Mathematic 405, Fall 2019: Assignment #4

Due: **Wednesday, October 2nd**

Instructions: Please ensure that your answers are legible. Also make sure that sufficient steps are shown. Page numbers refer to the course text.

Problem #1. Let I be a closed interval and J an open interval.

a) Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence with $x_n \in I$ show that $\lim_{n\to\infty} x_n \in I$.
b) Let $\{y_n\}_{n=1}^{\infty}$ be a convergent sequence with $y_n \in J$. Show by example that it may be possible for $\lim_{n\to\infty} y_n \notin J$.

Problem #2. Let $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence. For $k \in \mathbb{N}$ show that

$$\lim_{n\to\infty} x_n^k = \left(\lim_{n\to\infty} x_n\right)^k.$$

Problem #3. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences.

a) If $\{a_n\}_{n=1}^{\infty}$ is bounded and $\lim_{n\to\infty} b_n = 0$, then show $\{a_nb_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} a_nb_n = 0$.
b) Given an example where $\{a_n\}_{n=1}^{\infty}$ is unbounded and $\lim_{n\to\infty} b_n = 0$ and $\{a_nb_n\}_{n=1}^{\infty}$ is divergent.
c) Give an example where $\{a_n\}_{n=1}^{\infty}$ is bounded and $\lim_{n\to\infty} b_n$ exists and is $\neq 0$ and $\{a_nb_n\}_{n=1}^{\infty}$ is divergent.

Problem #4. Let $\{x_n\}_{n=1}^{\infty}$ be sequence with $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n-1} = x$. Show $\lim_{n\to\infty} x_n = x$.

Problem #5. Recall, the Fibonacci sequence is defined by $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.

a) Show that the sequence $x_n = \frac{f_{n+1}}{f_n}$ is well-defined and satisfies $x_1 = 1$ and $x_{n+1} = 1 + x_{n-1}^{-1}$.
b) Show that for $l \in \mathbb{N}$, the x_n satisfy $1 \leq x_1 \leq x_3 \leq \ldots \leq x_{2l+1} \leq x_{2l} \leq x_{2l-2} \leq \ldots \leq x_2 \leq 2$.
c) Show that $\{x_n\}_{n=1}^{\infty}$ is convergent and determine $\lim_{n\to\infty} x_n$. (Hint: Problem 4 is helpful).

Problem #6. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence. Show that

$$\limsup_{n\to\infty} x_n = -\liminf_{n\to\infty}(-x_n).$$

Problem #7. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences.

a) Show that $\{x_n + y_n\}_{n=1}^{\infty}$ is a bounded sequence.
b) Show that $\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$.
c) Show by example that it is possible for $\limsup_{n\to\infty}(x_n + y_n) < \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$.

Bonus Problem. (Do not need to complete) Dirichlet's approximation theorem says that for any irrational number α there are an infinite number of integers p and q so that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

a) Show that when α is irrational, for each rational $r \in \mathbb{Q}$ the set $\{(p,q) : r = p/q, \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}\}$ is finite. What happens when α is rational?
b) Using Dirichlet's approximation theorem, the fact that $|\sin x| \leq |x|$ and the fact that π is irrational, show that there is a subsequence $\{\sin n_k\}_{k=1}^{\infty}$ of $\{\sin n\}_{n=1}^{\infty}$ so that $\lim_{k\to 0} \sin(n_k) = 0$. (Hint: $|\sin(m\pi - x)| = |\sin(x)|$ for $m \in \mathbb{N}$).