Math 645, Fall 2017: Assignment #2

Due: **Thursday, September 28th**

Problem #1. Let M be an n-dimensional differentiable manifold. Define the space of frames of M at p, F_pM, to be the set of ordered bases of T_pM – i.e.,

$$F_pM = \{(v_1, \dots, v_n) : v_i \in T_pM \text{ and } \{v_1, \dots, v_n\} \text{ is a basis of } T_pM\}.$$

Let $FM = \bigcup_{p \in M} F_pM$ be the *frame bundle* of M. Define the following equivalence relation on FM, $(p, (v_1, \dots, v_n)) \sim (q, (w_1, \dots, w_n)) \iff p = q$ and $v_i = \sum_{j=1}^n A_{ij} w_j$ where $A_{ij} \in GL(n, \mathbb{R})$ has positive determinant. Let $\bar{M} = FM/\sim$ be the set of equivalence classes.

a) Prove that FM has a natural differentiable manifold structure and that the map $\Pi : FM \to M$ given by $(p, (v_1, \dots, v_n)) \mapsto p$ is smooth.
b) Show that $\bar{M}$ has a natural differentiable manifold structure and that the map $\pi : \bar{M} \to M$ defined by $[p, (v_1, \dots, v_n)] \mapsto p$ is a local diffeomorphism.
c) Show that $\bar{M}$ is an orientable n-dimensional manifold.
d) Show that $\bar{M}$ is connected if and only if M is connected and non-orientable. In this case, $\bar{M}$ is called the *oriented double cover* of M.

Problem #2. For $X \in \mathcal{X}(M)$, show that if $X_p \neq 0$, then there is a chart (U, V, ϕ) around p so that on U, $X = \frac{\partial}{\partial x^1}$. Show by example that if $Y \in \mathcal{X}(M)$ is another vector field with Y_p linearly independent from X_p, then it may not be possible to find a chart (U, V, ϕ) so that on U,

$$X = \frac{\partial}{\partial x^1} \text{ and } Y = \frac{\partial}{\partial x^2}.$$

Can you give a necessary condition on X and Y for the existence of such a chart?

Problem #3. Let M be differentiable manifold and $X, Y \in \mathcal{X}_0(M)$ be two vector fields with compact support. If $\phi_t : M \to M$ is the time t flow of X and ψ_t is the time t flow of Y show that

$$\frac{d}{dt}|_{t=0} \gamma_t(p) = [X, Y](p).$$

where

$$\gamma_t = \psi_{-\sqrt{t}} \circ \phi_{-\sqrt{t}} \circ \psi_{\sqrt{t}} \circ \phi_{\sqrt{t}}.$$

Problem #4. Show that for vector fields $X, Y_1, \dots, Y_k \in \mathcal{X}(M)$ and a $A \in \Gamma(M, T^{(0,k)}M)$,

$$(L_X A)(Y_1, \dots, Y_k) = X \cdot (A(Y_1, \dots, Y_n)) - \sum_{i=1}^k A(Y_1, \dots, Y_{i-1}, [X, Y_i], Y_{i+1}, \dots, Y_k).$$

Problem #5. Show the following:

a) If U and V are open subsets of $\mathbb{R}^n$ and $\phi : U \to V$ a diffeomorphism, then when $q = \phi(p)$,

$$(\phi^* dx^1 \wedge \dots \wedge dx^n)_p = (\det J_p(\phi)) dx^1_p \wedge \dots \wedge dx^n_p.$$

b) If $U \subset \mathbb{R}^n$ is an open set, then

$$\Lambda^n U = \{f dx^1 \wedge \dots \wedge dx^n : f \in C^\infty(U)\}.$$

That is, $\Lambda^n U$ is a the trivial rank one vector bundle.
c) For connected M, $\Lambda^n M$ is a rank one vector bundle that is trivial if and only if M is orientable.