A Mathematician's Lament

by Paul Lockhart

A musician wakes from a terrible nightmare. In his dream he finds himself in a society where music education has been made mandatory. "We are helping our students become more competitive in an increasingly sound-filled world." Educators, school systems, and the state are put in charge of this vital project. Studies are commissioned, committees are formed, and decisions are made— all without the advice or participation of a single working musician or composer.

Since musicians are known to set down their ideas in the form of sheet music, these curious black dots and lines must constitute the "language of music." It is imperative that students become fluent in this language if they are to attain any degree of musical competence; indeed, it would be ludicrous to expect a child to sing a song or play an instrument without having a thorough grounding in music notation and theory. Playing and listening to music, let alone composing an original piece, are considered very advanced topics and are generally put off until college, and more often graduate school.

As for the primary and secondary schools, their mission is to train students to use this language— to jiggle symbols around according to a fixed set of rules: "Music class is where we take out our staff paper, our teacher puts some notes on the board, and we copy them or transpose them into a different key. We have to make sure to get the clefs and key signatures right, and our teacher is very picky about making sure we fill in our quarter-notes completely. One time we had a chromatic scale problem and I did it right, but the teacher gave me no credit because I had the stems pointing the wrong way."

In their wisdom, educators soon realize that even very young children can be given this kind of musical instruction. In fact it is considered quite shameful if one's third-grader hasn't completely memorized his circle of fifths. "I'll have to get my son a music tutor. He simply won't apply himself to his music homework. He says it's boring. He just sits there staring out the window, humming tunes to himself and making up silly songs."

In the higher grades the pressure is really on. After all, the students must be prepared for the standardized tests and college admissions exams. Students must take courses in Scales and Modes, Meter, Harmony, and Counterpoint. "It's a lot for them to learn, but later in college when they finally get to hear all this stuff, they'll really appreciate all the work they did in high school." Of course, not many students actually go on to concentrate in music, so only a few will ever get to hear the sounds that the black dots represent. Nevertheless, it is important that every member of society be able to recognize a modulation or a fugal passage, regardless of the fact that they will never hear one. "To tell you the truth, most students just aren't very good at music. They are bored in class, their skills are terrible, and their homework is barely legible. Most of them couldn't care less about how important music is in today's world; they just want to take the minimum number of music courses and be done with it. I guess there are just music people and non-music people. I had this one kid, though, man was she sensational! Her sheets were impeccable— every note in the right place, perfect calligraphy, sharps, flats, just beautiful. She's going to make one hell of a musician someday."

Waking up in a cold sweat, the musician realizes, gratefully, that it was all just a crazy dream. "Of course!" he reassures himself, "No society would ever reduce such a beautiful and meaningful art form to something so mindless and trivial; no culture could be so cruel to its children as to deprive them of such a natural, satisfying means of human expression. How absurd!"

Meanwhile, on the other side of town, a painter has just awakened from a similar nightmare…

I was surprised to find myself in a regular school classroom— no easels, no tubes of paint. "Oh we don't actually apply paint until high school," I was told by the students. "In seventh grade we mostly study colors and applicators." They showed me a worksheet. On one side were swatches of color with blank spaces next to them. They were told to write in the names. "I like painting," one of them remarked, "they tell me what to do and I do it. It's easy!"

After class I spoke with the teacher. "So your students don't actually do any painting?" I asked. "Well, next year they take Pre-Paint-by-Numbers. That prepares them for the main Paint-by-Numbers sequence in high school. So they'll get to use what they've learned here and apply it to real-life painting situations— dipping the brush into paint, wiping it off, stuff like that. Of course we track our students by ability. The really excellent painters— the ones who know their colors and brushes backwards and forwards— they get to the actual painting a little sooner, and some of them even take the Advanced Placement classes for college credit. But mostly we're just trying to give these kids a good foundation in what painting is all about, so when they get out there in the real world and paint their kitchen they don't make a total mess of it."

"Um, these high school classes you mentioned…"

"You mean Paint-by-Numbers? We're seeing much higher enrollments lately. I think it's mostly coming from parents wanting to make sure their kid gets into a good college. Nothing looks better than Advanced Paint-by-Numbers on a high school transcript."

"Why do colleges care if you can fill in numbered regions with the corresponding color?"

"Oh, well, you know, it shows clear-headed logical thinking. And of course if a student is planning to major in one of the visual sciences, like fashion or interior decorating, then it's really a good idea to get your painting requirements out of the way in high school."

"I see. And when do students get to paint freely, on a blank canvas?"

"You sound like one of my professors! They were always going on about expressing yourself and your feelings and things like that—really way-out-there abstract stuff. I've got a degree in Painting myself, but I've never really worked much with blank canvasses. I just use the Paint-by-Numbers kits supplied by the school board."

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Sadly, our present system of mathematics education is precisely this kind of nightmare. In fact, if I had to design a mechanism for the express purpose of *destroying* a child's natural curiosity and love of pattern-making, I couldn't possibly do as good a job as is currently being done— I simply wouldn't have the imagination to come up with the kind of senseless, soul-crushing ideas that constitute contemporary mathematics education.

Everyone knows that something is wrong. The politicians say, "we need higher standards." The schools say, "we need more money and equipment." Educators say one thing, and teachers say another. They are all wrong. The only people who understand what is going on are the ones most often blamed and least often heard: the students. They say, "math class is stupid and boring," and they are right.

Mathematics and Culture

The first thing to understand is that mathematics is an art. The difference between math and the other arts, such as music and painting, is that our culture does not recognize it as such. Everyone understands that poets, painters, and musicians create works of art, and are expressing themselves in word, image, and sound. In fact, our society is rather generous when it comes to creative expression; architects, chefs, and even television directors are considered to be working artists. So why not mathematicians?

Part of the problem is that nobody has the faintest idea what it is that mathematicians do. The common perception seems to be that mathematicians are somehow connected with science— perhaps they help the scientists with their formulas, or feed big numbers into computers for some reason or other. There is no question that if the world had to be divided into the "poetic dreamers" and the "rational thinkers" most people would place mathematicians in the latter category.

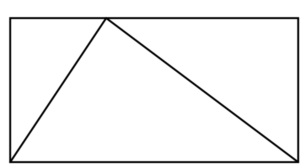
Nevertheless, the fact is that there is nothing as dreamy and poetic, nothing as radical, subversive, and psychedelic, as mathematics. It is every bit as mind blowing as cosmology or physics (mathematicians *conceived* of black holes long before astronomers actually found any), and allows more freedom of expression than poetry, art, or music (which depend heavily on properties of the physical universe). Mathematics is the purest of the arts, as well as the most misunderstood.

So let me try to explain what mathematics is, and what mathematicians do. I can hardly do better than to begin with G.H. Hardy's excellent description:

> A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with *ideas*.

So mathematicians sit around making patterns of ideas. What sort of patterns? What sort of ideas? Ideas about the rhinoceros? No, those we leave to the biologists. Ideas about language and culture? No, not usually. These things are all far too complicated for most mathematicians' taste. If there is anything like a unifying aesthetic principle in mathematics, it is this: *simple is beautiful*. Mathematicians enjoy thinking about the simplest possible things, and the simplest possible things are *imaginary*.

For example, if I'm in the mood to think about shapes— and I often am— I might imagine a triangle inside a rectangular box:

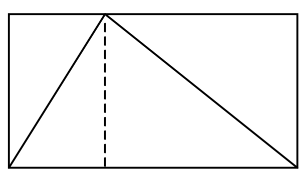


I wonder how much of the box the triangle takes up? Two-thirds maybe? The important thing to understand is that I'm not talking about this *drawing* of a triangle in a box. Nor am I talking about some metal triangle forming part of a girder system for a bridge. There's no ulterior practical purpose here. I'm just *playing*. That's what math is— wondering, playing, amusing yourself with your imagination. For one thing, the question of how much of the box the triangle takes up doesn't even make any *sense* for real, physical objects. Even the most carefully made physical triangle is still a hopelessly complicated collection of jiggling atoms; it changes its size from one minute to the next. That is, unless you want to talk about some sort of *approximate* measurements. Well, that's where the aesthetic comes in. That's just not simple, and consequently it is an ugly question which depends on all sorts of real-world details. Let's leave that to the scientists. The *mathematical* question is about an imaginary triangle inside an imaginary box. The edges are perfect because I want them to be— that is the sort of object I prefer to think about. This is a major theme in mathematics: things are what you want them to be. You have endless choices; there is no reality to get in your way.

On the other hand, once you have made your choices (for example I might choose to make my triangle symmetrical, or not) then your new creations do what they do, whether you like it or not. This is the amazing thing about making imaginary patterns: they talk back! The triangle takes up a certain amount of its box, and I don't have any control over what that amount is. There is a number out there, maybe it's two-thirds, maybe it isn't, but I don't get to say what it is. I have to *find out* what it is.

So we get to play and imagine whatever we want and make patterns and ask questions about them. But how do we answer these questions? It's not at all like science. There's no experiment I can do with test tubes and equipment and whatnot that will tell me the truth about a figment of my imagination. The only way to get at the truth about our imaginations is to use our imaginations, and that is hard work.

In the case of the triangle in its box, I do see something simple and pretty:



If I chop the rectangle into two pieces like this, I can see that each piece is cut diagonally in half by the sides of the triangle. So there is just as much space inside the triangle as outside. That means that the triangle must take up exactly half the box!

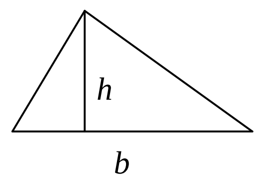
This is what a piece of mathematics looks and feels like. That little narrative is an example of the mathematician's art: asking simple and elegant questions about our imaginary creations, and crafting satisfying and beautiful explanations. There is really nothing else quite like this realm of pure idea; it's fascinating, it's fun, and it's free!

Now where did this idea of mine come from? How did I know to draw that line? How does a painter know where to put his brush? Inspiration, experience, trial and error, dumb luck. That's the art of it, creating these beautiful little poems of thought, these sonnets of pure reason. There is something so wonderfully transformational about this art form. The relationship between the triangle and the rectangle was a mystery, and then that one little line made it obvious. I couldn't see, and then all of a sudden I could. Somehow, I was able to create a profound simple beauty out of nothing, and change myself in the process. Isn't that what art is all about?

This is why it is so heartbreaking to see what is being done to mathematics in school. This rich and fascinating adventure of the imagination has been reduced to a sterile set of "facts" to be memorized and procedures to be followed. In place of a simple and natural question about shapes, and a creative and rewarding process of invention and discovery, students are treated to this:

Triangle Area Formula:

$A = 1/2\ b\ h$



"The area of a triangle is equal to one-half its base times its height." Students are asked to memorize this formula and then "apply" it over and over in the "exercises." Gone is the thrill, the joy, even the pain and frustration of the creative act. There is not even a *problem* anymore. The question has been asked and answered at the same time— there is nothing left for the student to do.

Now let me be clear about what I'm objecting to. It's not about formulas, or memorizing interesting facts. That's fine in context, and has its place just as learning a vocabulary does— it helps you to create richer, more nuanced works of art. But it's not the *fact* that triangles take up half their box that matters. What matters is the beautiful *idea* of chopping it with the line, and how that might inspire other beautiful ideas and lead to creative breakthroughs in other problems— something a mere statement of fact can never give you.

By removing the creative process and leaving only the results of that process, you virtually guarantee that no one will have any real engagement with the subject. It is like *saying* that Michelangelo created a beautiful sculpture, without letting me *see* it. How am I supposed to be inspired by that? (And of course it's actually much worse than this— at least it's understood that there *is* an art of sculpture that I am being prevented from appreciating).

By concentrating on *what*, and leaving out *why*, mathematics is reduced to an empty shell. The art is not in the "truth" but in the explanation, the argument. It is the argument itself which gives the truth its context, and determines what is really being said and meant. Mathematics is *the art of explanation*. If you deny students the opportunity to engage in this activity— to pose their own problems, make their own conjectures and discoveries, to be wrong, to be creatively frustrated, to have an inspiration, and to cobble together their own explanations and proofs— you deny them mathematics itself. So no, I'm not complaining about the presence of facts and formulas in our mathematics classes, I'm complaining about the lack of *mathematics* in our mathematics classes.

If your art teacher were to tell you that painting is all about filling in numbered regions, you would know that something was wrong. The culture informs you— there are museums and galleries, as well as the art in your own home. Painting is well understood by society as a medium of human expression. Likewise, if your science teacher tried to convince you that astronomy is about predicting a person's future based on their date of birth, you would know she was crazy— science has seeped into the culture to such an extent that almost everyone knows about atoms and galaxies and laws of nature. But if your math teacher gives you the impression, either expressly or by default, that mathematics is about formulas and definitions and memorizing algorithms, who will set you straight?

The cultural problem is a self-perpetuating monster: students learn about math from their teachers, and teachers learn about it from their teachers, so this lack of understanding and appreciation for mathematics in our culture replicates itself indefinitely. Worse, the perpetuation of this "pseudo-mathematics," this emphasis on the accurate yet mindless manipulation of symbols, creates its own culture and its own set of values. Those who have become adept at it derive a great deal of self-esteem from their success. The last thing they want to hear is that math is really about raw creativity and aesthetic sensitivity. Many a graduate student has come to grief when they discover, after a decade of being told they were "good at math," that in fact they have no real mathematical talent and are just very good at following directions. Math is not about following directions, it's about making new directions.

And I haven't even mentioned the lack of mathematical criticism in school. At no time are students let in on the secret that mathematics, like any literature, is created by human beings for their own amusement; that works of mathematics are subject to critical appraisal; that one can have and develop mathematical *taste*. A piece of mathematics is like a poem, and we can ask if it satisfies our aesthetic criteria: Is this argument sound? Does it make sense? Is it simple and elegant? Does it get me closer to the heart of the matter? Of course there's no criticism going on in school— there's no art being done to criticize!

Why don't we want our children to learn to do mathematics? Is it that we don't trust them, that we think it's too hard? We seem to feel that they are capable of making arguments and coming to their own conclusions about Napoleon, why not about triangles? I think it's simply that we as a culture don't know what mathematics is. The impression we are given is of something very cold and highly technical, that no one could possibly understand— a self-fulfilling prophesy if there ever was one.

It would be bad enough if the culture were merely ignorant of mathematics, but what is far worse is that people actually think they *do* know what math is about— and are apparently under the gross misconception that mathematics is somehow useful to society! This is already a huge difference between mathematics and the other arts. Mathematics is viewed by the culture as some sort of tool for science and technology. Everyone knows that poetry and music are for pure enjoyment and for uplifting and ennobling the human spirit (hence their virtual elimination from the public school curriculum) but no, math is *important*.

SIMPLICIO: Are you really trying to claim that mathematics offers no useful or practical applications to society?

SALVIATI: Of course not. I'm merely suggesting that just because something happens to have practical consequences, doesn't mean that's what it is *about*. Music can lead armies into battle, but that's not why people write symphonies. Michelangelo decorated a ceiling, but I'm sure he had loftier things on his mind.

SIMPLICIO: But don't we need people to learn those useful consequences of math? Don't we need accountants and carpenters and such?

SALVIATI: How many people actually use any of this "practical math" they supposedly learn in school? Do you think carpenters are out there using trigonometry? How many adults remember how to divide fractions, or solve a quadratic equation? Obviously the current practical training program isn't working, and for good reason: it is excruciatingly boring, and nobody ever uses it anyway. So why do people think it's so important? I don't see how it's doing society any good to have its members walking around with vague memories of algebraic formulas and geometric diagrams, and clear memories of hating them. It might do some good, though, to show them something beautiful and give them an opportunity to enjoy being creative, flexible, open-minded thinkers— the kind of thing a *real* mathematical education might provide.

SIMPLICIO: But people need to be able to balance their checkbooks, don't they?

SALVIATI: I'm sure most people use a calculator for everyday arithmetic. And why not? It's certainly easier and more reliable. But my point is not just that the current system is so terribly bad, it's that what it's missing is so wonderfully good! Mathematics should be taught as art for art's sake. These mundane "useful" aspects would follow naturally as a trivial by-product. Beethoven could easily write an advertising jingle, but his motivation for learning music was to create something beautiful.

SIMPLICIO: But not everyone is cut out to be an artist. What about the kids who aren't "math people?" How would they fit into your scheme?

SALVIATI: If everyone were exposed to mathematics in its natural state, with all the challenging fun and surprises that that entails, I think we would see a dramatic change both in the attitude of students toward mathematics, and in our conception of what it means to be "good at math." We are losing so many potentially gifted mathematicians— creative, intelligent people who rightly reject what appears to be a meaningless and sterile subject. They are simply too smart to waste their time on such piffle.

SIMPLICIO: But don't you think that if math class were made more like art class that a lot of kids just wouldn't learn anything?

SALVIATI: They're not learning anything now! Better to not have math classes at all than to do what is currently being done. At least some people might have a chance to discover something beautiful on their own.

SIMPLICIO: So you would remove mathematics from the school curriculum?

SALVIATI: The mathematics has already been removed! The only question is what to do with the vapid, hollow shell that remains. Of course I would prefer to replace it with an active and joyful engagement with mathematical ideas.

SIMPLICIO: But how many math teachers know enough about their subject to teach it that way?

SALVIATI: Very few. And that's just the tip of the iceberg...

Mathematics in School

There is surely no more reliable way to kill enthusiasm and interest in a subject than to make it a mandatory part of the school curriculum. Include it as a major component of standardized testing and you virtually guarantee that the education establishment will suck the life out of it. School boards do not understand what math is, neither do educators, textbook authors, publishing companies, and sadly, neither do most of our math teachers. The scope of the problem is so enormous, I hardly know where to begin.

Let's start with the "math reform" debacle. For many years there has been a growing awareness that something is rotten in the state of mathematics education. Studies have been commissioned, conferences assembled, and countless committees of teachers, textbook publishers, and educators (whatever they are) have been formed to "fix the problem." Quite apart from the self-serving interest paid to reform by the textbook industry (which profits from any minute political fluctuation by offering up "new" editions of their unreadable monstrosities), the entire reform movement has always missed the point. The mathematics curriculum doesn't need to be reformed, it needs to be *scrapped*.

All this fussing and primping about which "topics" should be taught in what order, or the use of this notation instead of that notation, or which make and model of *calculator* to use, for god's sake— it's like rearranging the deck chairs on the Titanic! Mathematics is *the music of reason*. To do mathematics is to engage in an act of discovery and conjecture, intuition and inspiration; to be in a state of confusion— not because it makes no sense to you, but because you *gave* it sense and you still don't understand what your creation is up to; to have a breakthrough idea; to be frustrated as an artist; to be awed and overwhelmed by an almost painful beauty; to be *alive*, damn it. Remove this from mathematics and you can have all the conferences you like; it won't matter. Operate all you want, doctors: *your patient is already dead*.

The saddest part of all this "reform" are the attempts to "make math interesting" and "relevant to kids' lives." You don't need to *make* math interesting— it's already more interesting than we can handle! And the glory of it is its complete *irrelevance* to our lives. That's why it's so fun!

Attempts to present mathematics as relevant to daily life inevitably appear forced and contrived: "You see kids, if you know algebra then you can figure out how old Maria is if we know that she is two years older than twice her age seven years ago!" (As if anyone would ever have access to that ridiculous kind of information, and not her age.) Algebra is not about daily life, it's about numbers and symmetry— and this is a valid pursuit in and of itself:

> Suppose I am given the sum and difference of two numbers. How can I figure out what the numbers are themselves?

Here is a simple and elegant question, and it requires no effort to be made appealing. The ancient Babylonians enjoyed working on such problems, and so do our students. (And I hope you will enjoy thinking about it too!) We don't need to bend over backwards to give mathematics relevance. It has relevance in the same way that any art does: that of being a meaningful human experience.

In any case, do you really think kids even *want* something that is relevant to their daily lives? You think something practical like *compound interest* is going to get them excited? People enjoy *fantasy*, and that is just what mathematics can provide— a relief from daily life, an anodyne to the practical workaday world.

A similar problem occurs when teachers or textbooks succumb to "cutesyness." This is where, in an attempt to combat so-called "math anxiety" (one of the panoply of diseases which are actually *caused* by school), math is made to seem "friendly." To help your students memorize formulas for the area and circumference of a circle, for example, you might invent this whole story about "Mr. C," who drives around "Mrs. A" and tells her how nice his "two pies are" ($C = 2\pi r$) and how her "pies are square" ($A = \pi r^2$) or some such nonsense. But what about the *real* story? The one about mankind's struggle with the problem of measuring curves; about Eudoxus and Archimedes and the method of exhaustion; about the transcendence of pi? Which is more interesting— measuring the rough dimensions of a circular piece of graph paper, using a formula that someone handed you without explanation (and made you memorize and practice over and over) or hearing the story of one of the most beautiful, fascinating problems, and one of the most brilliant and powerful ideas in human history? We're killing people's interest in *circles* for god's sake!

Why aren't we giving our students a chance to even hear about these things, let alone giving them an opportunity to actually do some mathematics, and to come up with their own ideas, opinions, and reactions? What other subject is routinely taught without any mention of its history, philosophy, thematic development, aesthetic criteria, and current status? What other subject shuns its primary sources— beautiful works of art by some of the most creative minds in history— in favor of third-rate textbook bastardizations?

The main problem with school mathematics is that there are no *problems.* Oh, I know what *passes* for problems in math classes, these insipid "exercises." "Here is a type of problem. Here is how to solve it. Yes it will be on the test. Do exercises 1-35 odd for homework." What a sad way to learn mathematics: to be a trained chimpanzee.

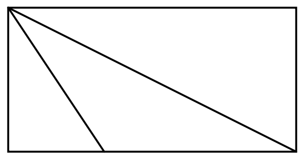
But a problem, a genuine honest-to-goodness natural human *question*— that's another thing. How long is the diagonal of a cube? Do prime numbers keep going on forever? Is infinity a number? How many ways can I symmetrically tile a surface? The history of mathematics is the history of mankind's engagement with questions like these, not the mindless regurgitation of formulas and algorithms (together with contrived exercises designed to make use of them).

A good problem is something you don't know *how* to solve. That's what makes it a good puzzle, and a good opportunity. A good problem does not just sit there in isolation, but serves as a springboard to *other* interesting questions. A triangle takes up half its box. What about a pyramid inside its three-dimensional box? Can we handle this problem in a similar way?

I can understand the idea of training students to master certain techniques— I do that too. But not as an end in itself. Technique in mathematics, as in any art, should be learned in context. The great problems, their history, the creative process— that is the proper setting. Give your students a good problem, let them struggle and get frustrated. See what they come up with. Wait until they are dying for an idea, *then* give them some technique. But not too much.

So put away your lesson plans and your overhead projectors, your full-color textbook abominations, your CD-ROMs and the whole rest of the traveling circus freak show of contemporary education, and simply do mathematics with your students! Art teachers don't waste their time with textbooks and rote training in specific techniques. They do what is natural to their subject— they get the kids painting. They go around from easel to easel, making suggestions and offering guidance:

"I was thinking about our triangle problem, and I noticed something. If the triangle is really slanted then it *doesn't* take up half it's box! See, look:



"Excellent observation! Our chopping argument assumes that the tip of the triangle lies directly over the base. Now we need a new idea."

"Should I try chopping it a different way?"

"Absolutely. Try all sorts of ideas. Let me know what you come up with!"

So how do we teach our students to do mathematics? By choosing engaging and natural problems suitable to their tastes, personalities, and level of experience. By giving them time to make discoveries and formulate conjectures. By helping them to refine their arguments and creating an atmosphere of healthy and vibrant mathematical criticism. By being flexible and open to sudden changes in direction to which their curiosity may lead. In short, by having an honest intellectual relationship with our students and our subject.

Of course what I'm suggesting is impossible for a number of reasons. Even putting aside the fact that statewide curricula and standardized tests virtually eliminate teacher autonomy, I doubt that most teachers even *want* to have such an intense relationship with their students. It requires too much vulnerability and too much responsibility— in short, it's too much work!

It is far easier to be a passive conduit of some publisher's "materials" and to follow the shampoo-bottle instruction "lecture, test, repeat" than to think deeply and thoughtfully about the meaning of one's subject and how best to convey that meaning directly and honestly to one's students. We are encouraged to forego the difficult task of making decisions based on our individual wisdom and conscience, and to "get with the program." It is simply the path of least resistance:

TEXTBOOK PUBLISHERS : TEACHERS ::

A) pharmaceutical companies : doctors
B) record companies : disk jockeys
C) corporations : congressmen
D) all of the above

The trouble is that math, like painting or poetry, is *hard creative work*. That makes it very difficult to teach. Mathematics is a slow, contemplative process. It takes time to produce a work of art, and it takes a skilled teacher to recognize one. Of course it's easier to post a set of rules than to guide aspiring young artists, and it's easier to write a VCR manual than to write an actual book with a point of view.

Mathematics is an *art*, and art should be taught by working artists, or if not, at least by people who appreciate the art form and can recognize it when they see it. It is not necessary that you learn music from a professional composer, but would you want yourself or your child to be taught by someone who doesn't even play an instrument, and has never listened to a piece of music in their lives? Would you accept as an art teacher someone who has never picked up a pencil or stepped foot in a museum? Why is it that we accept math teachers who have never produced an original piece of mathematics, know nothing of the history and philosophy of the subject, nothing about recent developments, nothing in fact beyond what they are expected to present to their unfortunate students? What kind of a teacher is that? How can someone teach something that they themselves don't do? I can't dance, and consequently I would never presume to think that I could teach a dance class (I could try, but it wouldn't be pretty). The difference is I *know* I can't dance. I don't have anyone telling me I'm good at dancing just because I know a bunch of dance words.

Now I'm not saying that math teachers need to be professional mathematicians— far from it. But shouldn't they at least understand what mathematics is, be good at it, and enjoy doing it?

If teaching is reduced to mere data transmission, if there is no sharing of excitement and wonder, if teachers themselves are passive recipients of information and not creators of new ideas, what hope is there for their students? If adding fractions is to the teacher an arbitrary set of rules, and not the outcome of a creative process and the result of aesthetic choices and desires, then *of course* it will feel that way to the poor students.

Teaching is not about information. It's about having an honest intellectual relationship with your students. It requires no method, no tools, and no training. Just the ability to be real. And if you can't be real, then you have no right to inflict yourself upon innocent children.

In particular, *you can't teach teaching*. Schools of education are a complete crock. Oh, you can take classes in early childhood development and whatnot, and you can be trained to use a blackboard "effectively" and to prepare an organized "lesson plan" (which, by the way, insures that your lesson will be *planned*, and therefore false), but you will never be a real teacher if you are unwilling to be a real person. Teaching means openness and honesty, an ability to share excitement, and a love of learning. Without these, all the education degrees in the world won't help you, and with them they are completely unnecessary.

It's perfectly simple. Students are not aliens. They respond to beauty and pattern, and are naturally curious like anyone else. Just talk to them! And more importantly, listen to them!

SIMPLICIO: All right, I understand that there is an art to mathematics and that we are not doing a good job of exposing people to it. But isn't this a rather esoteric, highbrow sort of thing to expect from our school system? We're not trying to create philosophers here, we just want people to have a reasonable command of basic arithmetic so they can function in society.

SALVIATI: But that's not true! School mathematics concerns itself with many things that have nothing to do with the ability to get along in society— algebra and trigonometry, for instance. These studies are utterly irrelevant to daily life. I'm simply suggesting that if we are going to include such things as part of most students' basic education, that we do it in an organic and natural way. Also, as I said before, just because a subject happens to have some mundane practical use does not mean that we have to make that use the focus of our teaching and learning. It may be true that you have to be able to read in order to fill out forms at the DMV, but that's not why we teach children to read. We teach them to read for the higher purpose of allowing them access to beautiful and meaningful ideas. Not only would it be cruel to teach reading in such a way— to force third graders to fill out purchase orders and tax forms— it wouldn't work! We learn things because they interest us now, not because they might be useful later. But this is exactly what we are asking children to do with math.

SIMPLICIO: But don't we need third graders to be able to do arithmetic?

SALVIATI: Why? You want to train them to calculate 427 plus 389? It's just not a question that very many eight-year-olds are asking. For that matter, most *adults* don't fully understand decimal place-value arithmetic, and you expect third graders to have a clear conception? Or do you not care if they understand it? It is simply too early for that kind of technical training. Of course it can be done, but I think it ultimately does more harm than good. Much better to wait until their own natural curiosity about numbers kicks in.

SIMPLICIO: Then what *should* we do with young children in math class?

SALVIATI: Play games! Teach them Chess and Go, Hex and Backgammon, Sprouts and Nim, whatever. Make up a game. Do puzzles. Expose them to situations where deductive reasoning is necessary. Don't worry about notation and technique, help them to become active and creative mathematical thinkers.

SIMPLICIO: It seems like we'd be taking an awful risk. What if we de-emphasize arithmetic so much that our students end up not being able to add and subtract?

SALVIATI: I think the far greater risk is that of creating schools devoid of creative expression of any kind, where the function of the students is to memorize dates, formulas, and vocabulary lists, and then regurgitate them on standardized tests—"Preparing tomorrow's workforce today!"

SIMPLICIO: But surely there is some body of mathematical facts of which an educated person should be cognizant.

SALVIATI: Yes, the most important of which is that mathematics is an art form done by human beings for pleasure! Alright, yes, it would be nice if people knew a few basic things about numbers and shapes, for instance. But this will never come from rote memorization, drills, lectures, and exercises. You learn things by doing them and you remember what matters to you. We have millions of adults wandering around with "negative *b* plus or minus the square root of *b* squared minus *4ac* all over *2a*" in their heads, and absolutely no idea whatsoever what it means. And the reason is that they were never given the chance to discover or invent such things for themselves. They never had an engaging problem to think about, to be frustrated by, and to create in them the desire for technique or method. They were never told the history of mankind's relationship with numbers— no ancient Babylonian problem tablets, no Rhind Papyrus, no *Liber Abaci*, no *Ars Magna*. More importantly, no chance for them to even get curious about a question; it was answered before they could ask it.

SIMPLICIO: But we don't have time for every student to invent mathematics for themselves! It took centuries for people to discover the Pythagorean Theorem. How can you expect the average child to do it?

SALVIATI: I don't. Let's be clear about this. I'm complaining about the complete absence of art and invention, history and philosophy, context and perspective from the mathematics curriculum. That doesn't mean that notation, technique, and the development of a knowledge base have no place. Of course they do. We should have both. If I object to a pendulum being too far to one side, it doesn't mean I want it to be all the way on the other side. But the fact is, people learn better when the product comes out of the process. A real appreciation for poetry does not come from memorizing a bunch of poems, it comes from writing your own.

SIMPLICIO: Yes, but before you can write your own poems you need to learn the alphabet. The process has to begin somewhere. You have to walk before you can run.

SALVIATI: No, you have to have something you want to run *toward*. Children can write poems and stories *as* they learn to read and write. A piece of writing by a six-year-old is a wonderful thing, and the spelling and punctuation errors don't make it less so. Even very young children can invent songs, and they haven't a clue what key it is in or what type of meter they are using.

SIMPLICIO: But isn't math different? Isn't math a language of its own, with all sorts of symbols that have to be learned before you can use it?

SALVIATI: Not at all. Mathematics is not a language, it's an adventure. Do musicians "speak another language" simply because they choose to abbreviate their ideas with little black dots? If so, it's no obstacle to the toddler and her song. Yes, a certain amount of mathematical shorthand has evolved over the centuries, but it is in no way essential. Most mathematics is done with a friend over a cup of coffee, with a diagram scribbled on a napkin. Mathematics is and always has been about ideas, and a valuable idea transcends the symbols with which you choose to represent it. As Gauss once remarked, "What we need are *notions*, not notations."

SIMPLICIO: But isn't one of the purposes of mathematics education to help students think in a more precise and logical way, and to develop their "quantitative reasoning skills?" Don't all of these definitions and formulas sharpen the minds of our students?

SALVIATI: No they don't. If anything, the current system has the opposite effect of dulling the mind. Mental acuity of any kind comes from solving problems yourself, not from being told how to solve them.

SIMPLICIO: Fair enough. But what about those students who are interested in pursuing a career in science or engineering? Don't they need the training that the traditional curriculum provides? Isn't that why we teach mathematics in school?

SALVIATI: How many students taking literature classes will one day be writers? That is not why we teach literature, nor why students take it. We teach to enlighten everyone, not to train only the future professionals. In any case, the most valuable skill for a scientist or engineer is being able to think creatively and independently. The last thing anyone needs is to be *trained*.

The Mathematics Curriculum

The truly painful thing about the way mathematics is taught in school is not what is missing— the fact that there is no actual mathematics being done in our mathematics classes— but what is there in its place: the confused heap of destructive disinformation known as "the mathematics curriculum." It is time now to take a closer look at exactly what our students are up against— what they are being exposed to in the name of mathematics, and how they are being harmed in the process.

The most striking thing about this so-called mathematics curriculum is its rigidity. This is especially true in the later grades. From school to school, city to city, and state to state, the same exact things are being said and done in the same exact way and in the same exact order. Far from being disturbed and upset by this Orwellian state of affairs, most people have simply accepted this "standard model" math curriculum as being synonymous with math itself.

This is intimately connected to what I call the "ladder myth"— the idea that mathematics can be arranged as a sequence of "subjects" each being in some way more advanced, or "higher" than the previous. The effect is to make school mathematics into a *race*— some students are "ahead" of others, and parents worry that their child is "falling behind." And where exactly does this race lead? What is waiting at the finish line? It's a sad race to nowhere. In the end you've been cheated out of a mathematical education, and you don't even know it.

Real mathematics doesn't come in a can— there is no such thing as an Algebra II *idea*. Problems lead you to where they take you. *Art is not a race*. The ladder myth is a false image of the subject, and a teacher's own path through the standard curriculum reinforces this myth and prevents him or her from seeing mathematics as an organic whole. As a result, we have a math curriculum with no historical perspective or thematic coherence, a fragmented collection of assorted topics and techniques, united only by the ease in which they can be reduced to step-by-step procedures.

In place of discovery and exploration, we have rules and regulations. We never hear a student saying, "I wanted to see if it could make any sense to raise a number to a negative power, and I found that you get a really neat pattern if you choose it to mean the reciprocal." Instead we have teachers and textbooks presenting the "negative exponent rule" as a *fait d'accompli* with no mention of the aesthetics behind this choice, or even that it is a choice.

In place of meaningful problems, which might lead to a synthesis of diverse ideas, to uncharted territories of discussion and debate, and to a feeling of thematic unity and harmony in mathematics, we have instead joyless and redundant exercises, specific to the technique under discussion, and so disconnected from each other and from mathematics as a whole that neither the students nor their teacher have the foggiest idea how or why such a thing might have come up in the first place.

In place of a natural problem context in which students can make decisions about what they want their words to mean, and what notions they wish to codify, they are instead subjected to an endless sequence of unmotivated and a priori "definitions." The curriculum is obsessed with jargon and nomenclature, seemingly for no other purpose than to provide teachers with something to test the students on. No mathematician in the world would bother making these senseless distinctions: 2 1/2 is a "mixed number," while 5/2 is an "improper fraction." They're *equal* for crying out loud. They are the same exact numbers, and have the same exact properties. Who uses such words outside of fourth grade?

Of course it is far easier to test someone's knowledge of a pointless definition than to inspire them to create something beautiful and to find their own meaning. Even if we agree that a basic common vocabulary for mathematics is valuable, this isn't it. How sad that fifth-graders are taught to say "quadrilateral" instead of "four-sided shape," but are never given a reason to use words like "conjecture," and "counterexample." High school students must learn to use the secant function, 'sec *x*,' as an abbreviation for the reciprocal of the cosine function, '1 / cos *x*,' (a definition with as much intellectual weight as the decision to use '&' in place of "and.") That this particular shorthand, a holdover from fifteenth century nautical tables, is still with us (whereas others, such as the "versine" have died out) is mere historical accident, and is of utterly no value in an era when rapid and precise shipboard computation is no longer an issue. Thus we clutter our math classes with pointless nomenclature for its own sake.

In practice, the curriculum is not even so much a sequence of topics, or ideas, as it is a sequence of notations. Apparently mathematics consists of a secret list of mystical symbols and rules for their manipulation. Young children are given '+' and '÷.' Only later can they be entrusted with '$\sqrt{\ }$,' and then '*x*' and '*y*' and the alchemy of parentheses. Finally, they are indoctrinated in the use of 'sin,' 'log,' '*f(x)*,' and if they are deemed worthy, '*d*' and '∫.' All without having had a single meaningful mathematical experience.

This program is so firmly fixed in place that teachers and textbook authors can reliably predict, years in advance, exactly what students will be doing, down to the very page of exercises. It is not at all uncommon to find second-year algebra students being asked to calculate $[\ f(x + h) - f(x)\ ] / h$ for various functions *f*, so that they will have "seen" this when they take calculus a few years later. Naturally no motivation is given (nor expected) for why such a seemingly random combination of operations would be of interest, although I'm sure there are many teachers who try to explain what such a thing might mean, and think they are doing their students a favor, when in fact to them it is just one more boring math problem to be gotten over with. "What do they want me to do? Oh, just plug it in? OK."

Another example is the training of students to express information in an unnecessarily complicated form, merely because at some distant future period it will have meaning. Does any middle school algebra teacher have the slightest clue why he is asking his students to rephrase “the number x lies between three and seven” as $|x - 5| < 2$? Do these hopelessly inept textbook authors really believe they are helping students by preparing them for a possible day, years hence, when they might be operating within the context of a higher-dimensional geometry or an abstract metric space? I doubt it. I expect they are simply copying each other decade after decade, maybe changing the fonts or the highlight colors, and beaming with pride when an school system adopts their book, and becomes their unwitting accomplice.

Mathematics is about problems, and problems must be made the focus of a students mathematical life. Painful and creatively frustrating as it may be, students and their teachers should at all times be engaged in the process— having ideas, not having ideas, discovering patterns, making conjectures, constructing examples and counterexamples, devising arguments, and critiquing each other’s work. Specific techniques and methods will arise naturally out of this process, as they did historically: not isolated from, but organically connected to, and as an outgrowth of, their problem-background.

English teachers know that spelling and pronunciation are best learned in a context of reading and writing. History teachers know that names and dates are uninteresting when removed from the unfolding backstory of events. Why does mathematics education remain stuck in the nineteenth century? Compare your own experience of learning algebra with Bertrand Russell’s recollection:

> “I was made to learn by heart: ‘The square of the sum of two numbers is equal to the sum of their squares increased by twice their product.’ I had not the vaguest idea what this meant and when I could not remember the words, my tutor threw the book at my head, which did not stimulate my intellect in any way.”

Are things really any different today?

SIMPLICIO: I don’t think that’s very fair. Surely teaching methods have improved since then.

SALVIATI: You mean *training* methods. Teaching is a messy human relationship; it does not require a method. Or rather I should say, if you need a method you’re probably not a very good teacher. If you don’t have enough of a feeling for your subject to be able to talk about it in your own voice, in a natural and spontaneous way, how well could you understand it? And speaking of being stuck in the nineteenth century, isn’t it shocking how the curriculum itself is stuck in the seventeenth? To think of all the amazing discoveries and profound revolutions in mathematical thought that have occurred in the last three centuries! There is no more mention of these than if they had never happened.

SIMPLICIO: But aren’t you asking an awful lot from our math teachers? You expect them to provide individual attention to dozens of students, guiding them on their own paths toward discovery and enlightenment, and to be up on recent mathematical history as well?

SALVIATI: Do you expect your art teacher to be able to give you individualized, knowledgeable advice about your painting? Do you expect her to know anything about the last three hundred years of art history? But seriously, I don't expect anything of the kind, I only wish it were so.

SIMPLICIO: So you blame the math teachers?

SALVIATI: No, I blame the culture that produces them. The poor devils are trying their best, and are only doing what they've been trained to do. I'm sure most of them love their students and hate what they are being forced to put them through. They know in their hearts that it is meaningless and degrading. They can sense that they have been made cogs in a great soul-crushing machine, but they lack the perspective needed to understand it, or to fight against it. They only know they have to get the students "ready for next year."

SIMPLICIO: Do you really think that most students are capable of operating on such a high level as to create their own mathematics?

SALVIATI: If we honestly believe that creative reasoning is too "high" for our students, and that they can't handle it, why do we allow them to write history papers or essays about Shakespeare? The problem is not that the students can't handle it, it's that none of the teachers can. They've never proved anything themselves, so how could they possibly advise a student? In any case, there would obviously be a range of student interest and ability, as there is in any subject, but at least students would like or dislike mathematics for what it really is, and not for this perverse mockery of it.

SIMPLICIO: But surely we want all of our students to learn a basic set of facts and skills. That's what a curriculum is for, and that's why it is so uniform— there are certain timeless, cold hard facts we need our students to know: one plus one is two, and the angles of a triangle add up to 180 degrees. These are not opinions, or mushy artistic feelings.

SALVIATI: On the contrary. Mathematical structures, useful or not, are invented and developed within a problem context, and derive their meaning from that context. Sometimes we want one plus one to equal zero (as in so-called 'mod 2' arithmetic) and on the surface of a sphere the angles of a triangle add up to more than 180 degrees. There are no "facts" per se; everything is relative and relational. It is the story that matters, not just the ending.

SIMPLICIO: I'm getting tired of all your mystical mumbo-jumbo! Basic arithmetic, all right? Do you or do you not agree that students should learn it?

SALVIATI: That depends on what you mean by "it." If you mean having an appreciation for the problems of counting and arranging, the advantages of grouping and naming, the distinction between a representation and the thing itself, and some idea of the historical development of number systems, then yes, I do think our students should be exposed to such things. If you mean the rote memorization of arithmetic facts without any underlying conceptual framework, then no. If you mean exploring the not at all obvious fact that five groups of seven is the same as seven groups of five, then yes. If you mean making a rule that 5 x 7 = 7 x 5, then no. Doing mathematics should always mean discovering patterns and crafting beautiful and meaningful explanations.

SIMPLICIO: What about geometry? Don't students prove things there? Isn't High School Geometry a perfect example of what you want math classes to be?

High School Geometry: Instrument of the Devil

There is nothing quite so vexing to the author of a scathing indictment as having the primary target of his venom offered up in his support. And never was a wolf in sheep's clothing as insidious, nor a false friend as treacherous, as High School Geometry. It is precisely *because* it is school's attempt to introduce students to the art of argument that makes it so very dangerous.

Posing as the arena in which students will finally get to engage in true mathematical reasoning, this virus attacks mathematics at its heart, destroying the very essence of creative rational argument, poisoning the students' enjoyment of this fascinating and beautiful subject, and permanently disabling them from thinking about math in a natural and intuitive way.

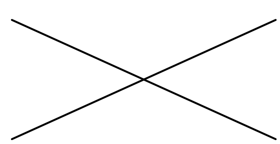
The mechanism behind this is subtle and devious. The student-victim is first stunned and paralyzed by an onslaught of pointless definitions, propositions, and notations, and is then slowly and painstakingly weaned away from any natural curiosity or intuition about shapes and their patterns by a systematic indoctrination into the stilted language and artificial format of so-called "formal geometric proof."

All metaphor aside, geometry class is by far the most mentally and emotionally destructive component of the entire K-12 mathematics curriculum. Other math courses may hide the beautiful bird, or put it in a cage, but in geometry class it is openly and cruelly tortured. (Apparently I am incapable of putting all metaphor aside.)

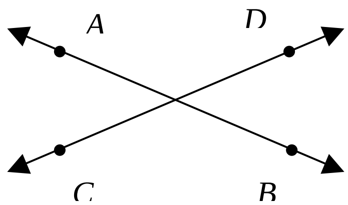
What is happening is the systematic undermining of the student's intuition. A proof, that is, a mathematical argument, is a work of fiction, a poem. Its goal is to *satisfy*. A beautiful proof should explain, and it should explain clearly, deeply, and elegantly. A well-written, well-crafted argument should feel like a splash of cool water, and be a beacon of light— it should refresh the spirit and illuminate the mind. And it should be *charming*.

There is nothing charming about what passes for proof in geometry class. Students are presented a rigid and dogmatic format in which their so-called "proofs" are to be conducted— a format as unnecessary and inappropriate as insisting that children who wish to plant a garden refer to their flowers by genus and species.

Let's look at some specific instances of this insanity. We'll begin with the example of two crossed lines:



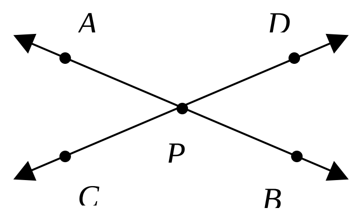
Now the first thing that usually happens is the unnecessary muddying of the waters with excessive notation. Apparently, one cannot simply speak of two crossed lines; one must give elaborate names to them. And not simple names like ‘line 1’ and ‘line 2,’ or even ‘*a*’ and ‘*b*.’ We must (according to High School Geometry) select random and irrelevant points on these lines, and then refer to the lines using the special “line notation.”



You see, now we get to call them *AB* and *CD*. And God forbid you should omit the little bars on top— ‘*AB*’ refers to the *length* of the line *AB* (at least I think that’s how it works). Never mind how pointlessly complicated it is, this is the way one must learn to do it. Now comes the actual statement, usually referred to by some absurd name like

PROPOSITION 2.1.1.

Let $\overline{A}\ \overline{B}$ and $\overline{C}\ \overline{D}$ intersect at *P*. Then $\angle APC \cong \angle BPD$.



In other words, the angles on both sides are the same. Well, duh! The configuration of two crossed lines is *symmetrical* for crissake. And as if this wasn’t bad enough, this patently obvious statement about lines and angles must then be “proved.”

Proof:

Statement	Reason
1. $m\angle APC + m\angle APD = 180$ $m\angle BPD + m\angle APD = 180$	1. Angle Addition Postulate
2. $m\angle APC + m\angle APD = m\angle BPD + m\angle APD$	2. Substitution Property
3. $m\angle APD = m\angle APD$	3. Reflexive Property of Equality
4. $m\angle APC = m\angle BPD$	4. Subtraction Property of Equality
5. $\angle APC \cong \angle BPD$	5. Angle Measurement Postulate

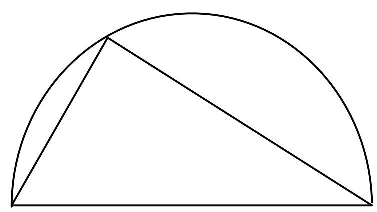
Instead of a witty and enjoyable argument written by an actual human being, and conducted in one of the world’s many natural languages, we get this sullen, soulless, bureaucratic form-letter of a proof. And what a mountain being made of a molehill! Do we really want to suggest that a straightforward observation like this requires such an extensive preamble? Be honest: did you actually even read it? Of course not. Who would want to?

The effect of such a production being made over something so simple is to make people doubt their own intuition. Calling into question the obvious, by insisting that it be “rigorously proved" (as if the above even constitutes a legitimate formal proof) is to say to a student, "Your feelings and ideas are suspect. You need to think and speak our way."

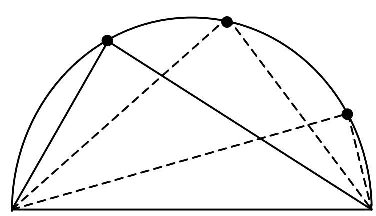
Now there is a place for formal proof in mathematics, no question. But that place is not a student's first introduction to mathematical argument. At least let people get familiar with some mathematical objects, and learn what to expect from them, before you start formalizing everything. Rigorous formal proof only becomes important when there is a *crisis*— when you discover that your imaginary objects behave in a counterintuitive way; when there is a paradox of some kind. But such excessive preventative hygiene is completely unnecessary here— nobody's gotten sick yet! Of course if a logical crisis should arise at some point, then obviously it should be investigated, and the argument made more clear, but that process can be carried out intuitively and informally as well. In fact it is the soul of mathematics to carry out such a dialogue with one's own proof.

So not only are most kids utterly confused by this pedantry— nothing is more mystifying than a proof of the obvious— but even those few whose intuition remains intact must then retranslate their excellent, beautiful ideas back into this absurd hieroglyphic framework in order for their teacher to call it "correct." The teacher then flatters himself that he is somehow sharpening his students' minds.

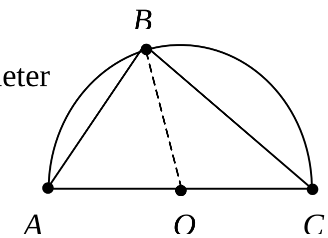
As a more serious example, let's take the case of a triangle inside a semicircle:



Now the beautiful truth about this pattern is that no matter where on the circle you place the tip of the triangle, it always forms a nice right angle. (I have no objection to a term like "right angle" if it is relevant to the problem and makes it easier to discuss. It's not terminology itself that I object to, it's pointless unnecessary terminology. In any case, I would be happy to use "corner" or even "pigpen" if a student preferred.)



Here is a case where our intuition is somewhat in doubt. It's not at all clear that this should be true; it even seems *unlikely*— shouldn't the angle change if I move the tip? What we have here is a fantastic math problem! Is it true? If so, *why* is it true? What a great project! What a terrific opportunity to exercise one's ingenuity and imagination! Of course no such opportunity is given to the students, whose curiosity and interest is immediately deflated by:



THEOREM 9.5. Let ΔABC be inscribed in a semicircle with diameter $\overline{AC}$.
Then $\angle ABC$ is a right angle.

Proof:

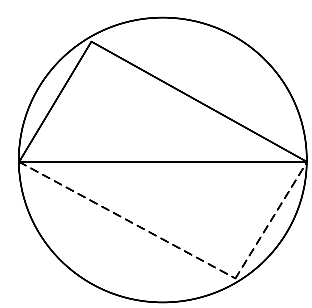
Statement	Reason
1. Draw radius *OB*. Then $OB = OC = OA$	1. Given
2. $m\angle OBC = m\angle BCA$ $m\angle OBA = m\angle BAC$	2. Isosceles Triangle Theorem
3. $m\angle ABC = m\angle OBA + m\angle OBC$	3. Angle Sum Postulate
4. $m\angle ABC + m\angle BCA + m\angle BAC = 180$	4. The sum of the angles of a triangle is 180
5. $m\angle ABC + m\angle OBC + m\angle OBA = 180$	5. Substitution (line 2)
6. $2\, m\angle ABC = 180$	6. Substitution (line 3)
7. $m\angle ABC = 90$	7. Division Property of Equality
8. $\angle ABC$ is a right angle	8. Definition of Right Angle

Could anything be more unattractive and inelegant? Could any argument be more obfuscatory and unreadable? This isn't mathematics! A proof should be an epiphany from the Gods, not a coded message from the Pentagon. This is what comes from a misplaced sense of logical rigor: *ugliness*. The spirit of the argument has been buried under a heap of confusing formalism.

No mathematician works this way. No mathematician has *ever* worked this way. This is a complete and utter misunderstanding of the mathematical enterprise. Mathematics is not about erecting barriers between ourselves and our intuition, and making simple things complicated. Mathematics is about removing obstacles to our intuition, and keeping simple things simple.

Compare this unappetizing mess of a proof with the following argument devised by one of my seventh-graders:

"Take the triangle and rotate it around so it makes a four-sided box inside the circle. Since the triangle got turned completely around, the sides of the box must be parallel, so it makes a parallelogram. But it can't be a slanted box because both of its diagonals are diameters of the circle, so they're equal, which means it must be an actual rectangle. That's why the corner is always a right angle."



Isn't that just delightful? And the point isn't whether this argument is any better than the other one *as an idea*, the point is that the idea comes across. (As a matter of fact, the idea of the first proof is quite pretty, albeit seen as through a glass, darkly.)

More importantly, the idea was the student's *own*. The class had a nice problem to work on, conjectures were made, proofs were attempted, and this is what one student came up with. Of course it took several days, and was the end result of a long sequence of failures.

To be fair, I did paraphrase the proof considerably. The original was quite a bit more convoluted, and contained a lot of unnecessary verbiage (as well as spelling and grammatical errors). But I think I got the feeling of it across. And these defects were all to the good; they gave me something to do as a teacher. I was able to point out several stylistic and logical problems, and the student was then able to improve the argument. For instance, I wasn't completely happy with the bit about both diagonals being diameters— I didn't think that was entirely obvious— but that only meant there was more to think about and more understanding to be gained from the situation. And in fact the student was able to fill in this gap quite nicely:

> "Since the triangle got rotated halfway around the circle, the tip must end up exactly opposite from where it started. That's why the diagonal of the box is a diameter."

So a great project and a beautiful piece of mathematics. I'm not sure who was more proud, the student or myself. This is exactly the kind of experience I want my students to have.

The problem with the standard geometry curriculum is that the private, personal experience of being a struggling artist has virtually been eliminated. The art of proof has been replaced by a rigid step-by step pattern of uninspired formal deductions. The textbook presents a set of definitions, theorems, and proofs, the teacher copies them onto the blackboard, and the students copy them into their notebooks. They are then asked to mimic them in the exercises. Those that catch on to the pattern quickly are the "good" students.

The result is that the student becomes a passive participant in the creative act. Students are making statements to fit a preexisting proof-pattern, not because they *mean* them. They are being trained to ape arguments, not to *intend* them. So not only do they have no idea what their teacher is saying, *they have no idea what they themselves are saying*.

Even the traditional way in which definitions are presented is a lie. In an effort to create an illusion of "clarity" before embarking on the typical cascade of propositions and theorems, a set of definitions are provided so that statements and their proofs can be made as succinct as possible. On the surface this seems fairly innocuous; why not make some abbreviations so that things can be said more economically? The problem is that definitions *matter*. They come from aesthetic decisions about what distinctions you as an artist consider important. And they are *problem-generated*. To make a definition is to highlight and call attention to a feature or structural property. Historically this comes out of working on a problem, not as a prelude to it.

The point is you don't start with definitions, you start with problems. Nobody ever had an idea of a number being "irrational" until Pythagoras attempted to measure the diagonal of a square and discovered that it could not be represented as a fraction. Definitions make sense when a point is reached in your argument which makes the distinction necessary. To make definitions without motivation is more likely to *cause* confusion.

This is yet another example of the way that students are shielded and excluded from the mathematical process. Students need to be able to make their own definitions as the need arises— to frame the debate themselves. I don't want students saying, "the definition, the theorem, the proof," I want them saying, "my definition, my theorem, my proof."

All of these complaints aside, the real problem with this kind of presentation is that it is *boring*. Efficiency and economy simply do not make good pedagogy. I have a hard time believing that Euclid would approve of this; I know Archimedes wouldn't.

SIMPLICIO: Now hold on a minute. I don't know about you, but I actually *enjoyed* my high school geometry class. I liked the structure, and I enjoyed working within the rigid proof format.

SALVIATI: I'm sure you did. You probably even got to work on some nice problems occasionally. Lot's of people enjoy geometry class (although lots more hate it). But this is not a point in favor of the current regime. Rather, it is powerful testimony to the allure of mathematics itself. It's hard to completely ruin something so beautiful; even this faint shadow of mathematics can still be engaging and satisfying. Many people enjoy paint-by-numbers as well; it is a relaxing and colorful manual activity. That doesn't make it the real thing, though.

SIMPLICIO: But I'm telling you, I *liked* it.

SALVIATI: And if you had had a more natural mathematical experience you would have liked it even more.

SIMPLICIO: So we're supposed to just set off on some free-form mathematical excursion, and the students will learn whatever they happen to learn?

SALVIATI: Precisely. Problems will lead to other problems, technique will be developed as it becomes necessary, and new topics will arise naturally. And if some issue never happens to come up in thirteen years of schooling, how interesting or important could it be?

SIMPLICIO: You've gone completely mad.

SALVIATI: Perhaps I have. But even working within the conventional framework a good teacher can guide the discussion and the flow of problems so as to allow the students to discover and invent mathematics for themselves. The real problem is that the bureaucracy does not allow an individual teacher to do that. With a set curriculum to follow, a teacher cannot lead. There should be no standards, and no curriculum. Just individuals doing what they think best for their students.

SIMPLICIO: But then how can schools guarantee that their students will all have the same basic knowledge? How will we accurately measure their relative worth?

SALVIATI: They can't, and we won't. Just like in real life. Ultimately you have to face the fact that people are all different, and that's just fine. In any case, there's no urgency. So a person graduates from high school not knowing the half-angle formulas (as if they do now!) So what? At least that person would come away with some sort of an idea of what the subject is really about, and would get to see something beautiful.

In Conclusion…

To put the finishing touches on my critique of the standard curriculum, and as a service to the community, I now present the first ever *completely honest* course catalog for K-12 mathematics:

The Standard School Mathematics Curriculum

LOWER SCHOOL MATH. The indoctrination begins. Students learn that mathematics is not something you do, but something that is done to you. Emphasis is placed on sitting still, filling out worksheets, and following directions. Children are expected to master a complex set of algorithms for manipulating Hindi symbols, unrelated to any real desire or curiosity on their part, and regarded only a few centuries ago as too difficult for the average adult. Multiplication tables are stressed, as are parents, teachers, and the kids themselves.

MIDDLE SCHOOL MATH. Students are taught to view mathematics as a set of procedures, akin to religious rites, which are eternal and set in stone. The holy tablets, or "Math Books," are handed out, and the students learn to address the church elders as "they" (as in "What do they want here? Do they want me to divide?") Contrived and artificial "word problems" will be introduced in order to make the mindless drudgery of arithmetic seem enjoyable by comparison. Students will be tested on a wide array of unnecessary technical terms, such as 'whole number' and 'proper fraction,' without the slightest rationale for making such distinctions. Excellent preparation for Algebra I.

ALGEBRA I. So as not to waste valuable time thinking about numbers and their patterns, this course instead focuses on symbols and rules for their manipulation. The smooth narrative thread that leads from ancient Mesopotamian tablet problems to the high art of the Renaissance algebraists is discarded in favor of a disturbingly fractured, post-modern retelling with no characters, plot, or theme. The insistence that all numbers and expressions be put into various standard forms will provide additional confusion as to the meaning of identity and equality. Students must also memorize the quadratic formula for some reason.

GEOMETRY. Isolated from the rest of the curriculum, this course will raise the hopes of students who wish to engage in meaningful mathematical activity, and then dash them. Clumsy and distracting notation will be introduced, and no pains will be spared to make the simple seem complicated. This goal of this course is to eradicate any last remaining vestiges of natural mathematical intuition, in preparation for Algebra II.

ALGEBRA II. The subject of this course is the unmotivated and inappropriate use of coordinate geometry. Conic sections are introduced in a coordinate framework so as to avoid the aesthetic simplicity of cones and their sections. Students will learn to rewrite quadratic forms in a variety of standard formats for no reason whatsoever. Exponential and logarithmic functions are also introduced in Algebra II, despite not being algebraic objects, simply because they have to be stuck in somewhere, apparently. The name of the course is chosen to reinforce the ladder mythology. Why Geometry occurs in between Algebra I and its sequel remains a mystery.

TRIGONOMETRY. Two weeks of content are stretched to semester length by masturbatory definitional runarounds. Truly interesting and beautiful phenomena, such as the way the sides of a triangle depend on its angles, will be given the same emphasis as irrelevant abbreviations and obsolete notational conventions, in order to prevent students from forming any clear idea as to what the subject is about. Students will learn such mnemonic devices as "SohCahToa" and "All Students Take Calculus" in lieu of developing a natural intuitive feeling for orientation and symmetry. The measurement of triangles will be discussed without mention of the transcendental nature of the trigonometric functions, or the consequent linguistic and philosophical problems inherent in making such measurements. Calculator required, so as to further blur these issues.

PRE-CALCULUS. A senseless bouillabaisse of disconnected topics. Mostly a half-baked attempt to introduce late nineteenth-century analytic methods into settings where they are neither necessary nor helpful. Technical definitions of 'limits' and 'continuity' are presented in order to obscure the intuitively clear notion of smooth change. As the name suggests, this course prepares the student for Calculus, where the final phase in the systematic obfuscation of any natural ideas related to shape and motion will be completed.

CALCULUS. This course will explore the mathematics of motion, and the best ways to bury it under a mountain of unnecessary formalism. Despite being an introduction to both the differential and integral calculus, the simple and profound ideas of Newton and Leibniz will be discarded in favor of the more sophisticated function-based approach developed as a response to various analytic crises which do not really apply in this setting, and which will of course not be mentioned. To be taken again in college, verbatim.

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And there you have it. A complete prescription for permanently disabling young minds— a proven cure for curiosity. What have they done to mathematics!

There is such breathtaking depth and heartbreaking beauty in this ancient art form. How ironic that people dismiss mathematics as the antithesis of creativity. They are missing out on an art form older than any book, more profound than any poem, and more abstract than any abstract. And it is *school* that has done this! What a sad endless cycle of innocent teachers inflicting damage upon innocent students. We could all be having so much more fun.

SIMPLICIO: Alright, I'm thoroughly depressed. What now?

SALVIATI: Well, I think I have an idea about a pyramid inside a cube...