Math 645, Fall 2017: Assignment #6

Due: **Thursday, October 26th**

Problem #1. Verify that the expression

$$K_p(x,y) = \frac{R_p(x,y,x,y)}{|x|_g^2|y|_g^2 - g_p(x,y)^2}$$

depends only on the two-plane in T_pM spanned by $x, y \in T_pM$.

Problem #2. Prove the 2nd Bianchi identity:

$$(D_T R)(X,Y,Z,W) + (D_Z R)(X,Y,W,T) + (D_W R)(X,Y,T,Z) = 0.$$

Hint: Compute using the geodesic frame introduced in Problem 5 of Homework 4.

Problem #3. Use the 2nd Bianchi identity to show the following (known as Schur's theorem): If (M,g) is an n-dimensional manifold with $n \geq 3$ and for all $p \in M$, $K_p(\sigma) = K(p)$ for each two-plane $\sigma \subset T_pM$ (i.e., the sectional curvature is independent of the two-plane and depends only on the point), then (M,g) has constant curvature (i.e., the sectional curvature is also independent of the point of M). Hint: The hypotheses means that $R_p = K(p)R_p^1$, where R_p^1 was defined in class. As $D_X R^1 = 0$, this means $D_X R = (X \cdot K)R^1$.

Problem #4. Show that if (M,g) is a three dimensional Riemannian manifold, then the Ricci tensor uniquely determines the Riemann curvature tensor. Hint: This is a purely algebraic fact.

Problem #5. Show that if (M,g) has vanishing Riemann curvature tensor, then the $\exp_p : T_pM \to M$ is a local isometry between (T_pM, g_p) and (M,g).