Problem 1. [12 points] Compute the following

(a) $\int_{-1}^{1} x(2x-1)^2\,dx = \int_{-1}^{1} x(4x^2-4x+1)\,dx = \int_{-1}^{1}(4x^3-4x^2+x)\,dx$

$$= \left(4\cdot\frac{x^4}{4} - 4\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_{-1}^{1}$$

$$= \left(1-\frac{4}{3}+\frac{1}{2}\right)-\left(1+\frac{4}{3}+\frac{1}{2}\right) = \boxed{-\frac{8}{3}}$$

(b) $\lim_{x\to 0} \frac{x^2}{1-\cos x}$ type $\frac{0}{0}$

$\parallel$ L'H

$\lim_{x\to 0} \frac{2x}{\sin x}$ type $\frac{0}{0}$

$\parallel$

$\lim_{x\to 0} \frac{2}{\cos x} = \frac{2}{1} = \boxed{2}$

(from class)

or at this point you could have recalled that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, so that

$$\lim_{x\to 0} \frac{2x}{\sin x} = \lim_{x\to 0} \frac{2}{\frac{\sin x}{x}} = \frac{2}{1} = 2.$$

(c) $\frac{d}{dx}\left(\int_4^{e^x} e^{-t}\,dt\right) = e^{-e^x} \cdot e^x = e^{x-e^x}$.

Reason: let $f(x) = \int_4^{e^x} e^{-t}\,dt$, $g(x) = \int_4^{x} e^{-t}\,dt$, $h(x) = e^x$.

Then $f(x) = g(h(x)) \underset{\text{Rule}}{\overset{\text{Chain}}{\Longrightarrow}} f'(x) = g'(h(x))\,h'(x)$

$g'(x) = e^{-x}$ by FTC I, & $h'(x) = e^x$

$\Longrightarrow f'(x) = e^{-(e^x)} \cdot e^x = e^{x-e^x}$.

Problem 2. [10 points] Let $f(x) = \int_{-\frac{\pi}{6}}^{x} (1 - 8\sin^3 t)\,dt$ on $I = (-\frac{\pi}{2}, \frac{\pi}{2})$.

(a) Find the critical numbers of f on I.

The critical numbers of f on I are the points $x \in I$ such that $f'(x)$ DNE or $f'(x) = 0$. By FTC I, f is dble everywhere & $f'(x) = 1 - 8\sin^3(x)$. So the critical #'s are the solutions to $0 = 1 - 8\sin^3(x) \Rightarrow \sin^3(x) = \frac{1}{8} \Rightarrow \sin(x) = \frac{1}{2}$. The only solution to this in $(-\frac{\pi}{2}, \frac{\pi}{2})$ is $x = \frac{\pi}{6}$.

Thus $\frac{\pi}{6}$ is the only critical number in I.

(b) Decide whether each critical number from (a) corresponds to a local max, a local min, or neither.

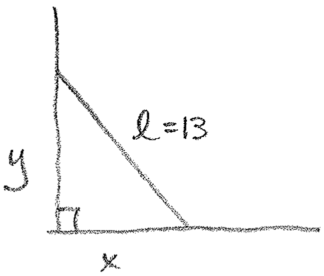
There are 2 ways to do it.

<u>2nd derivative test</u>: $f''(x) = \frac{d}{dx}(1 - 8\sin^3(x)) = -24\sin^2(x)\cos(x)$

$\Rightarrow f''(\frac{\pi}{6}) = -24(\frac{1}{2})^2(\frac{\sqrt{3}}{2}) < 0 \Rightarrow \boxed{\text{loc max at } \frac{\pi}{6}}$

<u>1st derivative test</u>: $f'(x) = 1 - 8\sin^3(x)$ is >0 for x near $\frac{\pi}{6}$ & $< \frac{\pi}{6}$, & $f'(x)$ is <0 for x near $\frac{\pi}{6}$ & $> \frac{\pi}{6}$ $\Rightarrow \boxed{\text{loc max at } \frac{\pi}{6}}$

Problem 3. [10 points] A 13-foot ladder is leaning against a vertical wall. If the bottom of the ladder is being pulled away from the wall at a constant rate of 2 feet per second, how fast is the top end of the ladder moving down the wall when it is 5 feet above the ground?



Know: $\frac{dx}{dt} = 2$

Want: $\left.\frac{dy}{dt}\right|_{y=5}$

Method 1: implicit differentiation

By Pythagorean Thm, $x^2 + y^2 = \ell^2$.

$\frac{d}{dt}$ both sides: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow y\frac{dy}{dt} = -x\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt}$

We have $\frac{dx}{dt} = 2$, & when $y = 5$, $x = \sqrt{\ell^2 - y^2} = \sqrt{169 - 25} = \sqrt{144} = 12$

$$\Rightarrow \left.\frac{dy}{dt}\right|_{y=5} = -\frac{12 \cdot 2}{5} = \boxed{-\frac{24}{5}}$$

The top of the ladder is moving downward at a speed of $\frac{24}{5}$ ft/s.

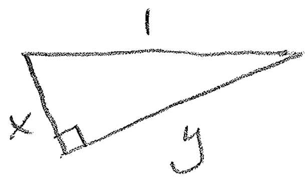
Method 2: y in terms of x

Again by Pyth. Thm, $y = \sqrt{\ell^2 - x^2} = \sqrt{169 - x^2} \Rightarrow \frac{dy}{dt} = \frac{-2x\frac{dx}{dt}}{2\sqrt{169 - x^2}} = \frac{-x\frac{dx}{dt}}{\sqrt{169 - x^2}}$.

When $y = 5$, we again find $x = 12 \Rightarrow \left.\frac{dy}{dt}\right|_{y=5} = \frac{-12 \cdot 2}{\sqrt{169 - 144}} = \boxed{\frac{-24}{5}}$.

So we again find that the top of the ladder is moving downward at a speed of $\frac{24}{5}$ ft/s.

Problem 4. [10 points] Find the maximum possible perimeter of a right triangle with hypotenuse of length 1. You must use calculus to justify your answer.



Know $x^2+y^2=1 \Rightarrow y=\sqrt{1-x^2}$

Want to maximize $P = 1+x+y = 1+x+\sqrt{1-x^2}$, $x\in[0,1]$

Need to find critical #'s: solve $0 = P'(x) = 1 + \frac{-2x}{2\sqrt{1-x^2}} \Rightarrow \frac{x}{\sqrt{1-x^2}} = 1$

$\Rightarrow x = \sqrt{1-x^2} \Rightarrow x^2 = 1-x^2 \Rightarrow 2x^2=1 \Rightarrow x^2=\frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$. Only the positive root is in $[0,1]$, so the only crit # is $\frac{1}{\sqrt{2}}$, which gives a perimeter of

$P(\frac{1}{\sqrt{2}}) = 1+\frac{1}{\sqrt{2}}+\sqrt{1-\frac{1}{2}} = 1+\frac{2}{\sqrt{2}} = 1+\sqrt{2}$.

To find the max, we also need to evaluate P at the endpoints $x=0$ & $x=1$:

$P(0) = 1+0+\sqrt{1-0} = 2$

$P(1) = 1+1+\sqrt{1-1} = 2$

By the closed interval method, the biggest possible perimeter is the largest value obtained at an endpt or a critical number, which in our case is $P(\frac{1}{\sqrt{2}}) = \boxed{1+\sqrt{2}}$.

Problem 5. [10 points]

(a) State Rolle's Theorem.

Let f be defined & continuous on $[a,b]$, dble on (a,b), & such that $f(a)=f(b)$.

Then $\exists\, c \in (a,b)$ s.t. $f'(c)=0$.

(Note that this is just the statement of the MVT given below, under the additional assumption that $f(a)=f(b)$.)

In the next two parts we'll prove the Mean Value Theorem, which asserts that if f is a function defined and continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

(b) Given f as in the statement of the Mean Value Theorem, let

$$g(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}\right)(x-a).$$

Explain why g satisfies the hypotheses of Rolle's Theorem.

Let $h(x) = \left(\frac{f(b)-f(a)}{b-a}\right)(x-a)$. Then h is cts on $[a,b]$ & dble on (a,b) since it's a polynomial fcn $\Rightarrow g(x) = f(x) - h(x)$ is cts on $[a,b]$ & dble on (a,b) b/c it's a difference of 2 such fcns.

Finally, $g(a) = f(a) - \left(\frac{f(b)-f(a)}{b-a}\right)\underbrace{(a-a)}_{0} = f(a)$

$g(b) = f(b) - \left(\frac{f(b)-f(a)}{\cancel{b-a}}\right)\cancel{(b-a)} = f(b) - (f(b)-f(a)) = f(a)$

Hence $g(a) = g(b)$.

(c) Apply Rolle's Theorem to g to deduce the conclusion of the Mean Value Theorem for f.

Note that $g'(x) = f'(x) - \frac{d}{dx}\left(\left(\frac{f(b)-f(a)}{b-a}\right)(x-a)\right) = f'(x) - \frac{f(b)-f(a)}{b-a}$

Now apply Rolle's Thm to g: $\exists\ c \in (a,b)$ s.t.

$0 = g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$.