Practice Midterm 2

1. (10 pts each) True or false; justify as much as you can.
a. If $f(x)$, $g(x)$ are continuous functions on [0,1] which agree at every rational, then $f = g$ on [0,1].
True. Given any $\varepsilon > 0$ and $y \in (0,1)$ choose $\delta > 0$ so that $|(f(x) - g(x)) - (f(y) - g(y)| < \varepsilon$ if $|x - y| \leq \delta$. Now take x to be a rational in $(y - \delta, y + \delta) \cap [0,1]$. Then $|f(y) - g(y)| < \varepsilon$.

b. If $|f(x)|$ is continuous at x_0 then $f(x)$ is continuous at x_0.
False. Take $f(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$
c. If f is a strictly monotone function on [0,1] with range an interval, then f is one to one.
True. In fact (see my notes on monotone functions) f is continuous and one to one.

d. Let f be continuous on $\mathbb{R}$. Then the inverse image of an open interval is an open interval.
False. The inverse image is open but not necessarily an interval. Take for example $f(x) = \sin x$.

e. If $f(x)$ is uniformly continuous on $\mathbb{R}$ and $\{x_n\}$ is a Cauchy sequence, then so is $\{f(x_n)\}$.
True. $|f(x_j) - f(x_k)| < \varepsilon$ if $|x_j - x_k| < \delta(\varepsilon)$. So if $j, k > N(\delta) = N(\varepsilon)$ then $|x_j - x_k| < \delta$, i.e the sequence $\{f(x_n)\}$ is Cauchy.

f. There exists a continuous bijection map $f : [0,1) \to \mathbb{R}$.
False. The image of $f([0, \frac{1}{2}]$ is compact, say contained in $[-N, N)$. Hence the inverse image of the points $-(N+1)$ and $N+1$ lie in $(\frac{1}{2}, 1)$. By the intermediate value theorem, the inverse image of the interval $(-(N+1), N+1)$ also lies in $(\frac{1}{2}, 1)$ so f cannot be one to one.

2. Let $f : [0,1] \to [0,1]$ be continuous. Show that the equation $f(x) = x$ has at least one solution in [0,1].

Let $h(x) = f(x) - x$. Then $h(0) = f(0) \geq 0$ and $h(1) = f(1) - 1 \leq 0$. By the intermediate value theorem there is an x such that $h(x) = 0$.

3. Let $f(x)$ be a C^1 function on $\mathbb{R}^+$ and satisfy $f'(x) > f(x)$, $f(0) = 0$. Show that $f(x) > 0$ for $x > 0$.
Let $h(x) = e^{-x} f(x)$. Then $h'(x) = e^{-x}(f'(x) - f(x)) > 0$ for $x > 0$ and $h(0) = 0$. Hence

$h(x) > 0$ for $x > 0$.

4. Let $f(x)$ be strictly increasing and continuous on $[0, \infty)$ with f(0)=0. Show that

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq ab \ .$$

When does equality hold? Hint: Draw a picture and interpret geometrically.
The first integral is the area under the graph of $y = f(x)$ from 0 to a and the second integral is the area bounded by the y axis and the graph from $y = 0$ to $y = b$ ("area under the graph of f^{-1} from 0 to b"). If $a \neq f^{-1}(b)$, then the left hand side is strictly greater than the right hand side interpreted as the area of the rectangle with base a and height b. If $a = f^{-1}(b)$, we have equality.

5. Let $f(x)$ be C^3 on an interval I . Suppose $a_0 < a_1 < a_2$ are points of I and $f(a_0) = f(a_1) = f(a_2) = f'(a_2) = 0$. Show there is a point $c \in I$ where $f'''(c) = 0$.
By the mean value theorem, there are points $b_1 \in (a_0, a_1)$, $b_2 \in (a_1, a_2)$ such that $f'(b_1) = f'(b_2) = 0$. Applying the mean value theorem again but this time to $f'(x)$, there are points $c_1 \in (b_1, b_2)$, $c_2 \in (b_2, a_2)$ such that $f''(c_1) = f''(c_2) = 0$. By the mean value theorem applied to $f''(x)$ we arat
6. Let $f(x)$ be continuous on $[0, \infty)$ and assume that $L = \lim_{x \to +\infty} f(x)$ exists and is finite. Show that f is bounded. (Recall $L = \lim_{x \to +\infty} f(x)$ means that give $\varepsilon > 0$, $\exists N = N(\varepsilon)$ such that $x > N$ implies $|f(x) - L| < \varepsilon$.)
By the definition of L, there exists N such that $x > N$ implies $|f(x) - L| < 1$. In particular $|f(x)| \leq L + 1$ on (N, ∞). Since $f(x)$ is continuous on $[0, N + 1]$, $|f(x)| \leq M$ on $[0, N + 1]$ for some M. Hence $f(x)$ is bounded by M+L+1 on $[0, \infty)$.

7. Let $f(x)$ be Riemann integrable on [0,1] and assume that $f(x) = 0$ when x is rational. Show that $\int_0^1 f(x)dx = 0$. Note that $f(x)$ is assumed bounded but nothing is assumed about the values of $f(x)$ when x is irrational.
Since f is assume Riemann integrable, given $\varepsilon > 0$, there is a partition P such that $S^+(f, P) - S^-(f, P)| < \varepsilon$. However any Cauchy sum $S(f, P) = \sum f(a_k)(x_{k+1} - x_k)$ satisfies $S^-(f, P) \leq S(f, P) \leq S^+(f, P)$. Therefore $| \int_0^1 f(x)dx - S(f, P)| < \varepsilon$. Now choose each a_k to be rational so $S(f, P) = 0$. Then $| \int_0^1 f(x)dx| < \varepsilon$ so $\int_0^1 f(x)dx = 0$.

8. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n}, \, n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$
Show that $f(x)$ is Riemann integrable.

Let $\varepsilon > 0$ be given and choose an integerr N so that $\frac{1}{N} < \varepsilon$. Let I_n be a closed interval of length $2^{-(n+1)}\varepsilon$ centered at the point $\frac{1}{n}$, $n = 1, 2, \ldots, N$. Let P be the partition of $[0, 1]$ consisting of the endpoints of the $I_n \cap [0, 1]$ and the points $0, \frac{\varepsilon}{2}, 1$. Then since $0 \leq f \leq 1$ and $f = 0$ on the open intervals $(\frac{1}{n}, \frac{1}{n+1})$, we have $S^+(f, P) - S^-(f, P)| < \varepsilon$. Hence f is Riemann integrable and as in problem 7 (here we choose a_k to be irrational in the Cauchy sum) $\int_0^1 f(x)dx = 0$.