1. All numbers in this problem refer to the notes. We will first show that $0 \cdot x = 0$ for any x in an ordered field. To see this, by (D) and (A4) of Definition 1.1.5 we have

$$0 \cdot x + 0 \cdot x = (0+0)x = 0 \cdot x$$

Using (ii) of Definition 1.1.1 we have

$$0 \cdot x + 0 \cdot x + 0 \cdot (-x) = 0 \cdot x + 0 \cdot (-x)$$

Using (D), (A4), (A5) we have

$$0 \cdot x + 0 \cdot (x + (-x)) = 0 \cdot (x + (-x)) \iff 0 \cdot x = 0$$

Now back to the original problem. Since $x < 0$ we have $(-x) + x < 0 + (-x)$ by (ii), so $0 < -x$ by (A4) and (A5). Since $y < z$ we have $y + (-y) < z + (-y)$ by (ii) and so $0 < z + (-y)$ by (A5). By (ii) of Definition 1.1.7 we then have $(-x)(z + (-y)) > 0$. Using (D) and (ii) of Definition 1.1.1 we have

$$(-x)z + (-x)(-y) > 0 \implies (-x)z + (-x)(-y) + x(-y) > (-x)(-y) + x(-y)$$

Using (D), (A5) and $0 \cdot x = 0$ we have

$$(-x)z + ((-x) + x)(-y) > x(-y) \implies (-x)z + 0 \cdot (-y) > x(-y) \implies (-x)z > x(-y)$$

Two applications of (ii) of Definition 1.1.1 gives

$$(-x)z + xz + xy > (-x)y + xz + xy$$

Using (A2) and (D) we have

$$((-x) + x)z + xy > ((-x) + x)y + xz$$

Finally using (A5) and $0 \cdot x = 0$ we get

$$0 \cdot z + xy > 0 \cdot y + xy \implies xy > xz$$

which is what we want to prove.

2. By definition of the supremum, we can find $a_1 \in A$ such that $s > a_1 > s - 1$. Since $a_1 \neq s$, there is $a_2 \in A$ such that $s > a_2 > a_1$. More generally, having defined $a_1, \ldots, a_n$, we let $a_{n+1} \in A$ be such that $s > a_{n+1} > a_n$. This gives a surjective map from $\mathbb{N}$ to $\{a_1, a_2, \ldots\} \subset A$, so A contains a countably infinite subset.

3. If E is empty then everything is vacously true, so suppose $E \neq \emptyset$. Every set $E \subset S$ is bounded below by 1 and above by ∞. If $\infty \notin E$, then E has an infimum is well-defined by well-ordering principle. E has a supremum if E is finite, and if E is infinite the supremum is ∞. If $\infty \in E$, then the infimum is again guaranteed by well-ordering principle, and the supremum is just ∞.

4. Since $x^2 \geq 0$ we have $x^2 \leq x^2 + y^2 = 0 \implies x^2 = 0 \implies x = 0$. Similarly $y = 0$.

5. When $n = 1$ we have $(1+x)^n = 1 + x$ so the inequality holds (and is actually a equality). Suppose the claim is true for n, then

$$(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x) = 1 + nx + x + nx^2 = 1 + (n+1)x + nx^2 > 1 + (n+1)x$$

so the claim holds true for $n+1$ as well.

6. In class we showed that there is a unique positive number $r \in \mathbb{R}$ such that $r^2 = 5$ and $r = \sup\{x \in \mathbb{R} \mid x^2 < 5\}$. Note that $\mathbb{Q}$ has the property that given two rational numbers $p < q$ there is a rational number s with $p < s < q$, so the same proof also shows that $r' = \{\sup x \in \mathbb{Q} \mid x^2 < 5\}$ satisfies $(r')^2 = 5$ ($\mathbb{Q}$ does not have the least upper bound property, so we must consider the set $\{x \in \mathbb{Q} \mid x^2 < 5\}$ as a subset of $\mathbb{R}$ in order to obtain the number r', which is not in $\mathbb{Q}$). Since r is the unique positive real number satisfying $r^2 = 5$, we conclude $r = r'$, which means the suprema are equal.

7. If $x \geq y$, then $\max\{x, y\} = x$ and

$$\frac{1}{2}(x + y + |x - y|) = \frac{1}{2}(x + y + x - y) = \frac{1}{2}(2x) = x$$

Similarly if $y \geq x$ then $\max\{x, y\} = y$ and

$$\frac{1}{2}(x + y + |x - y|) = \frac{1}{2}(x + y + y - x) = \frac{1}{2}(2y) = y$$

The second statement is proven similarly.

8. Given $\varepsilon > 0$, by definition there is $x_1 \in D$ such that $f(x_0) + g(x_0) > \sup_{x \in D}(f(x) + g(x)) - \varepsilon$. Then

$$\sup_{x \in D}(f(x) + g(x)) < f(x_0) + g(x_0) + \varepsilon \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x_0) + \varepsilon$$

Since this is true for any ε we see that

$$\sup_{x \in D}(f(x) + g(x)) \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x_0)$$

Let $D = \{1, -1\}$, $f(1) = 1$, $f(-1) = -1$ and $g(1) = -1$, $g(-1) = 1$, then $f + g$ is identically 0, so

$$0 = \sup_{x \in D}(f(x) + g(x)) \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x_0) = 1 + 1 = 2$$