Math 645, Fall 2017: Assignment #9

Due: **Thursday, December 7th**

Problem #1. Prove *Wirtinger's inequality*: If $f : [0, \pi] \to \mathbb{R}$ is C^2 and satisfies $f(0) = f(\pi) = 0$, then

$$\int_0^\pi f^2 dt \leq \int_0^\pi (f')^2 dt,$$

with equality if and only if $f(t) = c\sin(t)$ where $c \in \mathbb{R}$. Hint: Use Fourier series or the calculus of variations.

Problem #2. Let M be a complete simply connected Riemannian manifold. For each $p \in M$, denote by $\mathrm{conj}(p)$, the set of first conjugate points of p (i.e., for each $v \in T_pM$ with $v \neq 0$, consider the first conjugate point of the geodesic $c_v(s) = \exp_p(sv)$).

a) Show that if $p' \in \mathrm{conj}(p)$ satisfies $d_g(p, p') = L$, then there is a unit speed geodesic $\gamma : [0, L] \to M$ connecting p to p', so that there is a non-trivial Jacobi field J along γ so $J(0) = J(L) = 0$, and $I(J, J) = 0$.
b) If for each $p \in M$, $\mathrm{conj}(p)$ consists of one point p' with $d_g(p, p') = d_g(p, \mathrm{conj}(p)) = \pi$ and the sectional curvature of M satisfies $0 < \delta \leq K \leq 1$, then M is isometric to $(\mathbb{S}^n, g^S)$. Hint: Consider the geodesic, γ, and Jacobi field, J, from part a), using the Jacobi equation, the index property of J and the previous problem conclude that along the geodesic γ one has $K(\gamma', J) = 1$, where $K(\gamma', J)$ is the sectional curvature of the two plane spanned by γ' and J.

Problem #3. Let (M, g) be a Riemannian manifold. Let $\Omega \subset M$ be an open domain which is strongly convex (i.e. for every two points $p, q \in \Omega$ there is a minimizing geodesic contained in Ω connecting p to q) and so that $\partial\Omega$ is a smooth submanifold.

a) Show that the second fundamental form Π^N of $N = \partial\Omega$ with respect ot the outward pointing normal to Ω is non-negative in that $\Pi^N_p(v, v) \geq 0$ for all $v \in T_p\partial\Omega$ and $p \in \partial\Omega$.
b) The domain Ω in M is *convex* if $\partial\Omega$ is a submanifold curve whose whose second fundamental form with respect to the outward pointing normal to Ω is non-negative. Show by example that a domain may be convex but fail to be strongly convex.

Problem #4. Show that if $M \subset \mathbb{R}^{n+1}$ is a compact hypersurface (i.e., a codimension one submanifold), then there is a point $p \in M$ so that the second fundamental form of M is strictly positive (with respect to some choice of unit normal). Hint: Consider the smallest euclidean ball centered at the origin containing M.

Problem #5. Show that there can be no C^2 ismoetric embedding

$$f : (\mathbb{T}^2, g^T) \to (\mathbb{R}^3, g^E).$$

Hint: Use the previous exercise.