Mathematic 405, Fall 2019: Assignment #10

Due: **Wednesday, December 4th**

Instructions: Please ensure that your answers are legible. Also make sure that sufficient steps are shown. Page numbers refer to the course text.

Problem #1. Do Exercise 6.2.13 of Section 6.2 of Lebl.

Problem #2. Let $f_n : I \to \mathbb{R}$ be a sequence of Lipschitz continuous functions on an interval I (that is $|f(x) - f(y)| \leq K_n|x - y|$ for all $x, y \in I$ and K_n a constant depending on n) and suppose that f_n converge uniformly to a function $f : I \to \mathbb{R}$.

a) Show that if the K_n are uniformly bounded (independent of n), then the limit function is Lipschitz continuous.
b) Show that $f(x) = \sqrt{x}$ is not Lipschitz continuous on $[0, 1]$
c) Given an example of f_n that converge uniformly to $f(x) = \sqrt{x}$ on $[0, 1]$.

Problem #3. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of continuous functions with the property that $0 \leq f_{n+1}(x) \leq f_n(x)$ for all $x \in [a, b]$ and $n \geq 1$ and so f_n converges uniformly to the zero function on $[a, b]$. Show that if $S_N(x) = \sum_{n=1}^{N} (-1)^n f_n(x)$, then S_N converges uniformly to a continuous function $S : [a, b] \to \mathbb{R}$.

Problem #4. Use the previous exercise to show that $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges uniformly on $[-1, 0]$. Use this calculate the value of the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. Hint: How does the power series relate to the geometric power series $\sum_{n=1}^{\infty} x^n$?

Problem #5. Recall, a function $f : I \to \mathbb{R}$ is *real analytic* if I is an open interval and for every $x_0 \in I$, there is a power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ centered at x_0 with positive radius of convergence $R > 0$ so that $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ in $(x_0 - R, x_0 + R)$. We claimed in class that $f(x) = \frac{1}{1-x}$ was real analytic in $(-\infty, 1)$. Please complete the proof of this fact.

Problem #6. Consider the initial value problem (IVP)

$$\begin{cases} y' = \sqrt{|y|} \\ y(0) = 0 \end{cases}$$

a) Show that for any fixed $c \geq 0$ the function

$$f_c(x) = \begin{cases} 0 & x \leq c \\ \frac{1}{4}(x - c)^2 & x > c \end{cases}$$

is C^1 on $(-\infty, \infty)$ and satisfies the IVP. In other words, this IVP does not have a unique solution.

b) What is another solution to the IVP that is not of the form $f_c(x)$ for some $c \geq 0$.
c) Why does this not contradict Picard's theorem?