Name: ______________________________ **Section Number:**______________

110.108 CALCULUS I (Physical Sciences & Engineering)
FALL 2011
MIDTERM EXAMINATION SOLUTIONS
October 5, 2011

Instructions: The exam is **7** pages long, including this title page and a spare sheet at the end. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		20
2		15
3		20
4		20
5		25
TOTAL		100

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ **Date:**______________

Question 1. [20 points] For $f(x) = \dfrac{16x^2 - 9}{4x - 3}$, do the following:

(a) Use the ϵ-δ definition of a limit to show $\lim\limits_{x \to \frac{3}{4}} f(x) = 6$.

Strategy: The formal definition of a limit says that $\lim\limits_{x \to a} f(x) = L$ if, for any $\epsilon > 0$, there is a $\delta > 0$ where if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. To satisfy the definition is this case, we need to be able to calculate a δ in terms of ϵ (so that it will work for ANY value of ϵ), and then show that it indeed satisfies the definition. For this problem, you MUST be able to state the definition clearly. Simply calculating a δ as a function of ϵ is only a small part of the problem. Without any justification as to why it actually satisfies the definition, that calculation is not nearly enough.

Solution: First, notice that $f(x) = \dfrac{16x^2 - 9}{4x - 3} = \dfrac{(4x+3)\cancel{(4x-3)}}{\cancel{4x-3}} = 4x+3$ on the domain $\left\{ x \in \mathbb{R} \;\middle|\; x \neq \dfrac{3}{4} \right\}$. Thus $\lim\limits_{x \to \frac{3}{4}} f(x) = \lim\limits_{x \to \frac{3}{4}} 4x+3$, and we are left with using the definition of a limit to establish that $\lim\limits_{x \to \frac{3}{4}} 4x+3 = 6$. With $a = \frac{3}{4}$, $f(x) = 4x + 3$ and $L = 6$, the definition becomes: If for any $\epsilon > 0$, there is a $\delta > 0$ where if $0 < \left|x - \frac{3}{4}\right| < \delta$, then $|(4x + 3) - 6| < \epsilon$, then $\lim\limits_{x \to a} 4x + 3 = 6$. We work with the ϵ-inequality. Here

$$|(4x + 3) - 6| = |4x - 3| = 4\left|x - \frac{3}{4}\right| < \epsilon, \text{ becomes } \left|x - \frac{3}{4}\right| < \frac{\epsilon}{4}.$$

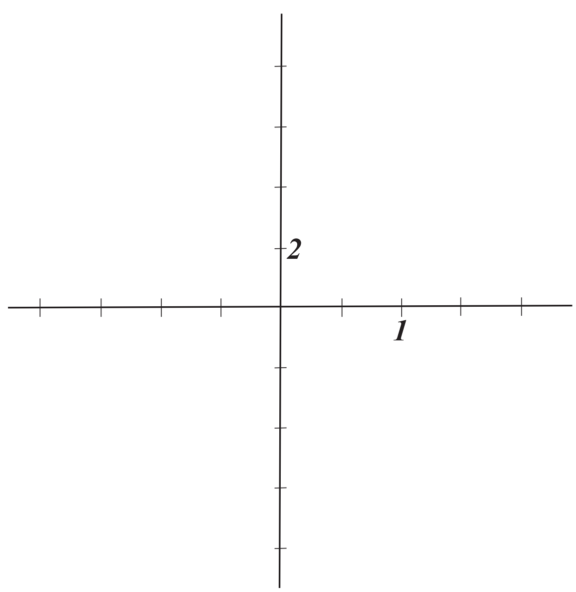
Noticing that in this last inequality, the left hand side is the same as that of the δ-inequality, we set $\delta = \frac{\epsilon}{4}$. This will then satisfy the definition. To show why this choice of δ works, we restrict input values to the interval defined by $0 < \left|x - \frac{3}{4}\right| < \delta$. Then

$$|f(x) - 6| = |(4x + 3) - 6| = 4\left|x - \frac{3}{4}\right| < 4\delta = 4\left(\frac{\epsilon}{4}\right) = \epsilon.$$

Thus the definition is satisfied, and $\lim\limits_{x \to \frac{3}{4}} 4x + 3 = 6$. And finally, since $4x + 3 = \frac{16x^2-9}{4x-3}$ everywhere except $x = \frac{3}{4}$, their limits are the same everywhere, and

$$\lim_{x \to \frac{3}{4}} \frac{16x^2 - 9}{4x - 3} = 6.$$

(b) Sketch the graph of $f(x)$ on the grid below.



Question 2. [15 points] Given

$$g(x) = \begin{cases} cx^2 + 2x & x \leq 2 \\ \dfrac{2c(x^2-4)}{x-2} & x > 2, \end{cases}$$

determine if there is a value of c which makes $g(x)$ continuous on $(-\infty, \infty)$. If so, then find the value of c.

Strategy: The functions defined on each piece of the domain are, respectively, a polynomial and a rational function. Both are continuous on their domains (the rational function has no places on $(-\infty, 2)$ where the denominator is 0). Hence $g(x)$ will be continuous on all of $(-\infty, \infty)$ if we can establish continuity at the point $x = 2$, where the two pieces meet. Thus we focus on that point.

Solution: First, notice that for any choice of c, $g(x)$ will be continuous from the left at $x = 2$ since the function there is a polynomial and the domain is a closed interval. Thus

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} cx^2 + 2x = c(2)^2 + 2(2) = 4c + 4$$

for any choice of c. The limit from the right at $x = 2$ is

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \frac{2c(x^2-4)}{x-2} = \lim_{x\to 2^+} \frac{2c(x+2)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x\to 2^+} 2c(x+2).$$

As a limit, the last one is of a function which is continuous at $x = 2$. Hence, the right hand limit at $x = 2$ exists and is

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \frac{2c(x^2-4)}{x-2} = 2c(2+2) = 8c.$$

The function $g(x)$ will be continuous at $x = 2$ if (1) these two side limits are equal (then the limit exists), and (2) they both equal the function value at $x = 2$. But for value of c, the function value will equal the left side limit. Hence we only need to find the value of c which makes the two side limits equal. Thus we need to satisfy

$$\lim_{x\to 2^-} g(x) = 4c + 4 = 8c = \lim_{x\to 2^+} g(x).$$

This works for $c = 1$, and thus the function

$$g(x) = \begin{cases} x^2 + 2x & x \leq 2 \\ \dfrac{2(x^2-4)}{x-2} & x > 2, \end{cases}$$

is a continuous function on $\mathbb{R}$.

Question 3. [20 points] Use the definition of a derivative to find $f'(1)$, where $f(x) = \dfrac{1}{(x+1)^2}$.

Strategy: The definition of the derivative evaluated at the point a comes in two flavors:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Either will work, as will the definition of a derivative function $f'(x)\big|_{x=a} = \left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right)\Big|_{x=a}$. although this last one will require a bit more work to first calculate the derivative function before plugging the value a. For our purposes we will use on the former ones.

Solution: For $a = 1$, the derivative can be calculated as

$$\begin{aligned}
f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{\frac{1}{(x+1)^2} - \frac{1}{(1+1)^2}}{x - 1} \\
&= \lim_{x\to 1} \frac{\frac{1}{(x+1)^2} - \frac{1}{4}}{x - 1} \\
&= \lim_{x\to 1} \frac{\frac{4-(x+1)^2}{4(x+1)^2}}{x - 1} \\
&= \lim_{x\to 1} \frac{4 - x^2 - 2x - 1}{4(x+1)^2(x-1)} \\
&= \lim_{x\to 1} \frac{-(x^2 + 2x - 3)}{4(x+1)^2(x-1)} \\
&= \lim_{x\to 1} \frac{-(x+3)\cancel{(x-1)}}{4(x+1)^2\cancel{(x-1)}} \\
&= \lim_{x\to 1} \frac{-(x+3)}{4(x+1)^2}.
\end{aligned}$$

This last limit is of a function that is continuous at $x - 1$ (it is a rational function where the denominator is 0 only at $x = -1$). Thus

$$f'(1) = \lim_{x\to 1} \frac{-(x+3)}{4(x+1)^2} = \frac{-(1+3)}{4(1+1)^2} = -\frac{4}{16} = -\frac{1}{4}.$$

Question 4. [20 points] Do the following:

(a) For $g(x) = \sqrt{1+xe^{-2x}}$, Find the equation of the line tangent to $g(x)$ at $x = 0$.

Strategy: By inspection, calculating this derivative will involve the Chain Rule (everything under the radical becomes the "inside" function), and the Product Rule (the xe^{-2x} part is a product of functions. Hence we will calculate $g'(0)$ in stages before determining the tangent line equation.

Solution: For the moment, we simply calculate $g'(x)$. First, note that the "outside" function is $\sqrt{x}$, whose derivative is $\frac{1}{2\sqrt{x}}$. Hence

$$g'(x) = \frac{d}{dx}\left[g(x)\right] = \frac{1}{2\sqrt{1+xe^{-2x}}} \cdot \frac{d}{dx}\left[1+xe^{-2x}\right].$$

The "inside" function includes a product of functions. To continue:

$$\begin{aligned} g'(x) &= \frac{1}{2\sqrt{1+xe^{-2x}}}\frac{d}{dx}\left[1+xe^{-2x}\right] = \frac{1}{2\sqrt{1+xe^{-2x}}}\left(0+\frac{d}{dx}\left[x\right]e^{-2x}+x\frac{d}{dx}\left[e^{-2x}\right]\right) \\ &= \frac{1}{2\sqrt{1+xe^{-2x}}}\left(e^{-2x}+x(-2)e^{-2x}\right). \end{aligned}$$

Thus, we have

$$g'(0) = \frac{1}{2\sqrt{1+(0)e^{-2(0)}}}\left(e^{-2(0)}+(0)(-2)e^{-2(0)}\right) = \frac{1}{2\sqrt{1}}(1) = \frac{1}{2}.$$

And when $x = 0$, we have $g(0) = \sqrt{1+(0)e^{-2(0)}} = 1$, we have our equation for the tangent line:

$$y - 1 = \frac{1}{2}x.$$

(b) Calculate $\left.\frac{d}{dx}\left[2^{1-\tan x}\right]\right|_{x=0}$.

Strategy: Here, we will again need the Chain Rule, with the "outside" function being 2^x and the "inside" function being $1 - \tan x = 1 - \frac{\sin x}{\cos x}$.. The derivative here is quite straightforward.

Solution: Again, we first calculate the derivative and then evaluate it at 0:

$$\frac{d}{dx}\left[2^{1-\tan x}\right] = 2^{1-\tan x}(\ln 2)\cdot\frac{d}{dx}\left[1-\tan x\right].$$

If you remember that $\frac{d}{dx}\left[\tan x\right] = \sec^2 x$, then you can proceed directly to

$$\begin{aligned} \frac{d}{dx}\left[2^{1-\tan x}\right] &= 2^{1-\tan x}(\ln 2)\cdot\frac{d}{dx}\left[1-\tan x\right] \\ &= 2^{1-\tan x}(\ln 2)\left(-\sec^2 x\right). \end{aligned}$$

(However, if you did not remember this, just write $\tan x$ as the ratio of $\sin x$ and $\cos x$ and use the Quotient Rule.) Now, evaluated at $x = 0$, we have

$$\left.\frac{d}{dx}\left[2^{1-\tan x}\right]\right|_{x=0} = 2^{1-\tan(0)}(\ln 2)\left(-\sec^2(0)\right) = 2^1(\ln 2)(-1) = -2\ln 2.$$

Question 5. [25 points] Let $y = \dfrac{x-1}{x^2-x+1}$. Do the following:

(a) Find all vertical and horizontal asymptotes, if any.

Strategy: The vertical asymptotes will correspond to places where the denominator of this rational function are 0 and the numerator is not. The horizontal asymptotes will be the limits at infinity and minus infinity. We calculate these here:

Solution: First, the denominator $x^2 - x + 1$ is never 0 for any real numbers. The quadratic expression does not factor (one can say that the equation $x^2 - x + 1 = 0$ has no solutions. If one were to use the quadratic formula to solve for the solutions, one would find that the part under the radical is negative. Hence there are no real solutions). This means that there can be no vertical asymptotes. As for the horizontal asymptotes, we evaluate the limit at infinity:

$$\begin{aligned}
\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{x-1}{x^2-x+1} &= \lim_{x\to\infty} \frac{x-1}{x^2-x+1}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right) \\
&= \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{1 - \frac{1}{x} + \frac{1}{x^2}} \\
&= \frac{\left(\lim\limits_{x\to\infty} \frac{1}{x}\right) - \left(\lim\limits_{x\to\infty} \frac{1}{x}\right)^2}{\left(\lim\limits_{x\to\infty} 1\right) - \left(\lim\limits_{x\to\infty} \frac{1}{x}\right) + \left(\lim\limits_{x\to\infty} \frac{1}{x}\right)^2},
\end{aligned}$$

using the Limit Laws and a clever form of 1 that makes rational functions easier to "see" near infinity. Remember using the Limit Laws only works when the individual limits all exist. But all of these limits are either of constants or the function $\frac{1}{x}$, which we know exist at infinity. Since $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$, we have then that

$$\lim_{x\to\infty} \frac{x-1}{x^2-x+1} = \frac{\left(\lim\limits_{x\to\infty} \frac{1}{x}\right) - \left(\lim\limits_{x\to\infty} \frac{1}{x}\right)^2}{\left(\lim\limits_{x\to\infty} 1\right) - \left(\lim\limits_{x\to\infty} \frac{1}{x}\right) + \left(\lim\limits_{x\to\infty} \frac{1}{x}\right)^2} = \frac{0-0^2}{1-0+0^2} = 0.$$

Hence the line $y = 0$ is a horizontal asymptote of the function.

One can do exactly the same calculation for the limit at $-\infty$, with the same result.

(b) Find the (x, y)-coordinates of all points on the graph of this function that have a horizontal tangent line (where $\dfrac{dy}{dx} = 0$).

Strategy: Setting the derivative to 0 is easy enough. Just looks for the places where the numerator is 0 but the denominator is not. As the denominator here is never 0, this makes the calculation that much easier.

Solution: First, the derivative, using the Quotient Rule:

$$\begin{aligned}
\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x-1}{x^2-x+1}\right] &= \frac{\frac{d}{dx}\left[(x-1)\right](x^2-x+1) - (x-1)\frac{d}{dx}\left[(x^2-x+1)\right]}{(x^2-x+1)^2} \\
&= \frac{1(x^2-x+1) - (x-1)(2x-1)}{(x^2-x+1)^2} = \frac{x^2-x+1-2x^2+3x-1}{(x^2-x+1)^2} \\
&= \frac{-x^2+2x}{(x^2-x+1)^2}.
\end{aligned}$$

Checking the numerator, we find that the derivative will be 0 when either $x = 0$ or $x = 2$.

And since for $x = 0$, $y = -1$, and when $x = 2$, $y = \frac{1}{3}$, the points on the graph where the tangent line is horizontal are $(0, -1)$, and $\left(2, \dfrac{1}{3}\right)$.

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