1. (a) Since f is differentiable at x_0, $M = \max\{|f'(x)| \mid x \in [x_0 - 1, x_0 + 1]\}$ is well-defined, and moreover there is $1 > \delta > 0$ such that $|x - x_0| < \delta$ implies

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq 2M \implies |f(x) - f(x_0)| < 2M\,|x - x_0|$$

Hence for $|x - x_0| < \delta$ we have by triangle inequality

$$|g(x) - f(x)| \leq |g(x) - f(x_0)| + |f(x) - f(x_0)| < |m|\,|x - x_0| + 2M\,|x - x_0|$$

so the claim holds with $C = |m| + 2M$.

(b) Suppose first that $m = f'(x_0)$, then

$$\frac{|f(x) - g(x)|}{|x - x_0|} = \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right|$$

Since f is differentiable at x_0, we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

so given any $\varepsilon > 0$ there is $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \varepsilon \implies |f(x) - g(x)| < \varepsilon\,|x - x_0|$$

Conversely suppose that given $\varepsilon > 0$ there is $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|f(x) - g(x)| < \varepsilon\,|x - x_0| \implies \left| \frac{f(x) - f(x_0)}{x - x_0} - m \right| < \varepsilon$$

This means that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = m$$

and uniqueness of limit shows that $m = f'(x_0)$.

2. We show that such an f must not be differentiable at 0. Suppose for a contradiction that there is an f differentiable everywhere on $(-2, 2)$. Consider the difference quotient

$$\frac{f(h) - f(0)}{h}$$

When $h > 0$, by the mean value theorem we have

$$\frac{f(h) - f(0)}{h} = f'(c) = -1$$

for some $c \in (0, h)$. On the other hand when $h < 0$ we have

$$\frac{f(h) - f(0)}{h} = f'(c) = 1$$

for some $c \in (h, 0)$, so the limit as $h \to 0$ cannot exist and this is a contradiction.

3. We show that $f'(x) = 0$ everywhere so that mean value theorem implies that f is constant. Indeed let $x \in \mathbb{R}$ and by assumption

$$|f'(x)| = \lim_{h \to 0} \left| \frac{f(x + h) - f(x)}{h} \right| \leq \lim_{h \to 0} \left| \frac{C\,|h|^{1+\varepsilon}}{h} \right| = \lim_{h \to 0} C\,|h|^{\varepsilon} = 0$$

so $f'(x) = 0$.

4. (a) By mean value theorem we have that for $x \in (0,1)$ there is some $c \in (0,x)$ such that

$$\frac{f(x)-f(0)}{x} = f'(c) \leq c < x$$

Since $f(0)=0$ this gives $f(x) \leq x^2$.

(b) Consider $g(x) = f(x) - \frac{x^2}{2}$. Then $g'(x) = f'(x) - x \leq 0$, so g is a decreasing function. Since $g(0) = 0$ we have $g(x) \leq 0$ for all $x \in [0,1]$, and it follows that $f(x) \leq \frac{x^2}{2}$.

5. (a) Suppose for a contradiction that there is $c' \in (a,b)$ such that $f(c') < f(c)$. By convexity we have for all $t \in [0,1]$ that

$$f(tc+(1-t)c') \leq tf(c) + (1-t)f(c') < f(c)$$

Since c is a relative minimum there is $\delta > 0$ such that $|d-c| < \delta$ implies $f(c) \leq f(d)$. Choose t close enough to 1 such that

$$|tc+(1-t)c'-c| = |1-t|\,|c-c'| < \delta$$

then it follows that

$$f(tc+(1-t)c') < f(c) \leq f(tc+(1-t)c')$$

which is a contradiction. So c is indeed an absolute minimum for f.

(b) Let $d \in [a,c)$ and $e \in (c,b]$. Let $t \in (0,1)$ be chosen so that $c = td + (1-t)e$. Then we have, since c is an absolute maximum,

$$f(c) \leq tf(d) + (1-t)f(e) \leq f(c)$$

The inequality can only hold if $f(d) = f(e) = f(c)$. Since d and e are arbitrary it follows that $f = f(c)$ is constant.

6. As the hint suggests for fixed $x, y \in (a,b)$ with $x < y$ we consider $g(t) = f(tx+(1-t)y) - tf(x) - (1-t)f(y)$. We compute

$$g'(t) = (x-y)f'(tx+(1-t)y) - f'(x) + f'(y)$$

and

$$g''(t) = (x-y)^2 f''(tx+(1-t)y) \geq 0$$

since $f''(x) \geq 0$, i.e. $g'(t)$ is an increasing function of t. Now we note that $g(0) = g(1) = 0$, and suppose for a contradiction that there is $t \in (0,1)$ such that $g(t) > 0$. By mean value theorem we have that there is $t_1 \in (0,t)$ such that

$$g'(t_1) = \frac{g(t)-g(0)}{t} > 0$$

and $t_2 \in (t,1)$ such that

$$g'(t_2) = \frac{g(1)-g(t)}{1-t} < 0$$

but this contradicts the fact that $g'(t)$ is increasing. Hence we must have $g(t) \leq 0$ and from there it follows that

$$g(t) \leq 0 \iff f(tx+(1-t)y) \leq tf(x) + (1-t)f(y)$$

Since x, y are arbitrary this finishes the proof.

7. We have

$$f(x) = \begin{cases} x^3 & x > 0 \\ -x^3 & x < 0 \end{cases}$$

So away from zero we have

$$f'(x) = \begin{cases} 3x^2 & x > 0 \\ -3x^2 & x < 0 \end{cases}$$

One checks that f' is continuous at 0, so f is differentiable with $f'(x) = 3\,\mathrm{sgn}(x)x^2$. Again one computes

$$f''(x) = \begin{cases} 6x & x > 0 \\ -6x & x < 0 \end{cases}$$

and checks that f'' is continuous at 0, so $f'' = 6\,|x|$. It is now standard to check that the absolute value function is not differentiable at 0.

8. It is helpful to note that f is an increasing function on $[0, 1]$ so that $m_i|_{[a_i,a_{i+1}]} = f(a_i)$ and $M_i|_{[a_i,a_{i+1}]} = f(a_{i+1})$. With these we have

$$L(P, f) = 0.1 \times f(0) + 0.3 \times f(0.1) + 0.6 \times f(0.4) = 0.0003 + 0.0384 = 0.0387$$
$$U(P, f) = 0.1 \times f(0.1) + 0.3 \times f(0.4) + 0.6 \times f(1) = 0.0001 + 0.0192 + 0.6 = 0.6293$$