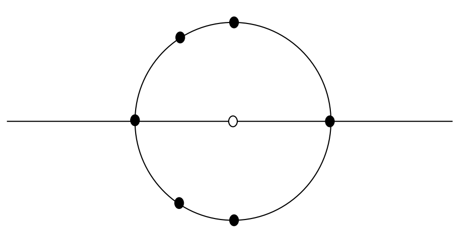


FIGURE 3. Irreducible components of L_σ.

In particular, for a 1-dimensional algebraic variety, the singular points all have an even number of branches emanating from them.

Theorem 6.2. [9] *Let $h\colon M \to M$ be an orientation preserving homeomorphism of a smooth orientable* 2 *manifold which preserves area. If p is an isolated fixed point of h, then the index of p is less than 2.*

proof. (of theorem 4.4) By Propositions 5.2 and 5.1 (except at the origin itself), L_σ transversally intersects each shell $\kappa^{-1}(\kappa(k))$, $k \sim -2$, in a finite discrete set. Recall that each shell is topologically a copy of S^2. Hence by the Lefschetz Fixed Point Theorem, there exists fixed points of positive index. By Theorem 6.2, these fixed points must have index 1. Note that on the $k = -2$ shell (again considering the sphere of directions at the origin) σ restricts to an element of the octahedral subgroup of $SO(3)$. Hence the eigenvalues of the Jacobian at any fixed point are order n roots of unity, for $n < 5$ (see Figure 4).



Eigenvalues of Octahedral group.

FIGURE 4. Set of multipliers for fixed points p on the shell $k = -2$.

Choose an irreducible component $\ell_\sigma \subset L_\sigma$ such that $\ell_\sigma \cap \{k = -2\} \neq \emptyset$. By Lemma 5.4, the eigenvalue λ varies continuously and in a nonconstant fashion with respect to p, for $p \in \ell_\sigma$. Note also that

$$\lambda_0 = \lim_{k \to -2} eig(Jac_\sigma(p))|_{\ell_\sigma} \subset U(1).$$

Suppose that this limit is an order 3 or 4 root of unity. Then for all p in ℓ_σ near the origin, the multipliers will still lie on $U(1)$, and these p will be nondegenerate elliptic fixed points. Hence by Corollary 4.6 the action is not ergodic here. Suppose now that $\lambda_0 = 1$. Note that the index is a continuous integer valued function on each connected component of $Nonsing(L_\sigma)$. We claim that for all p on ℓ_σ near the origin, the multipliers are on the unit circle. Suppose that there existed a p near the origin with multipliers near 1 but real. Then the fixed point would be hyperbolic. But a hyperbolic fixed point with positive eigenvalues has index -1. The only nondegenerate index 1 fixed point type with multipliers near 1 is elliptic. As L_σ near $k = -2$ will be transverse to the level sets, the multipliers at points $p \in \ell_\sigma$ near the origin will not have multipliers 1. Hence there will be an interval of nondegenerate elliptic fixed points here. Again, the action will not be ergodic here. Suppose now that $\lambda_0 = -1$. Then it is possible that the multipliers of points p near the origin can be real, as a hyperbolic fixed point with negative eigenvalues has index $+1$. However, if we square the automorphism, and consider the fixed point set $\ell_{\sigma^2} \subset L_{\sigma^2}$, λ_0 here is $+1$. A review of the case above shows that there exists an elliptic fixed point for the map σ^2. This point will be an elliptic period-2 orbit for σ. □

Appendix A. A non-Anosov ergodic action on $X(S_8, U(1))$.

In this appendix, we present an example of a symplectic automorphism of a torus which is not hyperbolic, yet acts ergodically. This example relies on the existence of unit modulus algebraic integers which are not roots of unity.

Consider the algebraic integer $z = \sqrt{2-\sqrt{2}} + i\sqrt{\sqrt{2}-1}$. z lies on the unit circle, but is not a root of unity. Its minimal polynomial

$$x^8 - 12x^6 + 6x^4 - 12x^2 + 1$$

has as its companion matrix Z, where

$$Z = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 12 & 0 & -6 & 0 & 12 & 0 \end{pmatrix} \in GL(\mathbb{Z}^8).$$

Note that although $Z \in SL(\mathbb{Z}^8) \subset GL(\mathbb{Z}^8)$, Z is not symplectic: A $n \times n$ matrix S is symplectic if it satisfies the equation $SJ_nS^t = J_n$, where ^{t} denotes transpose, and

$$J_n = \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ -I_{\frac{n}{2}} & 0 \end{pmatrix}.$$

However, $GL(\mathbb{Z}^n)$ embeds into $Sp(\mathbb{Z}^{2n})$ via the map

$$GL(\mathbb{Z}^n) \ni T \hookrightarrow \begin{pmatrix} T & 0 \\ 0 & (T^{-1})^t \end{pmatrix} \in Sp(\mathbb{Z}^{2n}).$$

A simple calculation will verify that the image of this embedding is indeed symplectic. In our case, it follows that z arises as an eigenvalue of the symplectic matrix

$$\begin{pmatrix} Z & 0 \\ 0 & (Z^{-1})^t \end{pmatrix} \in Sp(\mathbb{Z}^{16}).$$

The eigenvalues of this transformation all come from the eigenvalues of Z, the conjugates of the minimal polynomial above. Hence by Theorem 3.2, this transformation comes from a mapping class and acts ergodically on $X(S_8, U(1)) \cong T^{16}$.

Remark A.1. It is interesting to note that the algebraic conjugates of z above do not all lie on the unit circle $U(1)$. If an algebraic integer and all of its conjugates do lie on $U(1)$, then in fact the integer is a root of unity. This implies that the only elliptic automorphisms of T^n are periodic automorphisms. In the example above, the linear action of $\begin{pmatrix} Z & 0 \\ 0 & (Z^{-1})^t \end{pmatrix}$ on T^{16} possesses a sufficient number of hyperbolic "directions" to satisfy the ergodicity property.

References

1. Arnol'd, V., *Small denominators and problems of stability of motion in classical and celestial mechanics,* Russ. Math. Surveys **18** (1963), 85-191.
2. Goldman, W., *Ergodic theory on moduli spaces, Ann. Math* **146** (1997), 475-507.
3. Goldman, W., *Introduction to character varieties,* preprint.
4. Heubschmann, J., *Symplectic and Poisson structures of certain moduli spaces,* Pub. IRMA, Lille-1993, **33** No. X, preprint hep-th/9312112.
5. Katok, A., *Bernoulli diffeomorphisms on surfaces,* Ann. Math. **110** (1979), 529-547.
6. Lickorish, W., *A finite generating set for the homotopy group of a 2-manifold,* Pro. Camb. Phil. Soc. **60** (1964), 769-778.
7. Magnus, W., Karass, A. and Solitar, S., *Combinatorial Group Theory,* New York, Dover, 1976.
8. Mañe, R., *Ergodic theory and differentiable dynamics,* New York, Springer-Verlag, 1987.
9. Pelikan, S. and Slaminka, E., *A bound for the fixed point index of area-preserving homeomorphisms of two-manifolds,* Ergod. Th. Dyn. Sys. **7** (1987), 463-479.
10. Procesi, C., *The invariant theory of $n \times n$ matrices,* Adv. Math. **19** (1976), 306-381.
11. Sjamaar, R., and Lerman, E., *Stratified symplectic spaces and reduction,* Ann. Math. **134** (1991), 375-422.
12. Sullivan, D., *Combinatorial invariants of analytic spaces,* Proc. of Liverpool Singularities Symp. I, Lect. Notes in Math. **192**, New York, Springer-Verlag, 1971.
13. Whittemore, A., *On special linear characters of free groups of rank $n \geq 4$,* Proc. AMS **40**, No. 2 (1973), 383-388.

Department of Mathematics, University of Maryland, College Park, MD 20742 USA
E-mail address: rbrown@math.umd.edu