THE DEGREE OF THE SPECIAL LINEAR CHARACTERS OF A RANK TWO FREE GROUP

RICHARD J. BROWN

ABSTRACT. Given a word W in the free group on 2 letters F_2, Horowitz showed that the special linear character of W is an integer polynomial in the 3 characters of the basic words of F_2. Special linear characters are defined via the trace of their representations, and the polynomial character of an arbitrary W can be found by application of certain "trace relations", which allow one to write the character of a complicated word as a sum of products of the characters of simpler words. Even in the $n = 2$ case, where the polynomial is uniquely defined, this procedure can be difficult and tedious. In this note, we use the structure of a free group word in F_2 to compute the degree and the leading monomial of its $SL(2, \mathbb{C})$-character without actually computing the full polynomial.

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1. Introduction

For the free group on n generators, F_n, the set of all characters of special linear representations of F_n is a closed affine subset of $\mathbb{C}^{2^n-1}$, whose coordinates are given by the Horowitz generating set: the characters of special linear representations of a set of $2^n - 1$ basic words in F_n (See Horowitz [8]).

The character of a word $W \in F_n$ is the complex-valued function on the set of all representations of F_n,

$$w : Hom(F_n, SL(2, \mathbb{C})) \to \mathbb{C}, \quad w(\rho) = \operatorname{tr} \rho(W),$$

where "tr" means the standard trace for special linear matrices (for words in F_2 we will use upper case letters and their lower case version for characters). It was proposed by Fricke and Klein [5] and proved by Horowitz [8] that for any $W \in F_n$, w can be written as an integer polynomial in these Horowitz generators. Horowitz also established that the polynomial is unique up to some ideal, which is trivial for $n = 2$. While this result illustrates the algebraic nature of free group characters, the actual calculation of the character of an arbitrary free group word as a polynomial in the characters of a base set is tedious at best and quite difficult for long words, involving a delicate application of "trace relations" central to the study of the invariants of products of 2×2 matrices (a la Procesi [12]). In this paper, we exploit the characteristics of $W \in F_2$ to calculate the polynomial degree of its special linear character. Moreover, this computation reveals explicitly the leading monomial of the character.

Indeed, for $W \in F_2 = \langle X, Y \rangle$, the character w is an integer polynomial in the characters $x = \operatorname{tr} X$, $y = \operatorname{tr} Y$, and $z = \operatorname{tr} XY$. One can *normalize* W within its character class (the set of words in F_n that have the same character) through

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a process of cyclic reduction, cyclic permutation, and/or inversion, so that the character class of W is represented by a word of the form

$$\overline{W} = X^{W_1}Y^{W_2}\cdots X^{W_{n-1}}Y^{W_n}, \quad W_1 > 0. \tag{1.1}$$

Here each W_i, $i = 1,\ldots,n$, is a non-zero integer, n is either 1 or even, and we regard each letter with its corresponding exponent as a syllable (Compare Horowitz [8]). This form was used by Horowitz [8] to study nonconjugate representatives within $SL(2,\mathbb{C})$-character classes of F_2. This form also appears in the study of primitive elements of F_2 and what bases actually look like (see Cohen, et.al [4]). $\overline{W}$ is a minimal length representative within its character class, and is uniquely defined within a character class up to cyclic permutation and possibly inversion (See, for instance, Haralick, et.al [7]). The number of syllables in $\overline{W}$, denoted $s\ell(\overline{W})$, as well as the number of distinct maximal negative subwords ξ_W are quantities independent of the choice of normalized representative of W. Call $\deg(w)$ the polynomial degree of $w \in \mathbb{Z}[x,y,z]$. Horowitz noticed that when $n = 1$, the degree of the character of (for example) $\overline{W} = X^{W_1}$ is W_1, as w has the structure of a Chebyshev polynomial of the second kind in x (see below). Our main results herein show that a complete description of the highest degree term of the character of any $W \in F_2$ is easily computed using a normalized representative $\overline{W}$:

Theorem 1.1. *Suppose $W \in F_2$ has a normalized representative of the form $\overline{W} = X^{W_1}Y^{W_2}\cdots X^{W_{n-1}}Y^{W_n}$ with syllable length at least two. Then*

$$\deg(w) = \frac{s\ell(\overline{W})}{2} + \xi_W + \sum_{i=1}^{n}\left(|W_i| - 1\right).$$

Furthermore, w has a unique monomial of largest total degree.

Denote by $\epsilon_i(w)$ the total exponent of the variable $i \in \{x,y,z\}$ in the leading monomial of w for $W \in F_2$. Then the calculation of the total multidegree of w also reveals the exponents of each of the constituent coordinates x, y, and z in this leading monomial. We also show the following:

Theorem 1.2. *Suppose $W \in F_2$ has a normalized representative of the form $\overline{W} = X^{W_1}Y^{W_2}\cdots X^{W_{n-1}}Y^{W_n}$ with syllable length at least two. For $1 \le i \le n-1$, define*

$$r_i = \begin{cases} 1 & (-1)^{n-1}W_i > 0 \text{ and } W_iW_{i+1} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\epsilon_x(w) = \sum_{i=1}^{\frac{n}{2}} |W_{2i-1}| - \sum_{i=1}^{n-1} r_i, \quad \epsilon_y(w) = \sum_{i=1}^{\frac{n}{2}} |W_{2i}| - \sum_{i=1}^{n-1} r_i, \quad \text{and} \quad \epsilon_z(w) = \sum_{i=1}^{n-1} r_i.$$

In particular, $\dfrac{s\ell(\overline{W})}{2} - \xi_W = \displaystyle\sum_{i=1}^{n-1} r_i.$

In the study of the automorphisms of the character variety of F_n induced by the automorphisms of F_n, the inner automorphisms (those induced by conjugation within F_n), act trivially on characters. Hence there is an action of the outer automorphism group of F_n, $Out(F_n)$, on the character variety. For $n \le 3$, this action is evidently by polynomial automorphisms on the affine space $\mathbb{C}^{2^n-1}$ in which the character variety can be embedded (the $Out(F_n)$-action on characters does not in general lift to this ambient affine space, although individual elements do lift to polynomial automorphisms. See, for instance, McCool [10]). The dynamics of this action is of current interest, particularly when F_2 (and F_3) is isomorphic to the fundamental group of a compact surface (See Brown [1] and [2], Goldman [6], and Previte and Xia [11], for examples and details). Dynamical information such as the algebraic entropy (or the related dynamical degree) of this $Out(F_2)$ action on $\mathbb{C}^3$ relies on the growth in the degrees of the coordinate polynomials of the automorphisms under iteration. These coordinate polynomials are the characters of the basic words upon iteration of the automorphism. In Brown [1], we use the results of this paper to calculate the algebraic entropy of individual elements of $Out(F_2)$ as the asymptotic growth factor of the degrees of the coordinate polynomials of the automorphisms under iteration. This information is related to the topological entropy of the action on appropriate subvarieties of the character variety of F_2.

The paper is organized as follows: In Section 2, we discuss some preliminaries and establish notation for the paper. In Section 3, we show that the total multidegree of the character of a free group word is roughly dependent on the size of a normalized representative, where the size is defined using a triple of data. We then establish Theorem 1.1. In Section 4, we extend the total degree calculation to construct the leading monomial of the character, and prove Theorem 1.2. Unfortunately, due mainly to the fact that the ideal of polynomials in the basic characters that vanish on characters is not trivial for $n > 2$, this technique does not generalize readily to a computation of the leading monomial (and hence the degree) of the character of $W \in F_n$, $n > 2$. In Section 5, we discuss some of the reasons why the techniques of this paper fail for $n > 2$ and provide examples.

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2. Preliminaries

2.1. **The trace relation.** Employ the notation that a capital letter denote a free group word, and the small case version denote its corresponding special linear character. For words written as combinations of words, $W = UV$, this notation may be inconvenient, as in general $\operatorname{tr} UV \neq \operatorname{tr} U \operatorname{tr} V$. Hence, we also denote the character of $W \in F_n$ by $\langle W \rangle$, so that for $W \in F_2$,

$$w = \operatorname{tr} W = \langle W \rangle \in \mathbb{Z}[x, y, z],$$

where $x = \langle X \rangle$, $y = \langle Y \rangle$, and $z = \langle XY \rangle$ are the Horowitz generators (See [8]).

A fundamental "trace" relation can be constructed for 2×2 matrices with unit determinant: Any $A \in SL(2, \mathbb{C})$ solves the Cayley-Hamilton form of its own characteristic polynomial,

$$A^2 - (\operatorname{tr} A)\, A + I = 0, \tag{2.1}$$

where I is the 2×2 identity matrix. Multiplication on the left or right of A by any other matrix B doesn't alter the solution set:

$$B \left(A^2 - (\operatorname{tr} A)\, A + I \right) A^{-1} = BA - (\operatorname{tr} A)\, B + BA^{-1} = 0. \tag{2.2}$$

Moreover, since this is simply a sum of matrices, this equation can be solved on the level of each matrix entry. Thus the trace form of Equation (2.2)

$$\operatorname{tr} BA - \operatorname{tr} A \operatorname{tr} B + \operatorname{tr} BA^{-1} = 0$$

holds, which is more familiarly written as

$$\operatorname{tr} BA = \operatorname{tr} B \operatorname{tr} A - \operatorname{tr} BA^{-1},$$

or in the notation of this paper

$$\langle BA\rangle = \langle B\rangle\langle A\rangle - \langle BA^{-1}\rangle. \tag{2.3}$$

Many (more complex) trace relations can be derived from this, and it is known that all come from this relation (see Brunfiel-Hilden [3]). In fact, the ideal of polynomials in the Horowitz generators which vanish identically for all special linear characters of F_n is generated by Equation (2.2) (see [9]).

Example 2.1. Let $W = XYX$. Then

$$\begin{aligned} w = \langle W\rangle &= \langle XYX\rangle \\ &= \langle XY\rangle\langle X\rangle - \langle XYX^{-1}\rangle \\ &= \langle XY\rangle\langle X\rangle - \langle Y\rangle = xz - y. \end{aligned}$$

Example 2.2. Let $W = XX$. Then

$$\begin{aligned} w = \langle W\rangle &= \langle XX\rangle \\ &= \langle X\rangle\langle X\rangle - \langle XX^{-1}\rangle \\ &= \langle X\rangle^2 - \langle e\rangle = x^2 - 2. \end{aligned}$$

Example 2.3. Let $W = XYX^{-1}Y^{-1}$. Then

$$\begin{aligned} w = \langle W\rangle &= \langle XYX^{-1}Y^{-1}\rangle \\ &= \langle XYX^{-1}\rangle\langle Y\rangle - \langle XYX^{-1}Y\rangle \\ &= \langle Y\rangle^2 - \big(\langle XY\rangle\langle X^{-1}Y\rangle - \langle XX\rangle\big) \\ &= \langle Y\rangle^2 - \langle XY\rangle\left(\langle X\rangle\langle Y\rangle - \langle XY\rangle\right) + \big(\langle X\rangle^2 - 2\big) \\ &= -xyz + x^2 + y^2 + z^2 - 2. \end{aligned}$$

By observation, it is recognizable that a general procedure for calculating $\langle W\rangle$ for arbitrary $W \in F_n$ even for the case $n = 2$ is available, but can be difficult and tedious for words of large length.

2.2. **Chebyshev recursion.** Example 2.2 above is of particular import. Horowitz applied the trace relation

$$\operatorname{tr} UV = \operatorname{tr} U \operatorname{tr} V - \operatorname{tr} UV^{-1} \tag{2.4}$$

to the word $U^m = U^{m-1}U$, $m > 1$, to get

$$\operatorname{tr} U^m = \operatorname{tr} U^{m-1} \operatorname{tr} U - \operatorname{tr} U^{m-2}.$$

Denoting $\operatorname{tr} U^m = T_m(u)$, Horowitz recognized $T_m(u)$ as a Chebyshev polynomial of the second kind in the variable $u = \operatorname{tr} U$ which satisfies the recursion

$$T_m(u) = u \cdot T_{m-1}(u) - T_{m-2}(u), \tag{2.5}$$

where

$$T_0(u) = 2 \quad \text{and} \quad T_1(u) = u.$$

In fact, since $\operatorname{tr} U^{-1} = \operatorname{tr} U$, it follows that for any $m \in \mathbb{Z}$,

$$T_{|m|}(u) = \operatorname{tr} U^m.$$

Directly from this fact we see that $\deg(T_m(u)) = m \cdot \deg(u)$, and in particular

$$\deg\langle X^m\rangle = \deg\langle Y^m\rangle = \deg\langle (XY)^m\rangle = |m|\,. \tag{2.6}$$

The properties of Chebyshev polynomials of this type, like the fact that the parity of m determines the parity of the polynomial, will be useful in the discussion to follow.

2.3. **Normalized free group words.** As stated in the introduction, when discussing the special linear character of a free group word, we are free to alter the word via the processes of cyclic reduction, cyclic permutation and/or inversion, to change the representative of the character class which will be better suited for study. The process of finding a "good" word representative within a character class will be called *normalization*, and we will call the class representative a normalized word.

Let $W \in F_2$. Then, within its character class, a normalized form is

$$\overline{W} = X^{W_1}Y^{W_2}\cdots X^{W_{n-1}}Y^{W_n}, \quad W_1 > 0. \tag{2.7}$$

$\overline{W}$ is not unique, although it is cyclically reduced and hence of minimal word-length within its conjugacy class. Note that, as long as W is not a nontrivial power of a single generator, n is always even. Also, one might need to invert W to satisfy the condition that $W_1 > 0$. This normalized form was used by Horowitz [8] to determine non conjugate words in F_2 which have the same character. This form also shows up in the classification of bases of F_2 (See Cohen. et.al [4]).

3. The total multi-degree of a character

In this section, we exploit the structure of $\overline{W}$ to calculate the total multi-degree of the character of $W \in F_2$ as an element of $\mathbb{Z}[x, y, z]$. For words W such that $s\ell(\overline{W}) = 1$, where $\overline{W}$ is simply a positive power of a single generator, Equation (2.6) immediately implies $\deg(w) = W_1$. Presently, we generalize this degree calculation for $\overline{W}$ corresponding to arbitrary $W \in F_2$.

3.1. **Positive normalized words.** It seems intuitive that the degree of the character of $W \in F_2$ is roughly proportional to the "size" of its cyclically reduced word length. While this is true in general, it is especially true in the case where the normalized word has all positive exponents. For a normalized word $\overline{W} \in F_2$, a subword (not necessarily proper) is called *positive* if all of its exponents are positive. Similarly for a *negative* subword, although negative subwords of $\overline{W}$ are necessarily proper. A negative subword is called *maximal negative* if it is negative and if its adjacent letters in $\overline{W}$, including wraparound, are positive subwords.

For arbitrary $W \in F_2$, denote the size of $\overline{W}$ by the triple

$$\left(n, m = \max_i\{|W_i|\}, r = \#\{j|\,|W_j| = m\}\right).$$

This notion of size orders normalized words lexicographically. Note that r is always an integer between 1 and n, but that when $m = 1$, $r = n$.

Lemma 3.1. *Suppose $W \in F_2$ has a positive, normalized representative $\overline{W}$ with syllable length at least two; that is, $\overline{W} = X^{W_1}Y^{W_2}\cdots X^{W_{n-1}}Y^{W_n}$, where $n \geq 2$ and each $W_i > 0$. Then*

$$\deg(w) = \frac{s\ell(\overline{W})}{2} + \sum_{i=1}^{n}(W_i - 1)\,.$$

Furthermore, w has a unique monomial of largest total degree.

Proof. Consider the special case that $\overline{W}$ has size $(n, 1, n)$. Then

$$\overline{W} = XY\cdots XY = (XY)^{\frac{n}{2}}.$$

Hence, consistent with the notation from above using $Z = XY$, $\langle\overline{W}\rangle = \langle Z^{\frac{n}{2}}\rangle = T_{\frac{n}{2}}(z)$ is a second Chebyshev polynomial in $z = \operatorname{tr} XY$, and $\deg\langle W\rangle = \frac{n}{2}$ satisfies the lemma. Note also that w has a unique leading monomial here.

Using this as a base case, we prove the lemma using induction on the size of $\overline{W}$. Indeed, suppose the size of $\overline{W}$ is (n, m, r), where $m > 1$ and assume that the lemma holds for all normalized positive words of size less that $\overline{W}$. Through a cyclic permutation, we can assume that the last syllable has $W_n = \max_i\{W_i\}$ (we will assume here that the last syllable consists of the letter Y. The argument will follow directly with an exchange of the letters X and Y should the last letter be an X). Then we can write $\overline{W} = UY$ and

$$U = X^{W_1}Y^{W_2}\cdots X^{W_{n-1}}Y^{W_n - 1}.$$

Using the basic trace formula, Equation (2.3), we have

$$\langle\overline{W}\rangle = \langle U\rangle\langle Y\rangle - \langle UY^{-1}\rangle. \tag{3.1}$$

Since $\overline{W}$ is positive, the size of U is necessarily smaller than that of $\overline{W}$. Indeed, the size of U is either $(n, m, r-1)$ if $r > 1$, or $(n, m-1, t)$ if $r = 1$, where t is some integer between 1 and n. Hence by induction

$$\deg\langle U\rangle = \frac{n}{2} + \sum_{i=1}^{n}(W_i - 1) - 1.$$

As for the $\deg\langle UY^{-1}\rangle$, let $W_n > 2$. Then UY^{-1} also has either less maximal exponents (an equal m and a smaller r) or a smaller maximal exponent m. Thus, in this case, the size of UY^{-1} is also smaller than that of $\overline{W}$. And if $W_n = 2$, then UY^{-1} is not normalized, and $\overline{UY^{-1}}$ has syllable length strictly less than n, and hence is of smaller size than $\overline{W}$. Hence, again by induction

$$\deg\langle UY^{-1}\rangle = \begin{cases} \frac{n}{2} + \sum_{i=1}^{n}(W_i - 1) - 1 & \text{if } W_n = 2 \\ \frac{n}{2} + \sum_{i=1}^{n}(W_i - 1) - 2 & \text{otherwise} \end{cases}.$$

Note in the special case $n = 2$, $W_n = 2$, the $\deg\langle UY^{-1}\rangle = W_1$, in line with the discussion at the beginning of this section and Equation (2.6).

Lastly, in these calculations, $\deg\langle UY^{-1}\rangle < \deg(\langle U\rangle\langle Y^{-1}\rangle)$. Thus we can inductively conclude that w will satisfy the lemma and have a unique leading monomial (inherited from $\langle U\rangle\langle Y\rangle$). □

3.2. **General normalized words.** Due to the presence of the Horowitz generator $\langle XY\rangle = z$, the incidence of the subword $Z = XY \subset \overline{W}$ will be important to the calculation of the degree of $\langle W\rangle$. In turn, we may detect copies of $Z \in \overline{W}$ via the presence of negative subwords in $\overline{W}$. To start, recall that ξ_W denotes the number of distinct, maximal negative subwords in a normalized representative of W.

Remark 3.2. It is important to note here two aspects of the quantity ξ_W. First, the restriction that a normalized word W have positive first exponent eliminates the chance of a wraparound effect creating an ambiguity in the magnitude of ξ_W. For example, $W = X^{-1}YXY^{-1}$ is not normalized. A normalized form for W would be $XY^{-1}X^{-1}Y$. Here, $\xi_W = 1$. Second, including wraparound, there is exactly the same number of maximal negative subwords in any choice of $\overline{W}$ as there are maximal positive subwords. Hence neither cyclic permutation nor inversion will change the value of ξ_W. For this reason, we leave the bar notation off of the subscript here.

Theorem 1.1. *Suppose $W \in F_2$ has a normalized representative of the form $\overline{W} = X^{W_1}Y^{W_2}\cdots X^{W_{n-1}}Y^{W_n}$ with syllable length at least two. Then*

$$\deg(w) = \frac{s\ell(\overline{W})}{2} + \xi_W + \sum_{i=1}^{n}\left(|W_i| - 1\right).$$

Furthermore, w has a unique monomial of largest total degree.

Proof. We prove this inductively on the number of negative subwords. The $\xi_W = 0$ case is covered by Lemma 3.1. So let $\xi_W > 0$ and assume that the theorem holds for every word whose normalization has at most $\xi_W - 1$ maximal negative subwords. Write $\overline{W} = UVT$, where (1) V is a maximal negative word with $s\ell(V) = m > 0$, and (2) T and U are words of syllable lengths $k \geq 0$ (T may be trivial here, so that $k = 0$. We will assume that it is not) and $l > 0$, respectively. Note that the last letter of U and the first letter of T necessarily have positive exponent, and that $k + l + m = n$. Then $\xi_W = \xi_U + \xi_T + 1$.

Under the trace relation

$$\langle W\rangle = \langle UVT\rangle = \langle TUV\rangle = \langle TU\rangle\langle V\rangle - \langle TUV^{-1}\rangle, \tag{3.2}$$

$\xi_{TU} = \xi_{TUV^{-1}} = \xi_W - 1$ and $\xi_V = 0$ (any normalization of V is positive). So by assumption the degree formula holds for them. To calculate $\deg(w)$, consider two cases.

Case 1: m odd. Then V starts and ends with the same letter, so $s\ell(\overline{V}) = m - 1$. If we assume that V starts and ends with the letter X (this is equivalent to assuming that l is even), then a normalized representative for V is the positive word $\overline{V} = X^{-(V_1+V_m)}Y^{-V_{m-1}}\cdots X^{-V_3}Y^{-V_2}$.

Remark 3.3. An entirely parallel situation arises when V starts and ends with the letter Y. This is equivalent to assuming that l is odd. This leads to a choice of $\overline{V}$ with one set of combined exponents at the end. We omit this analogous case here for brevity. However, we will refer to this case at points in the proof where it seems relevant to note the difference from the developed case.

Using this $\overline{V}$ we have by Lemma 3.1,

$$\begin{aligned}\deg\langle V\rangle &= \frac{s\ell(\overline{V})}{2} + (|V_1+V_m|-1) + \sum_{i=2}^{m-1}(|V_i|-1)\\ &= \frac{m-1}{2} + 1 + \sum_{i=1}^{m}(|V_i|-1)\\ &= \frac{m+1}{2} + \sum_{i=1}^{m}(|V_i|-1).\end{aligned}$$

Now the syllable length of TU in this case is also odd, so that $s\ell(\overline{TU}) = k+l-1$. Note that the instance of $\overline{TU}$ will be slightly different depending on the parity of l, but in either case will have one set of combined exponents which will be additive due to the fact that the first exponent of U and the last exponent of T are positive as subwords of W. Via an analogous calculation to that above for $\deg\langle V\rangle$, but applying the inductive hypothesis instead of Lemma 3.1, we have

$$\deg\langle TU\rangle = \frac{k+l+1}{2} + \xi_{TU} + \sum_{i=1}^{k}(|T_i|-1) + \sum_{i=1}^{l}(|U_i|-1).$$

Combining these two, we get

$$\begin{aligned}\deg\left(\langle TU\rangle\langle V\rangle\right) &= \deg\langle TU\rangle + \deg\langle V\rangle\\ &= \frac{k+l+m}{2} + 1 + \xi_{TU} + \sum_{i=1}^{k}(|T_i|-1) + \sum_{i=1}^{l}(|U_i|-1) + \sum_{i=1}^{m}(|V_i|-1) \qquad (3.3)\\ &= \frac{s\ell(\overline{W})}{2} + \xi_W + \sum_{i=1}^{n}(|W_i|-1).\end{aligned}$$

In contrast, TUV^{-1} is either already normalized, or will be under a cyclic permutation. And since $\xi_{TUV^{-1}} = \xi_W - 1$, by induction the degree calculation holds, and

$$\deg\langle TUV^{-1}\rangle = \frac{s\ell(TUV^{-1})}{2} + \xi_{TUV^{-1}} + \sum_{i=1}^{s}(|T_i|-1) + \sum_{i=1}^{r}(|U_i|-1) + \sum_{i=1}^{m}(|V_i|-1).$$

Comparing this last equation with Equation (3.3), we see that $\deg\langle TUV^{-1}\rangle < \deg\left(\langle TU\rangle\langle V\rangle\right)$. Hence, in this case, $\deg(w) = \deg\left(\langle TU\rangle\langle V\rangle\right)$, the polynomial $\langle TU\rangle\langle V\rangle$ contributes the unique leading monomial, and the degree formula holds for W.

Case 2: m even. Here both TU and V are normalized after only a possible cyclic permutation, so $s\ell(TU) = k+l$ and we can use the degree formula directly:

$$\begin{aligned}\deg\langle TU\rangle &= \frac{k+l}{2} + \xi_{TU} + \sum_{i=1}^{k}(|T_i|-1) + \sum_{i=1}^{l}(|U_i|-1)\\ \deg\langle V\rangle &= \frac{m}{2} + \sum_{i=1}^{m}(|V_i|-1)\end{aligned}$$

so that

$$\deg\left(\langle TU\rangle\langle V\rangle\right) = \frac{k+l+m}{2} + \xi_{TU} + \sum_{i=1}^{k}(|T_i|-1) + \sum_{i=1}^{l}(|U_i|-1) + \sum_{i=1}^{m}(|V_i|-1). \tag{3.4}$$

However, in this case TUV^{-1} is not normalized. Normalizing will combine the exponents of the first letter of T, T_1 (necessarily positive) with the last letter of V^{-1}, $-V_1 > 0$, and the last letter of U, U_l (also positive) with the first letter of V^{-1}, $-V_m > 0$. Again with $\xi_{TUV^{-1}} = \xi_W - 1$, we have

$$\begin{aligned}
\deg\langle TUV^{-1}\rangle &= \frac{s\ell(\overline{TUV^{-1}})}{2} + \xi_{TUV^{-1}} + (|T_1 - V_1| - 1) + (|U_l - V_m| - 1) \\
&\quad + \sum_{i=2}^{k}(|T_i| - 1) + \sum_{i=1}^{l-1}(|U_i| - 1) + \sum_{i=2}^{m-1}(|V_i| - 1) \\
&= \frac{k+l+m-2}{2} + \xi_{TUV^{-1}} + 2 + \sum_{i=1}^{k}(|T_i| - 1) + \sum_{i=1}^{l}(|U_i| - 1) \\
&\quad + \sum_{i=1}^{m}(|V_i| - 1) \\
&= \frac{k+l+m}{2} + \xi_{TUV^{-1}} + 1 + \sum_{i=1}^{n}(|W_i| - 1) \\
&= \frac{s\ell(\overline{W})}{2} + \xi_W + \sum_{i=1}^{n}(|W_i| - 1).
\end{aligned} \tag{3.5}$$

In this case, comparing Equations (3.4) and (3.5), we get

$$\deg\langle TUV^{-1}\rangle > \deg\left(\langle TU\rangle\langle V\rangle\right),$$

so that $\deg\langle TUV^{-1}\rangle = \deg(w)$, $\langle TUV^{-1}\rangle$ contributes the unique leading monomial to w and the result holds. □

4. Exponent counts of the leading monomial

From the previous section, we see that for any $W \in F_2$ with a normalization $\overline{W} = X^{W_1}Y^{W_2}\cdots X^{W_{n-1}}Y^{W_n}$, $n \geq 2$, the polynomial degree of w is computed to be

$$\deg\langle W\rangle = \frac{s\ell(\overline{W})}{2} + \xi_W + \sum_{i=1}^{n}\left(|W_i| - 1\right).$$

In this section, we consolidate this information into a different form which exposes not only the degree of $\langle W\rangle$ but also its leading monomial.

Notice that

$$
\begin{aligned}
\deg\langle W\rangle &= \frac{s\ell(\overline{W})}{2} + \xi_W + \sum_{i=1}^{n}(|W_i|-1) \\
&= \frac{s\ell(\overline{W})}{2} + \xi_W + \sum_{i=1}^{n}|W_i| - n \\
&= \sum_{i=1}^{n}|W_i| + \xi_W - \frac{s\ell(\overline{W})}{2} \\
&= \sum_{i=1}^{n}|W_i| - \left(\frac{s\ell(\overline{W})}{2} - \xi_W\right). \qquad (4.1)
\end{aligned}
$$

The last term is a measure of how often the subwords XY or $(XY)^{-1} = Y^{-1}X^{-1}$ appear in $\overline{W}$. Since $\langle XY\rangle = z$, this will ultimately determine the presence and the exponent of the variable z in the leading monomial of $\langle W\rangle \in \mathbb{Z}[x,y,z]$. The exponents of the variables x and y will be recorded in the respective exponents W_i once we account for the z variable. Recall, for $i = x,y,z$, the value $\epsilon_i(w)$ is the exponent of the variable i in the leading monomial of w.

Lemma 4.1. *Suppose $W \in F_2$ has a positive, normalized representative of the form $\overline{W} = X^{W_1}Y^{W_2}\cdots X^{W_{n-1}}Y^{W_n}$ with syllable length at least two. Then*

$$
\epsilon_x(w) = \sum_{i=1}^{\frac{n}{2}} W_{2i-1} - \frac{n}{2}, \quad \epsilon_y(w) = \sum_{i=1}^{\frac{n}{2}} W_{2i} - \frac{n}{2}, \quad \text{and} \quad \epsilon_z(w) = \frac{n}{2}.
$$

Proof. In the case where the size of $\overline{W}$ is $(n,1,n)$, $w = T_{\frac{n}{2}}(z)$, and hence by Equation (2.6) the lemma holds and $\epsilon_x(w) = \epsilon_y(w) = 0$. So assume that $\overline{W}$ has size (n,m,r) with $m > 1$. By a cyclic permutation moving the letter with the largest exponent to the end of the word (and we assume here again as in Lemma 3.1 that the letter is a Y), write $\overline{W} = UY$. Also as in Lemma 3.1, assume the result holds for all words of size less than (n,m,r). Then using the basic trace formula, the leading monomial of $\langle W\rangle$ is the product of the leading monomial of $\langle U\rangle$ and y (see the degree calculations of the proof of the lemma). Here $U = X^{W_1}Y^{W_2}\cdots X^{W_{n-1}}Y^{W_n-1}$, is of a smaller size than $\overline{W}$ and $\epsilon_x(u) = \epsilon_x(w)$, $\epsilon_z(u) = \epsilon_z(w)$, and $\epsilon_y(u) = \epsilon_y(w) - 1$. Thus the result holds. □

Example 4.2. Let $W = XYXYXY = (XY)^3$. In this case,

$$
\sum_{i=1}^{3} W_{2i-1} = \sum_{i=1}^{3} W_{2i} = \frac{s\ell(\overline{W})}{2} = 3, \qquad \text{and } \xi_W = 0.
$$

Here $\frac{s\ell(\overline{W})}{2} - \xi_W = 3$, $\deg(w) = 3$, and $\epsilon_x(w) = \epsilon_y(w) = 0$. The character of W is

$$
w = \langle W\rangle = T_3(z) = z^3 - 3z,
$$

and the variable z is the only variable represented in the leading monomial.

In fact, the term $\frac{s\ell(\overline{W})}{2} - \xi_W$ may be calculated in another way: Record the occurrence of either the positive subword XY or the negative subword $Y^{-1}X^{-1}$ in the ith position of W by letting $r_i = r_i(W) = 1$ when either of these two conditions holds in the ith and $(i+1)$th positions and zero otherwise. As we will see, the sum total of the r_i's is $\frac{s\ell(\overline{W})}{2} - \xi_W$.

Remark 4.3. The criterion that a normalized $\overline{W}$ start with the letter X and a positive first exponent means that the sum total of the r_i's will be well defined for any choice of $\overline{W}$.

Theorem 1.2. *Suppose $W \in F_2$ has a normalized representative of the form $\overline{W} = X^{W_1}Y^{W_2}\cdots X^{W_{n-1}}Y^{W_n}$ with syllable length at least two. For $1 \le i \le n-1$, define*

$$r_i = \begin{cases} 1 & (-1)^{n-1}W_i > 0 \text{ and } W_iW_{i+1} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\epsilon_x(w) = \sum_{i=1}^{\frac{n}{2}} |W_{2i-1}| - \sum_{i=1}^{n-1} r_i, \quad \epsilon_y(w) = \sum_{i=1}^{\frac{n}{2}} |W_{2i}| - \sum_{i=1}^{n-1} r_i, \quad \text{and} \quad \epsilon_z(w) = \sum_{i=1}^{n-1} r_i.$$

In particular, $\dfrac{s\ell(\overline{W})}{2} - \xi_W = \displaystyle\sum_{i=1}^{n-1} r_i.$

Proof. As in the proof of Theorem 1.1, we again prove this theorem by induction on ξ_W. For the case $\xi_W = 0$, $\overline{W}$ is positive, and $r_i = 0$ if and only if i is even. Thus by Lemma 4.1 the theorem holds. So assume that the theorem holds for all words with the number of maximal negative words is less than ξ_W. As in Theorem 1.1, let $\overline{W} = UVT$, where V is a maximal negative subword, with $s\ell(V) = m > 0$, $s\ell(U) = l > 0$, and $s\ell(T) = k \ge 0$. We will follow the same convention as in the proof of Theorem 1.1: Under Equation (3.2), we will calculate the degrees of $\langle TU\rangle$, $\langle V\rangle$, and $\langle TUV^{-1}\rangle$ using the exponents of U, V and T and divide the analysis into two cases depending on the parity of m.

Case 1: m odd. Assume that V starts and ends with an instance of the letter X (i.e., let l be even: the analogous argument for V starting with a Y will again be omitted here, but referred to in the discussion. See Remark 3.3). Then, we can write

$$\overline{W} = UVT = (X^{U_1}Y^{U_2}\cdots X^{U_{l-1}}Y^{U_l})(X^{V_1}Y^{U_2}\cdots X^{V_m})(Y^{T_1}\cdots Y^{T_k}). \tag{4.2}$$

By Case 1 of the proof of Theorem 1.1, $\langle W\rangle$ inherits its leading monomial from $\langle TU\rangle\langle V\rangle$ via the basic trace formula. Here a normalized representative of V is the positive word

$$\overline{V} = X^{-(V_1+V_m)}Y^{-V_{m-1}}\cdots X^{-V_3}Y^{-V_2},$$

and by Lemma 4.1,

$$\epsilon_x(v) = \sum_{i=1}^{\frac{m+1}{2}} |V_{2i-1}| - \frac{m-1}{2}, \quad \epsilon_y(v) = \sum_{i=1}^{\frac{m-1}{2}} |V_{2i}| - \frac{m-1}{2}, \quad \text{and } \epsilon_z(v) = \frac{m-1}{2}.$$

In the case where V starts with the letter Y, the expressions for $\epsilon_x(v)$ and $\epsilon_y(v)$ are interchanged. And since $V_j = W_{l+j}$, we have

$$\begin{aligned}
\epsilon_x(v) &= \sum_{i=1}^{\frac{m+1}{2}} |W_{l+2i-1}| - \frac{m-1}{2} = \sum_{i=\frac{l}{2}+1}^{\frac{l+m+1}{2}} |W_{2i-1}| - \frac{m-1}{2}, \\
\epsilon_y(v) &= \sum_{i=1}^{\frac{m-1}{2}} |W_{l+2i}| - \frac{m-1}{2} = \sum_{i=\frac{l}{2}+1}^{\frac{l+m-1}{2}} |W_{2i}| - \frac{m-1}{2}, \\
\epsilon_z(v) &= \frac{m-1}{2} = \sum_{i=l+1}^{l+m-1} r_i(\overline{W}).
\end{aligned}$$

Now TU is also not normalized, but a normal representative is the reduced word representing UT,

$$\overline{TU} = X^{U_1} Y^{U_2} \cdots X^{U_{l-1}} Y^{U_l + T_1} X^{T_2} \cdots X^{T_{k-1}} Y^{T_k}. \tag{4.3}$$

Note here that $U_l > 0$ and $T_1 \geq 0$ (and greater than 0 when T is nontrivial), since V is a maximal negative subword of $\overline{W}$. Also note that $\xi_{TU} = \xi_W - 1$, so by assumption the theorem holds in this case, and

$$\begin{aligned}
\epsilon_x(\langle TU \rangle) &= \sum_{i=1}^{\frac{k-1}{2}} |T_{2i}| + \sum_{i=1}^{\frac{l}{2}} |U_{2i-1}| - \sum_{i=1}^{k+l-2} r_i(\overline{TU}), \\
\epsilon_y(\langle TU \rangle) &= |T_k| + \sum_{i=1}^{\frac{k-1}{2}} |T_{2i-1}| + \sum_{i=1}^{\frac{l}{2}} |U_{2i}| - \sum_{i=1}^{k+l-2} r_i(\overline{TU}), \\
&= \sum_{i=1}^{\frac{k+1}{2}} |T_{2i-1}| + \sum_{i=1}^{\frac{l}{2}} |U_{2i}| - \sum_{i=1}^{k+l-2} r_i(\overline{TU}), \\
\epsilon_z(\langle TU \rangle) &= \sum_{i=1}^{k+l-2} r_i(\overline{TU}). \qquad (4.4)
\end{aligned}$$

Here we have included notation expressing the dependence of the r_i's on $\overline{TU}$. Using Equation (4.2), $T_i = W_{l+m+i}$ for $1 \leq i \leq k$, $U_i = W_i$, for $1 \leq i \leq l$, and $l+m$ is odd. It follows from the inspection of Equations (4.2) and (4.3) that $r_i(\overline{TU}) = r_i(\overline{W})$ for $1 \leq i \leq l-1$ and $r_{(l-1)+i}(\overline{TU}) = r_{(l+m)+i}(\overline{W})$ for $1 \leq i \leq k-1$. In particular, since $U_l > 0$ and $T_1 \geq 0$ by hypothesis, $r_l(\overline{TU}) = r_{l+m+1}(\overline{W}) = 0$. Then

$$\begin{aligned}
\epsilon_x(\langle TU \rangle) &= \sum_{i=1}^{\frac{k}{2}} |W_{l+m+2i}| + \sum_{i=1}^{\frac{l}{2}} |W_{2i-1}| - \left(\sum_{i=1}^{l-1} r_i(\overline{W}) + \sum_{i=l+m+1}^{l+m+k-1} r_i(\overline{W}) \right), \\
\epsilon_y(\langle TU \rangle) &= \sum_{i=1}^{\frac{k}{2}} |W_{l+m+2i-1}| + \sum_{i=1}^{\frac{l}{2}} |W_{2i}| - \left(\sum_{i=1}^{l-1} r_i(\overline{W}) + \sum_{i=l+m+1}^{l+m+k-1} r_i(\overline{W}) \right), \\
\epsilon_z(\langle TU \rangle) &= \sum_{i=1}^{l-1} r_i(\overline{W}) + \sum_{i=l+m+1}^{l+m+k-1} r_i(\overline{W}).
\end{aligned}$$

Combining these calculations, we get

$$\begin{aligned}\epsilon_z(w) &= \epsilon_z(\langle TU\rangle\langle V\rangle) = \epsilon_z(\langle TU\rangle) + \epsilon_z(v) \\ &= \sum_{i=1}^{l-1} r_i(\overline{W}) + \sum_{i=l+1}^{l+m-1} r_i(\overline{W}) + \sum_{i=l+m+1}^{l+m+k-1} r_i(\overline{W}) = \sum_{i=1}^{l+m+k-1} r_i(\overline{W}) \end{aligned} \tag{4.5}$$

since $r_l(\overline{W}) = r_{l+m}(\overline{W}) = 0$, again because T_1, and U_l are positive.

And as for the other two variables, we get

$$\begin{aligned}\epsilon_x(w) &= \epsilon_x(\langle TU\rangle) - \epsilon_z(v) \\ &= \sum_{i=1}^{\frac{k}{2}} |W_{l+m+2i}| + \sum_{i=1}^{\frac{l}{2}} |W_{2i-1}| + \sum_{i=\frac{l}{2}+1}^{\frac{l+m+1}{2}} |W_{2i-1}| - \sum_{i=1}^{l+m+k-1} r_i(\overline{W}) \\ &= \sum_{i=\frac{l+m+1}{2}+1}^{\frac{l+m+k}{2}} |W_{2i-1}| + \sum_{i=1}^{\frac{l}{2}} |W_{2i-1}| + \sum_{i=\frac{l}{2}+1}^{\frac{l+m+1}{2}} |W_{2i-1}| - \sum_{i=1}^{l+m+k-1} r_i(\overline{W}) \\ &= \sum_{i=1}^{\frac{l+m+k}{2}} |W_{2i-1}| - \sum_{i=1}^{l+m+k-1} r_i(\overline{W})\end{aligned}$$

and a similar result for $\epsilon_y(w)$.

And finally for this case, note that $s\ell(\overline{W}) = s\ell(\overline{V}) + s\ell(\overline{TU}) + 2$, and $\xi_W = \xi_{TU} + 1$ (and $\xi_V = 0$). Hence Equation (4.5) and the inductive hypothesis imply that

$$\begin{aligned}\sum_{i=1}^{l+m+k-1} r_i(\overline{W}) &= \left(\frac{s\ell(\overline{TU})}{2} - \xi_{TU}\right) + \left(\frac{s\ell(\overline{V})}{2} - \xi_V\right) \\ &= \frac{s\ell(\overline{W}) - 2}{2} - (\xi_W - 1) = \frac{s\ell(\overline{W})}{2} - \xi_W.\end{aligned}$$

We re-note here (as in the beginning of this case) that we made the additional assumption that l was even. In the case that l is odd, similar calculations would yield an interchange of the expressions for $\epsilon_x(\langle TU\rangle)$ and $\epsilon_y(\langle TU\rangle)$. We leave these to the reader.

Case 2: m even. Then, by the basic trace formula, $\langle W\rangle$ and $\langle TUV^{-1}\rangle$ share the same leading monomial. Here TUV^{-1} is not normalized, and a normal representative is (compare Equation (4.3)) either

$$\overline{TUV^{-1}} = X^{U_1}Y^{U_2}\cdots X^{U_{l-1}}Y^{U_l - V_m}X^{-V_{m-1}}\cdots Y^{-V_2}X^{-V_1+T_1}Y^{T_2}\cdots Y^{T_k}, \tag{4.6}$$

or

$$\overline{TUV^{-1}} = X^{U_1}Y^{U_2}\cdots Y^{U_{l-1}}X^{U_l - V_m}Y^{-V_{m-1}}\cdots X^{-V_2}Y^{-V_1+T_1}X^{T_2}\cdots Y^{T_k},$$

depending on whether l is even or odd, respectively. Again, we will treat the former in detail and note the latter in passing. For clarity, in this case, note that

$$\overline{W} = UVT = X^{U_1}Y^{U_2}\cdots X^{U_{l-1}}Y^{U_l}X^{V_1}\cdots Y^{V_m}X^{T_1}\cdots Y^{T_k}.$$

Also note that $\xi_{TUV^{-1}} = \xi_W - 1$, so by induction the theorem applies to $\overline{TUV^{-1}}$, and we can calculate directly using the result. To start, using the normalized form given in Equation (4.6),

$$\begin{aligned}
\epsilon_z(\langle TUV^{-1}\rangle) &= \sum_{i=1}^{l+m+k-3} r_i(\overline{TUV^{-1}}) \\
&= \sum_{i=1}^{l-1} r_i(\overline{TUV^{-1}}) + \sum_{i=l}^{l+m-2} r_i(\overline{TUV^{-1}}) + \sum_{i=l+m-1}^{l+m+k-3} r_i(\overline{TUV^{-1}}). \qquad (4.7)
\end{aligned}$$

Now U, V, T and W are all normalized. Since the first $l-1$ syllables of $\overline{TUV^{-1}}$ are identical to those of U and W, and U_l and $U_l - V_m$ are positive, the first term in Equation (4.7) is precisely

$$\sum_{i=1}^{l-1} r_i(\overline{TUV^{-1}}) = \sum_{i=1}^{l-1} r_i(U) = \sum_{i=1}^{l-1} r_i(\overline{W}).$$

In the last term, all letters and exponents in the range of the index, save the first, correspond to those of the subword T. And since T_1 and $-V_1+T_1$ are each positive, we have

$$\begin{aligned}
\sum_{i=l+m-1}^{l+m+k-3} r_i(\overline{TUV^{-1}}) &= \sum_{i=(l+m-2)+1}^{(l+m-2)+k-1} r_i(\overline{TUV^{-1}}) \\
&= \sum_{i=1}^{k-1} r_i(T) = \sum_{i=1}^{k-1} r_{l+m+i}(\overline{W}) = \sum_{i=l+m+1}^{l+m+k-1} r_i(\overline{W}).
\end{aligned}$$

And for the middle term, note that

$$r_{(l-1)+i}(\overline{TUV^{-1}}) = r_{(l+m)-i}(\overline{W}) = r_{m-i}(V)$$

for $i = 1, \ldots, m-1$. Hence

$$\begin{aligned}
\sum_{i=l}^{l+m-2} r_i(\overline{TUV^{-1}}) &= \sum_{i=(l-1)+1}^{(l-1)+m-1} r_i(\overline{TUV^{-1}}) \\
&= \sum_{i=1}^{m-1} r_{m-i}(V) = \sum_{i=1}^{m-1} r_{(l+m)-i}(\overline{W}) = \sum_{i=l+1}^{l+m-1} r_i(\overline{W}).
\end{aligned}$$

Combining these terms, we get

$$\begin{aligned}
\sum_{i=1}^{l+m+k-3} r_i(\overline{TUV^{-1}}) &= \sum_{i=1}^{l-1} r_i(\overline{W}) + \sum_{i=l+1}^{l+m-1} r_i(\overline{W}) + \sum_{i=l+m+1}^{l+m+k-1} r_i(\overline{W}) \\
&= \sum_{i=1}^{l+m+k-1} r_i(\overline{W}) - (r_l + r_{l+m}).
\end{aligned}$$

But since both of these extra terms are 0, we have

$$\epsilon_z(w) = \epsilon_z(\langle TUV^{-1}\rangle) = \sum_{i=1}^{l+m+k-1} r_i(\overline{W}).$$

And since every exponent of W in this case is present in $\overline{TUV^{-1}}$ (this is true for l both odd and even, though we did not write down the form for W for the case where l is odd) and the combined exponents are all additive, the forms for $\epsilon_x(w)$ and $\epsilon_y(w)$ involve no clarifying descriptions; Indeed,

$$\begin{aligned}
\epsilon_x(w) &= \epsilon_x(\langle TUV^{-1}\rangle) = \epsilon_x(\langle UV^{-1}T\rangle) \\
&= \sum_{i=1}^{\frac{l}{2}} |U_{2i-1}| + \sum_{i=l+1}^{\frac{l+m}{2}} |V_{2i-1}| + \sum_{i=l+m+1}^{\frac{l+m+k}{2}} |T_{2i-1}| - \sum_{i=1}^{l+m+k-3} r_i(\overline{UV^{-1}T}) \\
&= \sum_{i=1}^{\frac{l+m+k}{2}} |W_{2i-1}| - \sum_{i=l}^{l+m+k-1} r_i(\overline{W})
\end{aligned}$$

with a similar expression for $\epsilon_y(w)$.

And finally, note that

$$\begin{aligned}
\sum_{i=1}^{l+m+k-3} r_i(\overline{TUV^{-1}}) &= \frac{s\ell(\overline{TUV^{-1}})}{2} - \xi_{TUV^{-1}} = \frac{s\ell(\overline{W})-2}{2} - (\xi_W - 1) \\
&= \frac{s\ell(\overline{W})}{2} - \xi_W = \sum_{i=1}^{l+m+k-1} r_i(\overline{W}).
\end{aligned}$$

□

Example 4.4. Let $W = XY^{-1}XY^{-1}XY^{-1}$. Here

$$\epsilon_z(w) = \sum_{i=1}^{5} r_i = 0,$$

so the leading monomial consists of the variables x and y. And since

$$\epsilon_x(w) = \sum_{i-1}^{3} |W_{2i-1}| = 3 = \sum_{i-1}^{3} |W_{2i}| = \epsilon_y(w),$$

we have $\deg(w) = 6$, and the leading monomial of w is x^3y^3. A straightforward calculation using the basic trace formula reveals

$$w = (xy - z)^3 - 3(xy - z).$$

Example 4.5. Let $W = X^2Y^3X^{-2}Y^{-1}X^{-2}Y^2$. Here $r_1 = r_4 = 1$ and the rest are 0. Essentially, now, we can decompose W as follows:

$$\epsilon_x(w) = \sum_{i=1}^{3} |W_{2i-1}| - \sum_{i=1}^{5} r_i = 2 + 2 + 2 - 2 = 4. \tag{4.8}$$

Continuing the calculation, we obtain the leading monomial to be $x^4y^4z^2$ and $\deg(w) = 10$. The actual character is

$$\begin{aligned}
w &= -2 + x^2 + 4y^2 + x^2y^2 - x^4y^2 - y^4 - 4x^2y^4 + x^4y^4 + x^2y^6 \\
&\quad + 2xyz - 4x^3yz + x^5yz - xy^3z + 6x^3y^3z - x^5y^3z - 2x^3y^5z \\
&\quad - x^2y^2z^2 - x^4y^2z^2 + x^2y^4z^2 + x^4y^4z^2 + x^3yz^3 - x^3y^3z^3.
\end{aligned}$$

5. The general case

The techniques of calculation in this paper do not easily generalize to higher rank free groups. In this section, we describe an interpretation of Theorem 1.2 in terms of an alternate presentation of F_2, which allows the leading monomial of a character to be "read off" from a normalized representative. Then we detail some of the problems inherent in attempting to extend these results to free groups of higher rank.

Example 5.1. In Example 4.5, $W = X^2Y^3X^{-2}Y^{-1}X^{-2}Y^2$. Decompose W as follows:

$$W = X(XY)Y^2X^{-2}(XY)^{-1}X^{-1}Y^2 = XZY^2X^{-2}Z^{-1}X^{-1}Y^2, \tag{5.1}$$

where $Z = XY$ corresponds to the Horowitz generator z. Then, by simply summing the absolute values of the corresponding exponents of each of the basic words, we again arrive at a leading monomial of $x^4y^4z^2$, and $\deg\langle W\rangle = 10$.

An analogous procedure produces the leading monomial of the character of any word in F_2. Make the process in Example 5.1 explicit by considering the redundant presentation via a Tietze transformation of F_2 given by

$$\widehat{F}_2 = \langle X, Y, Z | Z = XY\rangle.$$

The generating set for $\widehat{F}_2$ here has characters that correspond precisely to the Horowitz generators of F_2. Any normalized $W \in F_2$ will have a unique minimal length representative $\widehat{W} \in \widehat{F}_2$. Each occurrence of the generator Z in a normalized representative of $\widehat{W}$ corresponds to some $r_i = 1$, and results in a decrement of the sum of the exponents for each of the generators X and Y. Hence in this formulation, the exponents of the variables in the leading monomial are easily calculated without regard to negative subwords or syllable lengths.

However, constructing a redundant presentation of F_n, $n > 2$, so that the generators of $\widehat{F}_n$ correspond to the Horowitz generators of characters of words in F_n is more problematic. For one, minimal length representatives in $\widehat{F}_n$ of words in F_n are not unique in general. Let $F_3 = \langle X, Y, Z\rangle$ and consider

$$\widehat{F}_3 = \langle X, Y, Z, A, B, C, D | A = XY, B = XZ, C = YZ, D = XYZ = AZ = XC\rangle.$$

In this redundant presentation the generating set corresponds to the 7 Horowitz generators for characters of F_3:

$$\{\langle X\rangle, \langle Y\rangle, \langle Z\rangle, \langle XY\rangle, \langle XZ\rangle, \langle YZ\rangle, \langle XYZ\rangle\} = \{x, y, z, a, b, c, d\}.$$

Example 5.2. Let $W = Y^{-1}Z \in F_3$.

$$\widehat{W} = Y^{-1}Z \text{ and } \widehat{W} = Y^{-1}X^{-1}XZ = A^{-1}B.$$

Both versions of $\widehat{W}$ are of length 2. The character of $\widehat{W}$ is the same regardless of which minimal length word is chosen (using the basic trace formula once directly on $\langle AB^{-1}\rangle = ab - \langle AB\rangle$ is not enough, since $\langle AB\rangle$ is not a Horowitz generator. Here

$$\langle W\rangle = \langle\widehat{W}\rangle = \langle Y\rangle\langle Z\rangle - \langle YZ\rangle = yz - c.$$

Furthermore, even in the case where a unique minimal length representative in $\widehat{F}_3$ exists, its form may not readily indicate either the leading monomial or even the degree of w:

Example 5.3. Let $W = XZX^{-1}Y^{-1}$. In $\widehat{F}_3$, a minimal length representative of W is $\widehat{W} = BX^{-1}Y^{-1}$. While this would predict a degree 3 character, we have

$$\begin{aligned} \langle W \rangle &= \langle XZ \rangle \langle XY \rangle - \langle XZYX \rangle \\ &= \langle XZ \rangle \langle XY \rangle - \langle ZYX \rangle \langle X \rangle + \langle YZ \rangle. \end{aligned}$$

ZYX is not basic, and it is known (See Magnus [9], for example)

$$\langle ZYX \rangle = \langle X \rangle \langle YZ \rangle + \langle Y \rangle \langle XZ \rangle + \langle Z \rangle \langle XY \rangle - \langle X \rangle \langle Y \rangle \langle Z \rangle - \langle XYZ \rangle,$$

so that

$$\begin{aligned} \langle W \rangle &= ab - x(xc + yb + za - xyz - d) + c \\ &= x^2yz - x^2c - xyb - xza + xd + ab + c \end{aligned}$$

is of degree 4.

The last two examples do show a pattern in the original W and the leading monomial of w. However, as the following examples will show, this is not true in general.

A bigger problem arises in the lack of a unique monomial of highest total degree for characters of F_n, $n \geq 3$. The ring of $SL(2, \mathbb{C})$-characters of F_n is a quotient ring of the ring of integer polynomials in the $2^n - 1$ Horowitz generators by an ideal $\mathcal{I}_n$ which is principal for $n = 3$ (and trivial for $n < 3$; see, for example, Fricke and Klein [5] or Magnus [9]), and whose "size" grows rapidly with n. Due to the presence of this ideal, the degree of a character class is not well defined. Intuitively, one can choose the smallest total degree among the coset representatives of a character, but it is not clear that this representative will be unique, so that the exponent counts of the leading monomial is still not well defined over the class.

Moreover, as the final two examples illustrate, many F_3-characters have multiple monomials of equal highest total multidegree:

Example 5.4. Let $W = X^2YZ^3Y^{-1}X^{-1}YZ^2$. Then

$$\widehat{W} = X(XYZ)Z^2(XY)^{-1}(YZ)Z = XDZ^2A^{-1}CZ.$$

Here, as in the $\widehat{F}_2$ case of Example 5.1, and unlike Examples 5.2 and 5.3, the minimal length representative of W in $\widehat{F}_3$ does correctly indicate a highest total degree monomial in w. The full polynomial is

$$\begin{aligned} w \;=\; & xz^3acd - xyz^3d^2 + xyz^2ad - xz^2a^2c + xz^3bd + yz^3cd \\ & -z^3ac^2 + xyzd^2 - xz^2ab - xzacd - x^2z^2d + yz^2ac + yz^2b \\ & -y^2z^2d - z^3bc - xyad - xzbd + xz^2c + xa^2c - yzcd + zac^2 \\ & +xab + x^2d - yac + y^2d + zbc - xc - yb + za - d \end{aligned}$$

and one of the highest total degree monomials in w is xz^3acd which can be "read" from $\widehat{W}$.

However, here there are two monomials of total degree 7. This means that the leading monomial will ultimately depend on a chosen monomial ordering of the characters of the Horowitz generators. Given that $\mathcal{I}_3$ is generated by the single polynomial

$$\begin{aligned} p_3 \;=\; & xyzd - xya - xzb - xcd - yzc - ybd - zad \\ & +abc + x^2 + y^2 + z^2 + a^2 + b^2 + c^2 + d^2 - 4, \end{aligned}$$

the representative of the same coset of this character in $\mathcal{I}_3$ given by

$$w + (z^2 d)p_3$$

does have a unique leading monomial of degree 7, namely xz^3acd.

However, it appears that this cannot be done in general.

Example 5.5. Let $W = XYX^{-1}Z^{-1}$. Then

$$\begin{aligned}\langle W\rangle &= \langle XY\rangle\langle XZ\rangle - \langle XYZX\rangle \\ &= \langle XY\rangle\langle XZ\rangle - \langle XYZ\rangle\langle X\rangle + \langle YZ\rangle \\ &= xd - ab + c.\end{aligned}$$

There is no coset representative of this character with a unique monomial of total degree 2.

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Department of Mathematics, The Johns Hopkins University, 3400 North Charles Street, Baltimore, MD 21218-2686 USA

E-mail address: brown@math.jhu.edu