Math 645, Fall 2017: Assignment #7

Due: **Thursday, November 2nd**

Problem #1. Let (M, g) be a Riemannian manifold. Show that if $c \in \Omega_{pq}$ is a critical point for L_g, then there is a geodesic γ so that $\gamma([0,1]) = c([0,1])$. Show by example that the converse need not be true. Hint: Consider the curves $c_\epsilon : [0,1] \to (M \times \mathbb{R}, g \times g^E)$ defined by $c_\epsilon(t) = (c(t), \epsilon t)$ and observe that $c'_\epsilon(t) \neq 0$ whenever $\epsilon \neq 0$ and that if c is critical for length, then so is c_ϵ. What happens when $\epsilon \to 0$?

Problem #2. Let (M, g) be a Riemannian manifold.

a) Show a curve $c : [a,b] \to M$ is a geodesic if and only if it is locally E_g minimizing. I.e., for every $t \in (a,b)$, there is $\delta > 0$ so $c|_{[t-\delta, t+\delta]}$ has least energy among all curves connection $c(t-\delta)$ to $c(t+\delta)$ with domain $[t-\delta, t+\delta]$.
b) Show that if M is compact then there is an $\epsilon > 0$ depending on the metric g so that if $c : \mathbb{S}^1 \to M$ is a closed curve with $E_g(c) < \epsilon$, then c is null-homotopic. Hint: Use that normal coordinates give a collection of contractible charts and that energy controls length.
c) Show by example that the assumption that M be compact is necessary.

Problem #3.

a) Show that when n is odd any element of $A \in SO(n)$ has an eigenvector with eigenvalue 1.
b) Show that if (M, g) is oriented, then the parallel transport preserves the natural orientation induced on each T_pM.
c) Use the above to show that (M, g) is an even dimensional oriented Riemannian manifold and $c : [0,1] \to M$ is a closed geodesic (i.e. c is geodesic, $c(0) = c(1)$ and $c'(0) = c'(1)$), then there is a $u \in T_{c(0)}$ orthogonal to $c'(0)$ so that the parallel transport, P along c satisfies $P(u) = u$.
d) Use the second variation formula to conclude that for every even dimensional oriented Riemannian manifold with positive sectional curvature, any closed geodesic must be unstable in that some variation decreases length (Hint: consider the variation obtained by parallel transporting the vector u.)
e) (Extra Credit): Use the fact that every free-homotopy class of a compact non-simply connected Riemannian manifold contains a geodesic that minimizes length in the class, to prove Synge's theorem: If (M, g) is a closed orientable Riemannian manifold with positive sectional curvature, then M is simply connected.

Problem #4. Let (M, g) be a Riemannian manifold of dimension n. Let $\omega_n = vol_{g_{euc}}(B_1)$ be the volume of the unit (euclidean) ball in $\mathbb{R}^n$. Show that for any point $p \in M$ and $r > 0$ small,

$$\frac{vol_g(\mathcal{B}_r(p))}{\omega_n r^n} = 1 - \frac{S(p)}{6(n+2)} r^2 + o(r^2)$$

where here $S(p)$ is the scalar curvature of g at the point p and $\mathcal{B}_r$ is the normal ball of radius r about p.

Problem #5. Let (M, g) be a Riemannian manifold and X a Killing vector field of (M, g) (recall a vector field is Killing if its flow is an isometry of g you considered them in Problem 4 of Homework 3).

a) Show that the restriction of X to any geodesic gives a Jacobi field.

b) Show that if M is complete, then X is uniquely determined by knowing X_p and $(D_u X)_p$ for some $p \in M$ and all $u \in T_pM$.