Math 645, Fall 2020: Assignment #3

Due: **Tuesday, September 29th**

Problem #1. Verify that a $\phi \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ that satisfies $\phi(0) = 0$, is an isometry of $(\mathbb{R}^n, \bar{g})$ if and only if ϕ can be identified with an element of $O(n) = \{A \in \mathbb{R}^{n\times n} : A^\top A = I_n\}$.

Problem #2. Show that if a smooth manifold M admits an atlas $\{(U_\alpha, \phi_\alpha)\}$ so that all transition functions $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ are Euclidean isometries, then M admits a flat Riemannian metric g.

Problem #3. Let (M, g) and (N, h) be Riemannian manifolds and f a smooth positive function on M. Define a $(0, 2)$-tensor on $M \times N$ by

$$(g \times_f h)_p(X_p, Y_p) = g_{\pi_M(p)}(T_p\pi_M(X_p), T_p\pi_M(Y_p)) + f^2(\pi_M(p))h_{\pi_N(p)}(T_p\pi_N(X_p), T_p\pi_N(Y_p))$$

where here π_N, π_M are the natural projections.

a) Show that $g \times_f h$ is a Riemannian metric on $M \times N$. it is the *warped product metric* introduced in class.

b) Let $(M, g) = (\mathbb{R}^+, \bar{g})$ and $(N, h) = (\mathbb{S}^n, \mathring{g})$ and denote by $r \in C^\infty(M)$ the standard coordinate on $\mathbb{R}^+$. Consider the following family of warped product metrics

$$c_\lambda = g \times_{\lambda r} h.$$

Show that $(\mathbb{R}^+ \times \mathbb{S}^1, c_\lambda)$ is locally isometric to $(\mathbb{R}^2\backslash\{0\}, \bar{g})$ for all $\lambda > 0$.

Problem #4. Let $(M, g) = ((0, 2\pi), \bar{g})$ and $(N, h) = (\mathbb{S}^n, \mathring{g})$ show that

$$\bar{g} \times_{\sin(r)} \mathring{g}$$

is isometric to $(\mathbb{S}^{n+1} \setminus \{S, N\}, \mathring{g})$. Here $S, N \in \mathbb{S}^{n+1}$ are antipodal points.

Problem #5. Show that every one dimensional Riemannian manifold (M, g) is flat.