Homework 1 Sample Solutions

Problem 1.4(b). Let $g_a(x) = f(x) + a$. Sketch the bifurcation diagram corresponding to the family of differential equations $x' = g_a(x)$.

Solution. I chose this problem because most students gave no indication of how they found the bifurcation diagram, and the method works not only for this problem, but for problem 3(c) as well.

To begin, we wish to find the set of points (a, x) such that $g_a(x) = a + f(x) = 0$. Solving for a, $a = -f(x)$. We can think of this as a function $a(x)$. Of course, our diagram will be oriented so that x depends on a. To compensate, we can simply flip the graph of $a(x)$ along the $a = x$ line (doing so sends points (x, a) to (a, x)). Once we have the equilibrium curve, we simply find whether x' is positive or negative in each region to complete the diagram. This process is illustrated in the sketches below. $\square$

Problem 1.6. Find the general solution of the logistic differential equation with constant harvesting,

$$x' = x(1 - x) - h,$$

for all values of the parameter $h > 0$.

Solution. This is a rather long exercise, but it can be done completely using the methods of Calculus II. First of all, we separate variables:

$$-\int \frac{dx}{x^2 - x + h} = \int dt = t + C$$

assuming $x^2 - x + h > 0$. Now to compute the left integral, we need to factor $x^2 - x + h$ if possible. For $h \leq \frac{1}{4}$, the polynomial factors. For $h > \frac{1}{4}$ it's irreducible. Hence we have the following cases:

If $h < \frac{1}{4}$, then $x^2 - x + h = (x - (\frac{1+\sqrt{1-4h}}{2}))(x - (\frac{1-\sqrt{1-4h}}{2}))$ using the quadratic formula. Since these factors are distinct, we can use partial fractions, which yields

$$\frac{1}{x^2 - x + h} = \frac{\frac{1}{\sqrt{1-4h}}}{x - (\frac{1-\sqrt{1-4h}}{2})} - \frac{\frac{1}{\sqrt{1-4h}}}{x - (\frac{1+\sqrt{1-4h}}{2})}$$

Note that we assumed $1 - 4h > 0$ already. Now we can integrate these easily, which yields

$$\frac{1}{\sqrt{1-4h}} \ln \left| \frac{x - (\frac{1-\sqrt{1-4h}}{2})}{x - (\frac{1+\sqrt{1+4h}}{2})} \right| = -t + C$$

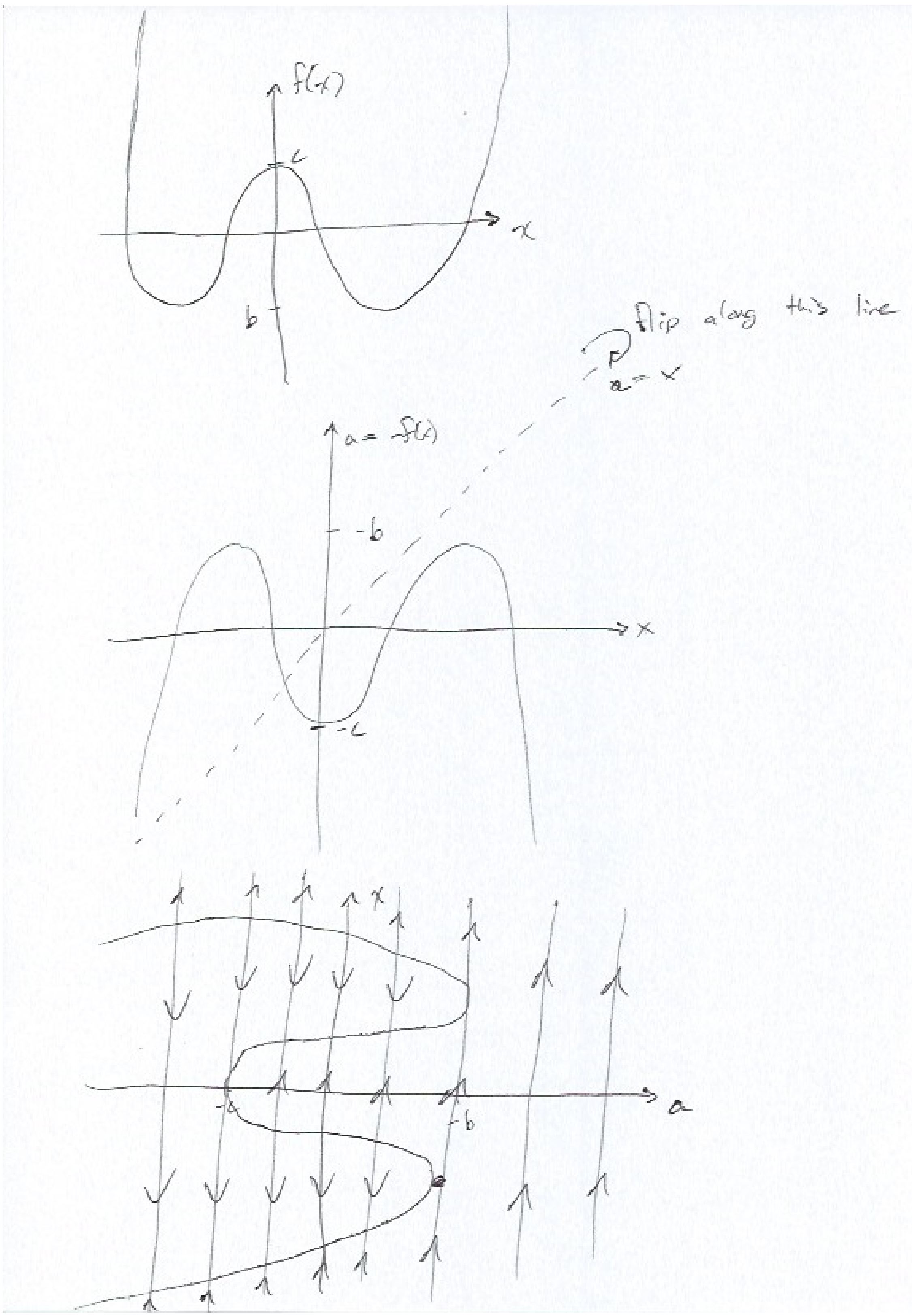

Figure 1: Bifurcation Diagram for Problem 1.4(b)

Algebraic manipulation yields the solutions

$$x = \frac{\frac{1-\sqrt{1-4h}}{2} - (\frac{1+\sqrt{1-4h}}{2})Ke^{-\sqrt{1-4h}t}}{1 - Ke^{-\sqrt{1-4h}t}}$$

where $K \in \mathbb{R} \setminus \{0\}$. We also have equilibrium solutions $x \equiv \frac{1 \pm \sqrt{1-4h}}{2}$ in this case. These correspond to $K = 0$ and $K =$ "∞" (I wouldn't recommend writing $K = \infty$ on an exam though).

If $h = \frac{1}{4}$ then we have the following:

$$\int \frac{dx}{x^2 - x + h} = \int \frac{dx}{(x - \frac{1}{2})^2} = \frac{-1}{x - \frac{1}{2}} = -t + C$$

Solving yields

$$x = \frac{1}{t + C} + \frac{1}{2}$$

where $C \in \mathbb{R}$. We also have the equilibrium solution $x \equiv \frac{1}{2}$ corresponding to $C =$ "∞".

Finally, if $h > \frac{1}{4}$ then we must complete the square and use a different integration trick.

$$\int \frac{dx}{x^2 - x + h} = \int \frac{dx}{(x - \frac{1}{2})^2 + (h - \frac{1}{4})} = \frac{1}{\sqrt{h - \frac{1}{4}}} \tan^{-1}\left(\frac{x - \frac{1}{2}}{\sqrt{h - \frac{1}{4}}}\right) = -t + C$$

Solving yields

$$x = \sqrt{h - 1/4}\tan\left(-(\sqrt{h - 1/4})t + C\right) + \frac{1}{2}$$

where $C \in \mathbb{R}$. There are no equilibria in this case. □

Problem 7(a). Consider the nonautonomous differential equation

$$x' = \begin{cases} x - 4 & : \text{if } t < 5 \\ 2 - x & : \text{if } t \geq 5 \end{cases}$$

Find a solution of this equation satisfying $x(0) = 4$. Describe the qualitative behavior of this solution.

Solution. Note that if $x(0) = 4$, then at $t = 0$, $x' = 4 - 4 = 0$. Thus for $t < 5$, our solution is the equilibrium solution $x \equiv 4$. However, at $t = 5$, the behavior changes! This is what through a few students off. We now have to solve a *new* initial value problem, namely $x(5) = 4$ (forced by continuity) and $x' = 2 - x$ for $t \geq 5$. This is straightforward and we obrtain the solution

$$x = \begin{cases} 4 & : \text{if } t < 5 \\ 2 + 2e^{5-t} & : \text{if } t \geq 5 \end{cases}$$

Thus the solution is constant for $t \leq 5$ and then approaches the stable equilibrium $x \equiv 2$ as $t \to \infty$. Part (b) is similar. Just solve the initial value problem in the region $t < 5$, then use continuity to get a new initial value (at $t = 5$) for the initial value problem in the region $t \geq 5$. Then glue these solutions together to get one global continuous solution (the solution will not be continuously differentiable though. I'm not sure whether we exclude such solutions in this class or not). □