MAPPING CLASS ACTIONS ON MODULI SPACES

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Abstract. It is known that the mapping class group of a compact surface S, $MCG(S)$, acts ergodically with respect to symplectic measure on each symplectic leaf of the Poisson moduli space of flat $SU(2)$-bundles over S, $X(S, SU(2))$. In our study of how individual mapping classes act on X, we show that ergodicity does not restrict to that of cyclic subgroups of $MCG(S_{1,1})$, for $S_{1,1}$ a punctured torus. The action of a mapping class on $X(S_{1,1}, SU(2))$ induces a continuous deformation of discrete Hamiltonian dynamical systems on the 2-sphere S^2. We discuss some of the dynamical phenomena associated to this action. We then present a method for extending this result to the actions of certain pseudo-Anosov classes on the moduli spaces of closed surfaces of genus $g > 1$.

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1. Introduction

For G a compact Lie group, the space M of G-characters of representations of the fundamental group of a compact Kähler manifold S into G is basic semi-algebraic, and a Poisson space. In this case, M is birationally isomorphic to the space of flat G-bundles over S, $X(S, G)$, and is a general model space for many of the geometric structure spaces associated to S (see for instance [10]). Any diffeomorphism φ of S induces an action on M, which depends only on the isotopy class of φ. Hence there is an action of the mapping class group $MCG(S)$ on M. This action preserves the Poisson structure, and acts as symplectic measure preserving transformations on each symplectic leaf. In [7], Goldman shows that $MCG(S)$ acts ergodically on each symplectic leaf of M, for S a compact surface, and G compact with $U(1)$ and $SU(2)$ factors. And in [12], Pickrell and Xia extend this result to cover any compact, connected G.

In this research announcement, we embark on a study of the dynamical properties of the action of individual elements $\sigma \in MCG(S)$ on M, for $G = SU(2)$. In particular, can cyclic subgroups of $MCG(S)$ act ergodically?

The results of Goldman and Pickrell-Xia imply that the ergodic decomposition associated to the action of $MCG(S)$ on M is equal to the symplectic foliation induced by the Poisson structure. However, in the case of a torus with one boundary component, $S_{1,1}$, the ergodic decomposition of any $\sigma \in MCG(S_{1,1})$ is actually finer than this symplectic foliation: There is a a positive measure set of leaves where the action is not ergodic on $X(S_{1,1}, SU(2))$. A study of the dynamics across the leaves displays the complexity of these actions.

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The action of elements of $MCG(S_{1,1})$ on $X(S_{1,1}, SU(2))$ induces a smooth deformation of discrete Hamiltonian dynamical systems on the 2-sphere S^2. For a hyperbolic mapping class, this 1-parameter family of S^2 maps ranges from a singular Bernoulli map with maximal topological entropy on the set of reducible characters, to a finite order isometry of S^2. It is known that all periodic points of a hyperbolic action are hyperbolic in a neighborhood of the reducible characters, yet nondegenerate elliptic points persist well into a neighborhood of the S^2 isometry. Much of the dynamical behavior associated to nonlinear hamiltonian systems in general, like period-doubling cascades, and creation and higher-order bifurcations, are present within these deformations. Positive topological entropy is a generic property of area-preserving S^2 self-maps. Yet it is currently not known what the entropy is within the two extremes of these deformations.

A closed, orientable positive genus-g surface S_g covers the standard 2-torus as an orbifold with one order-g branch point $\mathbb{T}_1$. A representation ψ of $\pi_1(S_{1,1})$ into G identifies with a representation of $\pi_1(\mathbb{T}_1)$ when ψ evaluated over an element parallel to the boundary is finite of order g. Hence, this covering determines an embedding of $X(\mathbb{T}_1, G) \subset X(S_{1,1}, G)$ into $X(S_g, G)$ which is compatible with the Poisson structure. Lifting a mapping class through this covering results in an action on $X(S_g, G)$ which is compatible with this embedding. We determine the subgroups of $MCG(S_{1,1})$ which lift to subgroups of $MCG(S_g)$. Anosov mapping classes of the torus lift to pseudo-Anosov elements of S_g. We show that non-hyperbolic points exist for a large number of pseudo-Anosov mapping class actions on $X(S_g, SU(2))$. We conjecture that the action of $\sigma \in MCG(S_g)$ on $X(S_g, SU(2))$ is not ergodic.

2. Preliminaries

Let $S_{g,n}$ be an orientable genus-g surface with n (possibly 0) boundary components, and G a compact, connected Lie group. The set $Hom(\pi_1(S_{g,n}), G)$ is an algebraic subset of G^{2g+n} formed by evaluation on a set of generators of $\pi_1(S_{g,n})$. G acts on G^{2g+n} by diagonal conjugation, leaving invariant $Hom(\pi_1(S_{g,n}), G)$. The resulting quotient space

$$X(S_{g,n}, G) = Hom(\pi_1(S_{g,n}), G)/G$$

is Poisson. The symplectic leaves correspond to fixed conjugacy class of holonomy over $\partial S_{g,n}$ and are algebraic varieties with finite symplectic volume [8]. Since G is compact, the leaf space $[G]^n$, defined as the product of the sets of conjugacy classes of G over each element of the boundary of $S_{g,n}$, is a standard measure space. The mapping class group $MCG(S_{g,n}) = \pi_0 Diff(S_{g,n})$ acts on $X(S_{g,n}, G)$, preserving the Poisson structure, and the symplectic volume on each leaf.

As evidenced by Goldman and Pickrell-Xia above, this action is ergodic with respect to symplectic measure on every leaf. We address the question of whether a single element $\sigma \in MCG(S_{g,n})$ can act ergodically. To this end, we start with a classification of ergodic mapping class actions on $U(1)$-characters of closed surfaces. We then use this as a model to study the case when $G = SU(2)$).

3. Anosov mapping classes of the punctured torus

Let S_g denote a closed, genus-g surface. For a compact, abelian Lie group G,

$$X(S_g, G) = Hom(\pi_1(S_g), G)/G = H^1(\pi_1(S_g); G) \cong G^{2g}.$$

It is sufficient to consider only the case $G = U(1)$. Here $X(S_g; U(1))$ identifies with a torus. The mapping class group factors through the integral symplectic group. In the classical theory of linear automorphisms of the torus, we have the following result (see, for example, [9]):

Theorem 3.1. *A linear automorphism of a torus acts ergodically iff none of its eigenvalues is a root of unity.*

This is a complete classification of the ergodic elements of $MCG(S_g)$: Those with image in $Sp(\mathbb{Z}^{2g})$ with non-periodic spectra. Note here that this is different from an action of a pseudo-Anosov element of $MCG(S_g)$ on $X(S_g, G)$. Papadapolous [11] has shown that in the epimorphism $MCG(S_g) \to Sp(\mathbb{Z}^{2g})$, the inverse image of any integral symplectic matrix contains a pseudo-Anosov mapping class. In particular, there are pseudo-Anosov homeomorphism classes which map to $I \in Sp(\mathbb{Z}^{2g})$ (i.e., which act trivially on $X(S_g, G)$). Also note that in this case, for $g > 1$, there exist non-hyperbolic (non-Anosov) toral automorphisms which act ergodically on $X(S_g; U(1))$ [1].

Consider now the space of $SU(2)$-characters of the 2-torus $\mathbb{T}^2$. The set of 2 generator abelian subgroups of $SU(2)^2$ naturally identifies with $U(1)^2$, but conjugation within $SU(2)$ is not trivial; conjugation by the generator of the Weyl group is of order 2. Here, $X(\mathbb{T}^2, SU(2))$ is the quotient of a 2-torus by an involution. Topologically, this quotient is a 2-sphere S^2, with four order-2 branch points corresponding to the places where the involution is not free. The mapping class group factors through $\mathbb{P}SL(\mathbb{Z}^2)$, and it is precisely the hyperbolic elements which act ergodically:

Theorem 3.2. *$\sigma \in MCG(\mathbb{T}^2)$ acts ergodically on $X(\mathbb{T}^2, SU(2))$ iff σ is hyperbolic (Anosov).*

Remove an open disk from the torus and denote the resulting genus-1 surface with 1 boundary component $S_{1,1}$. Note $\pi_1(S_{1,1})$ is free on two generators. $X(S_{1,1}, SU(2))$ identifies with a compact semi-algebraic subset of $\mathbb{R}^3$, parameterized by the characters of representations evaluated on the two generators of $\pi_1(S_{1,1})$ and the character of their product. This result originally goes back to Fricke [6]. Compare also Fried's construction for word maps on $S^3 \times S^3$ [5]. The symplectic leaves corresponding to fixed boundary holonomy are the level sets of the cubic polynomial κ which is the character of the representation evaluated on the element $c \in \pi_1(S_{1,1})$ homotopic to $\partial S_{1,1}$. $X(S_{1,1}, SU(2))$ is thus comprised of the level sets which satisfy the inequality

$$-2 \leq trace(\phi([b,a])) \leq 2, \quad \phi \in Hom(\pi_1(S_{1,1}), SU(2)).$$

The level set corresponding to trace 2 boundary holonomy is the $SU(2)$-character variety of the closed torus $\mathbb{T}^2$ and consists of the reducible characters. This is the set of abelian representation classes of the free group on two generators into $SU(2)$. Topologically, this set is a 2-sphere. More precisely, it is the quotient of a 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ by an involution which fixes the half integer lattice in $\mathbb{R}^2$. The level set corresponding to trace -2 (boundary holonomy $-Id$) is the origin, and the intermediate level sets of κ form a set of concentric spheres. Mapping classes $S_{1,1}$ identify with mapping classes of a closed torus. Hence $MCG(S_{1,1})$ acts as a volume preserving $\mathbb{P}SL(\mathbb{Z}^2)$ action on $\mathbb{R}^3$ leaving invariant κ and acting as area preserving transformations on each level set. The fixed point set of any mapping class action is a 1-dimensional algebraic variety which intersects non-trivially and almost everywhere transversally on each level set of $X(S_{1,1}, SU(2))$ (see [4]). The fixed point type of a fixed point can be determined by the eigenspaces tangent to these level sets.

Theorem 3.3. *For any $\sigma \in MCG(S_{1,1})$, there exists a positive measure set of leaves in $X(S_{1,1}, SU(2))$ on which σ does not act ergodically.*

The proof of this, documented in [1], relies on the fact that on these leaves lie non-degenerate elliptic fixed points.

4. Deformations of discrete Hamiltonian dynamical systems on S^2

A symplectic diffeomorphism of a manifold M describes a discrete Hamiltonian dynamical system on M. In dimension 2, this diffeomorphism need only be area preserving. Replace the origin in $\mathbb{R}^3$ with the sphere of directions, so that $X(S_{1,1}, SU(2)) \cong S^2 \times [-2, 2]$. Then a mapping class $\sigma \in MCG(S_{1,1})$ induces a map

$$\sigma \colon S^2 \times [-2, 2] \to S^2,$$

where for $r \in [-2, 2]$, $\sigma|_{\kappa^{-1}(r)} = \sigma(\cdot, r)$.

Hyperbolic elements of $\mathbb{P}SL(\mathbb{Z}^2)$ act ergodically (indeed, the action is Bernoulli) on the $\kappa = 2$ shell. Recall that that this shell is the set of reducible characters, and identifies with the quotient of $\mathbb{T}^2$ by an involution. The $\mathbb{P}SL(\mathbb{Z}^2)$ action on this shell arises from an $SL(\mathbb{Z}^2)$ action on $\mathbb{T}^2$ which factors through to the quotient. Hence a hyperbolic element acts as a Bernoulli map on $\kappa = 2$ and is structurally stable. Hence, on the shells in a neighborhood of the $\kappa = 2$ shell, the maps will remain Bernoulli. Moreover, the topological entropy can readily be computed on the $\kappa = 2$ shell and is $h_{top}(\sigma) = \log \lambda$ where λ is the spectral radius of σ.

On the sphere of directions replacing the origin of $X(S_{1,1}, SU(2))$ (the $\kappa = -2$ shell), the action is a finite rigid rotation of S^2 of order at most 4. It is shown in [1] that elliptic fixed points exist on every level set of κ in a neighborhood of this shell. Hence the action is not ergodic on these level sets. However, there is complexity. It is known [14] that the generic area-preserving map of S^2 has positive topological entropy, due to a result by Pixton [13] that on a residual set of area preserving maps of S^2, all hyperbolic points have transverse homoclinic points (this implies the existence of an embedded horseshoe, which contributes to entropy). Numerical studies of the action of hyperbolic elements of $\mathbb{P}SL(\mathbb{Z}^2)$ indicate the presence of hyperbolic points on all shells outside of the origin. Hence the action of a hyperbolic mapping class on the shells near the origin exhibit nonergodic behavior with positive topological entropy. It is not currently known what the topological entropy is on these shells outside of the two extremes.

For σ hyperbolic, the fixed point set $L_\sigma = Fix(\sigma)$ is a 1-dimensional algebraic variety, non-trivially transverse to almost every level set of κ. The singular points of L_σ (points of index greater than 2) are precisely the places where the intersection is not transverse. If we parameterize the Hamiltonian deformation by κ, these singular points are bifurcation points as viewed along an irreducible component of L_σ, and lead to changes in fixed point type. In [4], some of the dynamical behavior of these mapping class actions is studied via an analysis of the structure of L_σ. In particular, the Anosov toral automorphism given by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is highlighted.

5. Mapping class lifts and extensions of their actions on moduli

A closed, orientable, positive genus-g surface S_g covers a torus as an orbifold with 1 order-g branch point. This dihedral covering is formed via the composition of two cyclic coverings and is detailed in Brown [2]. Let $\mathcal{O}$ be a compact surface orbifold with n singular points. By excising a small isolated neighborhood of each singularity of $\mathcal{O}$, one may associate to the orbifold a compact, genus-g surface $S_{g,n}$. Then $X(\mathcal{O}, G)$ embeds into $X(S_{g,n}, G)$ as the subset of symplectic leaves of $X(S_{g,n}, G)$ corresponding to representations whose boundary holonomy is finite of order the same as the index of each singularity of $\mathcal{O}$. By definition, elements of $MCG(S_{g,n})$ correspond to diffeomorphisms which pointwise fix each component of the boundary. Orbifold isomorphisms of $\mathcal{O}$ which fix the singular set will then correspond to mapping classes of the associated $S_{g,n}$. Thus one may identify surface orbifold isomorphisms to mapping classes of $S_{g,n}$. The associated surface of the torus with one order-g branch point is $S_{1,1}$, a torus with one boundary component.

A mapping class of a topological space is said to lift to that of a covering space iff the automorphisms commute with the covering map. For mapping classes of $S_{1,1}$ which lift, Anosov classes lift to non-periodic, irreducible elements of $MCG(S_g)$; the pseudo-Anosov classes.

Theorem 5.1 ([2]). *For $g > 1$, any hyperbolic toral automorphism has a finite positive power which lifts to a quadratic, orientable pseudo-Anosov element of $MCG(S_g)$.*

A covering map $\rho : M \to N$ induces an injection of the fundamental group of the cover into that of the base. By pull back through this cover, $X(N, G)$ embeds into $X(M, G)$. Let $\sigma \in MCG(N)$. Denote by $\widetilde{\sigma} \in MCG(M)$ a lift of σ. By definition, $\widetilde{\sigma}$ is a lift of σ if $\rho \circ \widetilde{\sigma} = \sigma \circ \rho$. $\widetilde{\sigma}$ acts on $X(M, G)$ and restricts to the action of σ on the embedded $X(N, G)$. In addition, the covering group associated to ρ acts on $X(M, G)$, fixing the embedded $X(N, G)$ and commuting with $\widetilde{\sigma}$. In this way, the tangent space to a fixed point of a lifted mapping class decomposes into invariant spaces, the eigenspaces of the covering transformation.

In the present case, let $\mathbb{T}_2$ denote the 2-torus as an orbifold with 2 order-g branch points, and $\mathbb{T}_1$ the torus with one order-g branch point. Let $\tau : \mathbb{T}_2 \to \mathbb{T}_1$ denote the regular double covering of $\mathbb{T}_2$ over $\mathbb{T}_1$ formed by unwrapping $\mathbb{T}_1$ around one of its generators. Then it is shown in [3]:

Theorem 5.2. *Let $\sigma \in MCG(\mathbb{T}_1)$ be hyperbolic. Then the action of $\tau^*\sigma$ on $X(\mathbb{T}_2, SU(2))$ possesses non-hyperbolic points.*

These non-hyperbolic points are not elliptic, however, but rather 1-elliptic, referring to the dimension of the symplectic subspace where the multipliers are non-real [3].

The cover $\rho : S_g \to \mathbb{T}_1$ referred to above is comprised of the cyclic orbifold covering of S_g over $\mathbb{T}_2$ composed with τ. For g even, the singular representation corresponding to $-id$ boundary holonomy in $X(\mathbb{T}_1, SU(2))$ is smooth in $X(S_g, SU(2))$. A mapping class which lifts will fix this point, and the eigenvalues of the Jacobian will include plus or minus 1 of order 2.

Corollary 5.3. *Any $\widetilde{\sigma} \in MCG(S_g)$ which arises as a lift of an Anosov toral automorphism possesses a non-hyperbolic fixed point.*

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