1. We will show that the max function $M(x)$ is continuous. The case for min $m(x)$ is similar. Let $x \in S$. Suppose first that $f(x) > g(x)$. Then by Problem 7 in PS5, there is $\delta > 0$ such that $|y-x| < \delta$ implies $f(y) - g(y) > 0$. Consequently if $|y-x| < \delta$ we have $M(y) = f(y)$, and it follows that M is continuous at x since f is continuous. Similarly we can deal with the case when $g(x) > f(x)$.

   Thus it remains to consider the case when $f(x) = g(x) = L$. Let $\varepsilon > 0$. Since f and g are both continuous, there are δ_1, δ_2 respectively such that $|y-x| < \delta_1 \implies |f(y) - L| < \varepsilon$ and $|y-x| < \delta_2 \implies |g(y) - L| < \varepsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. It follows that if $|y-x| < \delta$ we have

   $$|M(y) - M(x)| = |M(y) - L| = \begin{cases} |g(y) - L| & g(y) > f(y) \\ |f(y) - L| & g(y) \le f(y) \end{cases} < \varepsilon$$

   So M is continuous at x.

2. Consider $g(x) = f(x) - x$. Then $g(0) = f(0) \ge 0$. If $g(0) = 0$ then $f(0) = 0$ and we are done since we can take $c = 0$. So we may assume $g(0) > 0$. Moreover $g(1) = f(1) - 1 \le 0$. If $g(1) = 0$ then we are also done since $f(1) = 1$, so we may assume $g(1) < 0$. But then intermediate value theorem implies there is $c \in (0,1)$ such that $g(c) = 0 \implies f(c) = c$.

3. (a) Take for example $f(x) = \frac{1}{2}$ for $x \in [0,1]$.

   (b) Suppose for a contradiction that f is onto. Let $x_n \in [0,1]$ be such that $f(x_n) = \frac{1}{n}$. By Bolzano-Weierstrass we may extract a subsequence $\{x_{n_k}\}_{k=1}^\infty$ such that $x_{n_k} \to x \in [0,1]$. Since f is continuous it follows that

   $$0 = \lim_{k\to\infty} \frac{1}{n_k} = \lim_{k\to\infty} f(x_{n_k}) = f(x)$$

   which is a contradiction since 0 is not in the range of f.

4. (a) We show that f is discontinuous at 0. Let $\varepsilon = \frac{1}{2}$. For any $\delta > 0$ choose $n \in \mathbb{N}$ large enough so that

   $$x = \frac{1}{2n\pi + \frac{\pi}{2}} < \delta$$

   and it follows that

   $$f(x) = \sin(2n\pi + \frac{\pi}{2}) = 1 \implies |f(x) - f(0)| = 1 > \frac{1}{2}$$

   so no such δ in the definition of continuity exists. This means f is discontinuous at 0.

   (b) f is continuous away from 0 as a composition of continuous function, so f automatically has the intermediate value property if $0 < a < b$ or $a < b < 0$. It remains to verify the property when $a < 0 < b$. Let y be such that $f(a) < y < f(b)$. Let $\theta \in [-\pi/2, \pi/2]$ be the unique angle such that $\sin(\theta) = f(a)$. Choose $n \in \mathbb{N}$ large enough such that

   $$0 < a' = \frac{1}{2n\pi + \theta} < b$$

   Thus we have found a point $a' \in \mathbb{R}$ with $0 < a' < b$ and $f(a') = f(a)$. Intermediate value theorem now implies that there is $c \in (a', b) \subset (a, b)$ such that $f(c) = y$.

5. (a) When n is odd, $p_A(t)$ is an odd degree polynomial. This means that we can find $t_1 < 0$ such that $p_A(t_1) < 0$ (since the leading term is odd-degree), and $t_2 > 0$ such that $p_A(t_2) > 0$. Intermediate value theorem implies there is some $t_0 \in (t_1, t_2)$ such that $p_A(t_0) = 0$.

   (b) When n is even we can find $t_1 < 0$ such that $p_A(t_1) > 0$ and $t_2 > 0$ such that $p_A(t_2) > 0$. Since $p_A(0) < 0$, intermediate value theorem implies that we can find $t_0 \in (t_1, 0)$ and $t_0' \in (0, t_2)$ such that $p_A(t_0) = p_A(t_0') = 0$.

6. (a) Consider $f(x) = \frac{1}{x}$ and $x_n = \frac{1}{n}$. Clearly $\{x_n\}_{n=1}^\infty$ is Cauchy, but $\{f(x_n)\}_{n=1}^\infty = \{n\}_{n=1}^\infty$ is clearly not Cauchy.

(b) By Theorem 2.4.5 in the notes, a sequence of real number is Cauchy if and only if it converges. Hence we may assume $x_n \to x \in \mathbb{R}$ as $n \to \infty$. Since g is continuous we have $g(x_n) \to g(x)$, so $\{g(x_n)\}_{n=1}^{\infty}$ is also Cauchy.

7. For $h > 0$ we consider the difference quotient

$$\left|\frac{f(h)-f(0)}{h-0}\right| = \frac{f(h)}{h} = \begin{cases} h & h \in \mathbb{Q} \\ 0 & h \notin \mathbb{Q} \end{cases}$$

In both cases as $h \to 0$ we have

$$\lim_{h\to 0} \frac{f(h)-f(0)}{h-0} = 0$$

so f is differentiable at $x = 0$. To show that f is discontinuous everywhere else, we first consider $x \in \mathbb{Q}$. Then $f(x) = x^2 \neq 0$. Let $\varepsilon = \frac{x^2}{2}$. For any $\delta > 0$ choose $y \in (x-\delta, x+\delta) \setminus \mathbb{Q}$ (such y exists since $\mathbb{Q}$ is countable). It follows that

$$|f(x) - f(y)| = x^2 > \frac{x^2}{2}$$

so f is not continuous at x. If $x \in \mathbb{R} \setminus \mathbb{Q}$. Let $\varepsilon = \frac{x^2}{8}$, and given $\delta > 0$ choose $y \in (x - \min\{\delta, |x|/2\}, x + \min\{\delta, |x|/2\}) \cap \mathbb{Q}$ (such y exists since $\mathbb{Q}$ is dense in $\mathbb{R}$). It follows that

$$|f(x) - f(y)| = y^2 > \frac{x^2}{4} > \frac{x^2}{8}$$

so f is not continuous at x either.

8. (a) Clearly f is differentiable away from 0 since the function is a product of differentiable function. At $x = 0$, for $h > 0$ we consider

$$\left|\frac{f(h)-f(0)}{h-0}\right| = \left|\frac{h^2\sin(1/h)}{h}\right| = |h\sin(1/h)|$$

Note that

$$\lim_{h\to 0} |h\sin(1/h)| = \lim_{x\to\infty} \frac{\sin(x)}{x} = 0$$

by squeeze theorem. So we have

$$\lim_{h\to 0} \frac{f(h)-f(0)}{h-0} = 0$$

meaning f is differentiable at 0 with $f'(0) = 0$.

(b) By the chain rule we compute directly that away from 0

$$f'(x) = 2x\sin(1/x) - \cos(1/x)$$

If $f'(x)$ were continuous at $x = 0$ we would have

$$\lim_{x\to 0} 2x\sin(1/x) - \cos(1/x) = f'(0) = 0$$

but by Problem 4 we see that the limit $\cos(1/x)$ as $x \to 0$ does not exist, so $f'(x)$ is not continuous at 0.