1. When $n = 1$ we have $n^3 + 5n = 6$ which is obviously divisible by 6. Now suppose the assertion is true for n, then

$$(n+1)^3 + 5(n+1) = n^3 + 3n^2 + 3n + 1 + 5n + 5 = n^3 + 5n + 3n(n+1) + 6$$

By inductive hypothesis, it remains to prove that $3n(n+1)$ is divisible by 6. But this is easy as there must be an even number between two consecutive natural numbers, so 2 divides $n(n+1)$, which implies 6 divides $3n(n+1)$.

2. Suppose for a contradiction that there is $p, q \in \mathbb{N}$ such that $x = p/q$ in lowest terms (that is, the greatest common divisor of p and q is 1). Then

$$\frac{p^2}{q^2} = 6 \implies p^2 = 6q^2$$

This means p has to be even. Writing $p = 2p_0$ and inserting it in the above equation yields

$$4p_0^2 = 6q^2 \implies 2p_0^2 = 3q^2$$

This means q has to be even as well, contradicting the fact that x is expressed in lowest terms.

3. As the hint suggests, let F_n be the set of factors of n which is greater than 1. Note F_n is not empty since $n \in F_n$. By well-ordering principle there is a least element, say p, in F_n. We claim that p is a prime. Indeed, suppose that p is not a prime, then we can write $p = p_0 q_0$ where $p_0, q_0 > 1$ are factors of p. It follows that $p_0 < p$ and $p_0 \in F_n$, a contradiction to our choice of p.

4. Consider the map $F : 2^A \to P(A)$ constructed in the following way: given $f \in 2^A$, F sends f to the set

$$\{a \in A \mid f(a) = 2\}$$

That is, the subset of A whose element are those mapped to 1 by f. F is an injection: given two distict maps f and g there is $a_0 \in A$ such that $f(a_0) \neq g(a_0)$. Say $f(a_0) = 2$ and $g(a_0) = 1$, then $a_0 \in F(f)$ but $a_0 \neq F(g)$, so $F(f) \neq F(g)$. F is also a surjection: given any $A_0 \subset A$ we let $f_0 \in 2^A$ be defined as

$$f_0(a) = \begin{cases} 2 & a \in A_0 \\ 1 & a \notin A_0 \end{cases}$$

One easily checks that $F(f_0) = A_0$. So F is the desired bijection.

5. Let A be a subset of $\mathbb{N}$ that is not finite. By the well-ordering principle there is a least element of A, say a_1. Since A is not finite, so is $A \setminus \{a_1\}$ and there is another least element of $A \setminus \{a_1\}$, say a_2. Continuing this way, we can define a map $f : \mathbb{N} \to A$ inductively by assigning $f(1) = a_1$, and, having defined $f(1), \ldots, f(n)$, $f(n+1) = a_{n+1}$, the least element of the (infinite) set $A \setminus \{f(1), \ldots, f(n)\}$, for $n \geq 1$. One easily checks that f is a bijection and so A is countable.

   In general given any countable set B, let $g : B \to \mathbb{N}$ be a bijection. A subset of $B_0 \subset B$ is in bijection with $g(B_0)$, which is finite or countable by the above paragraph.

6. Let $f_n : \mathbb{N} \to A_n$ be the corresponding bijections. Let

$$G = \{2^n 3^m \mid n, m \in \mathbb{N}\}$$

G is evidently countable as a subset of $\mathbb{N}$ by Problem 5. Now define $f : G \to A$ by

$$f(2^n 3^m) = f_n(m)$$

One checks easily that f is indeed a surjection, so A is at most countable. Finally note that A_1 is countable and $A_1 \subset A$, so A is countable (that is, A cannot be finite).

7. Let $f_B : \mathbb{N} \to B$ be a bijection, then

$$A \times B = \bigcup_{n \in \mathbb{N}} A \times \{f_B(n)\}$$

which is countable by Problem 6 (applied to $A_n = A \times \{f_B(n)\}$ which is countable).

To see $\mathbb{Q}$ is countable, let $f : \mathbb{Z} \times \mathbb{N} \setminus \{0\} \to \mathbb{Q}$ be given by $f(p, q) = p/q$. f is a surjection so

$$|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{N} \setminus \{0\}|$$

the right hand side of which is countable by part a).

8. We claim that $P_{finite}(\mathbb{N})$ is countable. Denote by $P_n(\mathbb{N})$ the subsets of $\mathbb{N}$ of exactly n elements, then

$$P_{finite}(\mathbb{N}) = \bigcup_{n \in \mathbb{N}} P_n(\mathbb{N})$$

By Problem 6, it remains to show that each $P_n(\mathbb{N})$ is countable. To see this, denote by $P_n(\mathbb{N}_m)$ the subsets of $\mathbb{N}_m = \{1, \ldots, m\}$ of exactly n elements (so when $n > m$, $P_n(\mathbb{N}_m) = \emptyset$). Then we can write

$$P_n(\mathbb{N}) = \bigcup_{m \in \mathbb{N}} P_n(\mathbb{N}_m)$$

which is countable by Problem 6, since this time every set on the right hand side is finite.