Mathematic 405, Fall 2019: Assignment #5

Due: **Wednesday, October 16th**

Instructions: Please ensure that your answers are legible. Also make sure that sufficient steps are shown. Page numbers refer to the course text.

Problem #1. Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series. Prove that

$$\left|\sum_{n=1}^{\infty} x_n\right| \leq \sum_{n=1}^{\infty} |x_n|.$$

Problem #2. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence with partial sums $s_k = \sum_{i=1}^{k} x_i$. The sequence is said to be *Cesàro summable* if $C = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} s_k$ exists. The value C is called the *Cesàro sum* of the sequence.

a) Show that if $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence with $\lim_{n\to\infty} a_n = A$, then the sequence $\{b_n\}_{n=1}^{\infty}$ defined by $b_n = \frac{1}{n}\sum_{k=1}^{n} a_k$ is also convergent and $\lim_{n\to\infty} b_n = A$.
b) Show that if $S = \sum_{n=1}^{\infty} x_n$ converges, then the sequence is Cesàro summable and $S = C$. Hint: use a).
c) Show that $x_n = (-1)^n$ is Cesàro summable, even though $\sum_{n=1}^{\infty} (-1)^n$ is not convergent.

Problem #3. Let $A \subset S \subset \mathbb{R}$. Show that if c is a cluster point of A, then c is a cluster point of S.

Problem #4. Let c_1 be a cluster point of A and $c_2 \in B$ be a cluster point of B. Suppose $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are functions with $\lim_{x\to c_1} f(x) = c_2$ and $\lim_{x\to c_2} g(x) = L = g(c_2)$. Show $\lim_{x\to c_1} g(f(x)) = L$.

Problem #5. Suppose $S \subset \mathbb{R}$ and c is a cluster point of S. Suppose $f : S \to \mathbb{R}$ is bounded. Show that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in S\backslash\{c\}$ and $\lim_{n\to\infty} x_n = c$ such that $\{f(x_n)\}_{n=1}^{\infty}$ converges.

Problem #6. Using the $\epsilon - \delta$-definition of continuity directly prove that $f(x) = \frac{1}{x}$ is continuous on $(0, \infty)$.

Problem #7. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(c) > 0$, then there is a $\delta > 0$ so that $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$.

Problem #8. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *Lipschitz* if there is a constant $L > 0$ so $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}$. Show that f is continuous.