Solutions Midterm Exam 2 — Nov. 11, 2015

1. (20 points) Determine the tangent line at $(1, -1)$ to the curve $e^{x+y} = y^2$.

We compute the slope of the tangent line by implicit differentiation. First, we take the logarithm of both sides and obtain $x+y = 2\ln|y|$ (we could also just just use the chain rule). Differentiating, gives

$$1 + \frac{dy}{dx} = 2\frac{1}{y}\frac{dy}{dx}.$$

We substitute $(x, y) = (1, -1)$ to obtain

$$1 + \frac{dy}{dx} = -2\frac{dy}{dx}$$

so $\frac{dy}{dx} = -\frac{1}{3}$. Hence, the tangent line is $y = -\frac{1}{3}(x-1) - 1$.

2. Give examples of functions with the given properties. You do not need to justify your answers.

   (a) (5 points) Absolute maximum value on $[0, 2]$ at $x = 2$.

   $f(x) = x$

   (b) (5 points) No absolute minimum value on $[-1, \infty)$.

   $f(x) = 1 - x$

   (c) (5 points) Continuous on $[-1, 1]$ with no local maximum and one local minimum in $[-1, 1]$.

   $f(x) = x^2$

   (d) (5 points) Continuous on $[-1, 1]$, no local extrema and one critical number in $(-1, 1)$.

   $f(x) = x^3$

3. Let $f(x) = xe^{-1/x}$.

(a) (10 points) Determine the intervals of increase and decrease and all local extrema.

We compute using the product and chain rules that

$$f'(x) = e^{-1/x} + x\frac{d}{dx}e^{-1/x} = 1 + xe^{-1/x}\frac{d}{dx}\left(-\frac{1}{x}\right) = \frac{x+1}{x}e^{-1/x}.$$

This has a zero at $x = -1$ and is discontinuous at $x = 0$ and is otherwise continuous so the only places the sign can change are at $x = 0$ and $x = -1$. Once checks that, when $x < -1$ we have $f'(x) > 0$ and when $-1 < x < 0$ we have $f'(x) < 0$ and when $x > 0$ we have $f'(x) > 0$. Hence, by the I/D test, f is increasing on $(-\infty, -1)$ and $(0, \infty)$ and decreasing on $(-1, 0)$. As f is continuous at $x = -1$, but is discontinuous at $x = 0$ the mean value theorem tells us that in fact f is increasing on $(-\infty, -1]$ and $(0, \infty)$ and decreasing on $[-1, 0)$.

Observe that f is not defined at $x = 0$ and (in fact, it has a vertical asymptote there) and so we conclude that the only local extremum is at $x = -1$. This is a local maximum by the first derivative test.

(b) (10 points) Determine where the graph is concave up and where it is concave down and all inflection points.

We compute that $f''(x) = \left(\frac{1}{x} - \frac{x+1}{x^2} + \frac{x+1}{x^3}\right) = \frac{1}{x^3}e^{-1/x}$. This is never zero and discontinuous at $x = 0$ so $x = 0$ is only place the sign can change. We have that $f''(x) > 0$ for $x > 0$ and $f''(x) < 0$ for $x < 0$. Hence, by the concavity test the graph is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$ and there are no inflection points as the only place the sign changes is not in the domain.

4. (20 points) Let $f(x) = 2x^3 + x^2 - 8x + 2$. Determine the absolute maximum value and absolute minimum value of f on $[0, 2]$

We may apply the closed interval method as f is continuous and $[0, 2]$ is a closed bounded interval. We first find the critical points of f in $(0, 2)$. To that end we observe that $f'(x) = 6x^2 + 2x - 8 = 2(3x + 4)(x - 1)$ so the only critical point is $x = 1$ (the value $x = -\frac{4}{3}$ is not in the interval). We evaluate f at the endpoints to obtain $f(0) = 2$ and $f(2) = 2 * 8 + 4 - 16 + 2 = 6$. Evaluating at the critical point we obtain $f(1) = 2 + 1 - 8 + 2 = -3$. Hence, the absolute maximum value is $f(2) = 6$ achieved at $x = 2$ and the absolute minimum is $f(1) = -3$ achieved at $x = 1$.

5. Let $f(x) = 2x + \sin(x) - e^{3x}$.

(a) (10 points) Determine $F(x)$, the antiderivative of $f(x)$ that satisfies $F(0) = 0$.

We compute that the general antiderivative is $F(x) = x^2 - \cos(x) - \frac{1}{3}e^{3x} + C$. We have $F(0) = -1 - \frac{1}{3} + C = 0$ that is $C = \frac{4}{3}$.

(b) (10 points) Calculate $\lim_{x\to 0} \frac{F(x)}{\ln(x+1)}$ where F is the function given in part a).

We observe that F is differentiable near $x = 0$ as is $G(x) = \ln(x+1)$ and that $G'(x) = \frac{1}{x+1} \neq 0$ for $x \neq -1$. In particular, $\lim_{x\to 0} F(x) = F(0) = 0$ and $\lim_{x\to 0} G(x) = G(0) = \ln(1) = 0$. Hence, we have a limit of indeterminate type $\frac{0}{0}$ and so we may apply L'Hospital's rule to obtain

$$\lim_{x\to 0} \frac{F(x)}{G(x)} = \lim_{x\to 0} \frac{F'(x)}{G'(x)} = \frac{2x + \sin(x) - e^{3x}}{\frac{1}{x+1}} = \lim_{x\to 0} (x+1)(2x + \sin(x) - e^{3x}).$$

This is the limit of a continuous function and so direct substitution gives that that

$$\lim_{x\to 0} \frac{F(x)}{G(x)} = (0+1)(2 * 0 + \sin(0) - e^{3*0}) = -1.$$