Midterm 2 Math 405 November 18, 2013

Show all work in a clear, concise and legible style.

Each problem is worth 25 points.

1. Let $f(x)$ be a bounded monotone increasing continuous function on $[a, b)$. Show that f extends to a continuous on [a,b] in the following steps:
a. Let $\{x_n\}$ be a sequence converging to b. Show that $L = \lim_{n\to\infty} f(x_n)$ exists.

The sequence $f(x_n)$ is monotone increasing and bounded so converges to a finite limit L.

b. Now suppose $\{y_n\}$ is another sequence converging to b with $M = \lim_{n\to\infty} f(y_n)$. Show that $M \leq L$. By symmetry $L \leq M$ and hence $L = M$.

Given $\varepsilon > 0$ choose $N = N(\varepsilon)$ so large that $M - \varepsilon \leq f(y_n) \leq M$ for $n > N$. Fix $n > N$; then $M - \varepsilon \leq f(y_n) \leq f(x_k) \leq L$ for k sufficiently large since the sequence $\{x_n\}$ converges to b. Hence $M \leq L + \varepsilon$ and so $M \leq L$. By symmetry M=L and so $\lim_{x\to b^-} f(x) = L$ so f is continuous on [a,b].

2. Determine the constants k_1, k_2 so that the function

$$h(x) = \begin{cases} k_1 x - 5 & \text{if } x < 2 \\ 3 - k_2 x^2 & \text{if } x \geq 2 \end{cases}$$

is differentiable at $x = 2$. *Be sure to fully justify.*

We want to choose k_1, k_2 so that $2k_1 - 5 = 3 - 4k_2$ and $k_1 = -4k_2$. Solving gives $k_2 = -2$, $k_1 = 8$ which makes $h(2) = 11$. This makes $h(x)$ continuous at $x = 2$ and we can write the difference quotient for $x \neq 2$

$$\frac{h(x) - h(2)}{x - 2} = \frac{h(x) - 11}{x - 2} = \begin{cases} 8 & \text{if } x < 2 \\ 2(x + 2) & \text{if } x \geq 2 \end{cases}$$

Taking the limit as $x \to 2$ we that that $h(x)$ is differentiable at $x = 2$ with $h'(2) = 8$.

3. Let f be a twice continuously differentiable (i.e C^2) function on $\mathbb{R}$.
a. State Taylor's theorem about the approximation of $f(x)$ near a point x_0 by a second order polynomial. Use Taylor's theorem to show that if $f'' < 0$, the graph of $f(x)$ lies on one side (below) its tangent line (the graph of its best linear approximation $l(x)$) in a small

neighborhood of any x_0.

Taylors theorem: Let f be a C^2 function in a neighborhood of x_0. Then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + o(|x - x_0|^2) \quad \text{as } x \text{ tends to } x_0 .$$

In particular if $f''(x_0) < 0$ then if $|x - x_0|$ is small enough so that $\frac{1}{2}f''(x_0)(x - x_0)^2 + o(|x - x_0|^2) < 0$, i.e $\frac{1}{2}f''(x_0) + o(1) < 0$ then $f(x) \leq f(x_0) + f'(x_0)(x - x_0)$ in a small neighborhood of x_0 with equality only at $x = x_0$.

b. Still assuming that $f'' < 0$, show that the graph of $f(x)$ globally lies under its tangent line.
Suppose for contradiction that the graph $y = f(x)$ touches the tangent line $y = f(x_0) + f'(x_0)(x - x_0)$ at some point $x_1 \neq x_0$. Let $g(x) = f(x) - (f(x_0) + f'(x_0)(x - x_0))$. Then $g(x_0) = g(x_1) = 0$ and $g(x) < 0$ on the interval between x_0 and x_1 (we may assume that x_1 is the "first such point"). Then $g(x)$ has a minimum at at point c in the interval and $g'(c) = 0$, $g''(c) \geq 0$. Hence $f''(c) \geq 0$ a contradiction and also we see that $f(x)$ lies above its tangent line which is parallel to $y = f(x_0) + f'(x_0)(x - x_0)$.

4. Let $f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 4 & \text{if } x = 1 \\ 3 - x & \text{if } 1 < x \leq 2 \end{cases}$

State the Cauchy criterion for Riemann integrability and use it to show that f is Riemann integrable on [0,2]. You may use the theorem that a continuous function on a closed interval $[a, b]$ is Riemann integrable.

The Cauchy criterion states that a function f is Riemann integrable on $[a, b]$ if and only if give $\varepsilon > 0$ there is a partition P of $[a, b]$ such that $S^+(f, P) - S^-(f, P) < \varepsilon$.

Given $\varepsilon > 0$ consider the interval $I = (1 - \frac{\varepsilon}{24}, 1 + \frac{\varepsilon}{24})$. Then on I, $S^+(h, I) - S^-(h, I) < \frac{\varepsilon}{3}$ since the oscillation of h is less than 4 and the length of I is $\frac{\varepsilon}{12}$.
The functions x and $3 - x$ are continuous and so Riemann integrable so there is a partition P_1 of $[0, 1 - \frac{\varepsilon}{24}]$ and a partition P_2 of $[1 + \frac{\varepsilon}{24}, 2]$ so that $S^+(h, P_j) - S^-(h, P_j) < \frac{\varepsilon}{3}$, $j = 1, 2$. Now take P to be the partition 0f $[0, 2]$ formed by the endpoints of P_1 and P_2 combined. Then $S^+(h, P) - S^-(h, P) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.