Problem 1. [15 points] Compute the following limits.

(a) $\lim_{x\to\infty} \frac{4x^3 - 4x + 1}{3x^3 - 4x^2 + x + 6} = \lim_{x\to\infty} \frac{4 - \frac{4}{x^2} + \frac{1}{x^3}}{3 - \frac{4}{x} + \frac{1}{x^2} + \frac{6}{x^3}} = \frac{\lim_{x\to\infty}\left(4 - \frac{4}{x^2} + \frac{1}{x^3}\right)}{\lim_{x\to\infty}\left(3 - \frac{4}{x} + \frac{1}{x^2} + \frac{6}{x^3}\right)} = \boxed{\frac{4}{3}}$

(b) $\lim_{x\to 1} \frac{x^2 - 3x - 2}{x^2 + x - 1} = \frac{1-3-2}{1+1-1} = \boxed{-4}$ by direct substitution

(c) $\lim_{x\to\infty} (1+3x)^{1/\ln(x)} = \lim_{x\to\infty} e^{\ln\left((1+3x)^{1/\ln(x)}\right)} = \lim_{x\to\infty} e^{\frac{1}{\ln(x)}\cdot \ln(1+3x)}$

$= e^{\lim_{x\to\infty} \frac{\ln(1+3x)}{\ln(x)}}$ ← lim of type $\frac{\infty}{\infty}$

b/c e^x is cts

$= e^{\lim_{x\to\infty} \frac{\frac{3}{1+3x}}{\frac{1}{x}}}$ (L'Hopital)

$= e^{\lim_{x\to\infty} \frac{3x}{1+3x}}$

$= e^1 = \boxed{e}$

(d) $\lim_{x\to 2} \frac{\ln(x)-\ln(2)}{x-2}$

Solution 1: By def, this is $\frac{d}{dx}(\ln(x))\Big|_{x=2} = \frac{1}{x}\Big|_{x=2} = \boxed{\frac{1}{2}}$

Solution 2: Lim is of of type $\frac{0}{0}$, so by L'Hopital it's $= \lim_{x\to 2} \frac{\frac{1}{x}}{1} = \boxed{\frac{1}{2}}$

(e) $\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} \sin^3(t)\,dt = \lim_{h\to 0} \frac{1}{h}\left[\int_0^{x+h} \sin^3(t)\,dt - \int_0^x \sin^3(t)\,dt\right]$

$= \frac{d}{dx}\left(\int_0^x \sin^3(t)\,dt\right)$

$= \boxed{\sin^3(x)}$ by FTC I.

Problem 2. [15 points] Compute the following.

(a) $\int_1^2 \frac{1}{x}\,dx$ $= \ln(x)\Big|_1^2 = \ln(2) - \cancel{\ln(1)}^{0} = \boxed{\ln(2)}$

(b) $\int -\frac{t^2}{(t^3+6)^3}\,dt$ $\qquad u = t^3+6 \quad du = 3t^2\,dt$

$= -\int \frac{1}{u^3} \cdot \frac{du}{3} = -\frac{1}{3}\frac{u^{-2}}{(-2)} + C$

$= \boxed{\frac{1}{6(t^3+6)^2} + C}$

(c) $\int_{-\pi/2}^{\pi/2} \cos\theta \cos(\pi \sin\theta)\, d\theta$

$u = \pi \sin\theta \quad du = \pi \cos\theta \, d\theta$

$u(-\pi/2) = -\pi \quad u(\pi/2) = \pi$

$$\| $$

$$\int_{-\pi}^{\pi} \cos(u) \cdot \frac{du}{\pi} = \frac{\sin(u)}{\pi}\Big|_{-\pi}^{\pi} = 0 - 0 = \boxed{0}.$$

(d) $\int \frac{1-2v}{1+2v}\, dv$

$u = 1+2v \quad du = 2\,dv$

$u-1 = 2v$

$$\| $$

$$\int \frac{1-(u-1)}{u} \frac{du}{2} = \int \frac{2-u}{2u}\, du = \int \left(\frac{1}{u} - \frac{1}{2}\right) du$$

$$= \ln|u| - \frac{u}{2} + C$$

$$= \ln|1+2v| - \frac{1+2v}{2} + C$$

(OK to leave it like this, or can continue to get $\ln|1+2v| - v + C'$, by lumping the $\frac{1}{2}$ into the constant)

(e) The average value of $f(x) = \dfrac{1}{\sqrt{5-x^2}}$ on $[0, \sqrt{5}]$

The average val is $\dfrac{1}{\sqrt{5}-0}\displaystyle\int_0^{\sqrt{5}} \dfrac{1}{\sqrt{5-x^2}}\,dx = \dfrac{1}{\sqrt{5}}\int_0^{\sqrt{5}} \dfrac{1}{\sqrt{5}}\cdot\dfrac{1}{\sqrt{1-\frac{x^2}{5}}}\,dx$

$u = \dfrac{x}{\sqrt{5}}$

$du = \dfrac{dx}{\sqrt{5}}$

$u(0) = 0$

$u(\sqrt{5}) = 1$

$$= \frac{1}{\sqrt{5}}\int_0^1 \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{\sqrt{5}}\sin^{-1}(u)\Big|_0^1 = \frac{1}{\sqrt{5}}\left(\frac{\pi}{2}-0\right) = \boxed{\frac{\pi}{2\sqrt{5}}}$$

Remark To make the numbers come out nicely, I strictly speaking goofed the formulation of this problem: f isn't defined on all of $[0, \sqrt{5}]$ since it's not defined at $\sqrt{5}$, & therefore the integral isn't defined, at least as far as this class is concerned. You'll see in Calc II that you can make sense of the integral in this case, & that the answer is exactly as above.

Problem 3. [10 points] Find $\lim_{x\to 1} \dfrac{5x^2 - 15x + 10}{3x - 3}$ and give a δ-ϵ proof that you're right.

$$\lim_{x\to 1} \frac{5x^2-15x+10}{3x-3} = \lim_{x\to 1} \frac{5(x^2-3x+2)}{3(x-1)} = \lim_{x\to 1} \frac{5\cancel{(x-1)}(x-2)}{3\cancel{(x-1)}} = -\frac{5}{3}$$

Step 1 To prove that the lim is $-\frac{5}{3}$, given $\varepsilon > 0$, we need a $\delta > 0$ s.t.

$$0<|x-1|<\delta \Rightarrow \left|\frac{5x^2-15x+10}{3x-3}+\frac{5}{3}\right| < \varepsilon$$

$$\left|\frac{5x^2-15x+10}{3x-3}+\frac{5}{3}\right| = \frac{5}{3}\left|\frac{\cancel{(x-1)}(x-2)}{\cancel{x-1}}+1\right| = \frac{5}{3}|x-1|$$

$\frac{5}{3}|x-1| \leftarrow$ This will be $<\varepsilon \iff |x-1| < \frac{3}{5}\varepsilon$

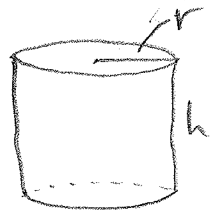
Suggests we take $\delta = \frac{3}{5}\varepsilon$.

Step 2 Let $\varepsilon > 0$, & take $\delta = \frac{3}{5}\varepsilon$. Assume $0<|x-1|<\delta$. Then

$$\left|\frac{5x^2-15x+10}{3x-3} - \left(-\frac{5}{3}\right)\right| = \frac{5}{3}\left|\frac{x^2-3x+2}{x-1}+1\right| = \frac{5}{3}\left|\frac{\cancel{(x-1)}(x-2)}{\cancel{x-1}}+1\right| = \frac{5}{3}|x-2+1|$$

$$= \frac{5}{3}|x-1| < \frac{5}{3}\cdot\frac{3}{5}\varepsilon = \varepsilon,$$ as desired.

Problem 4. [10 points] A cylindrical canister is to be made with cardboard sides and metal top and bottom. If cardboard costs \$1 per square foot and metal costs \$2 per square foot, what are the height and radius of the cylinder of greatest volume that can be made with \$12?



Want to maximize $V = \pi r^2 h$.

Our constraint is that

the cost $12 = 1 \cdot \underbrace{2\pi r h}_{\text{area of side}} + 2 \cdot \underbrace{2\pi r^2}_{\text{area of top + area of bottom}}$

$$\Rightarrow 2\pi r h = 12 - 4\pi r^2$$

$$\Rightarrow h = \frac{12 - 4\pi r^2}{2\pi r} = \frac{6 - 2\pi r^2}{\pi r}$$

$$\Rightarrow V(r) = \cancel{\pi} r^{\cancel{2}} \cdot \frac{6 - 2\pi r^2}{\cancel{\pi} \cancel{r}} = 6r - 2\pi r^3, \quad r > 0.$$

To find the max value, we solve for the critical points:

Set $0 = V'(r) = 6 - 6\pi r^2 \Rightarrow 6\pi r^2 = 6 \Rightarrow r = \sqrt{\frac{1}{\pi}}$

For $r < \sqrt{\frac{1}{\pi}}$, $V'(r) > 0$, & for $r > \sqrt{\frac{1}{\pi}}$, $V'(r) < 0$

$\xRightarrow[\text{from class}]{\text{Thm}}$ V has abs max at $\boxed{r = \sqrt{\frac{1}{\pi}}}$.

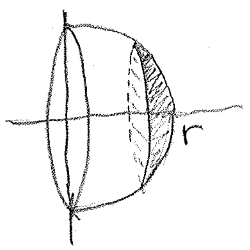
Correspondingly we get $h = \dfrac{6 - 2\pi\left(\sqrt{\frac{1}{\pi}}\right)^2}{\pi\sqrt{\frac{1}{\pi}}} = \boxed{\dfrac{4}{\sqrt{\pi}}}$

Problem 5. [10 points] It's holiday time, and Morris, the Math Department Administrator, is whipping up a batch of his famous eggnog.

(a) Morris is pouring the eggnog into a hemispherical bowl of radius r. Show that when the eggnog in the bowl has depth h, it occupies a volume $V = \pi(rh^2 - \frac{1}{3}h^3)$. (HINT: Depending on how you set up the calculation, the formula $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ may be helpful.)

There are (at least) a couple different ways to set this up. Both involve turning the bowl on its side.

Method 1:



Here we're rotating a quarter-circle of radius r about the x-axis. The equation for the curve is $y = \sqrt{r^2 - x^2}$. To get the volume of the eggnog, which has depth h, we integrate from $x = r-h$ to $x = r$:

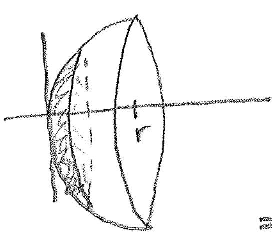
$$V = \int_{r-h}^{r} \pi y^2\,dx = \int_{r-h}^{r} \pi(r^2 - x^2)\,dx = \left(\pi r^2 x - \pi\frac{x^3}{3}\right)\Big|_{r-h}^{r}$$

$$= \pi \cancel{r^3} - \pi\frac{r^3}{3} - \left(\pi r^2(\cancel{r} - h) - \pi\frac{(r-h)^3}{3}\right)$$

$$= -\frac{\pi r^3}{3} + \pi r^2 h + \pi \cdot \frac{r^3 - 3r^2h + 3rh^2 - h^3}{3} \leftarrow \text{using the hint!}$$

$$= \boxed{\pi\left(rh^2 - \frac{h^3}{3}\right)}.$$

Method 2: This time we move the circle so that it's centered at $(r, 0)$, & we integrate from 0 to h. The equation for the circle is



$$y^2 + (x-r)^2 = r^2 \Rightarrow y = \sqrt{r^2 - (x-r)^2}$$

$$\Rightarrow V = \int_0^h \pi y^2\,dx = \pi\int_0^h (r^2 - (x-r)^2)\,dx = \pi\int_0^h [\cancel{r^2} - (x^2 - 2xr + \cancel{r^2})]\,dx$$

$$= \pi\left[-\frac{x^3}{3} + 2r\cdot\frac{x^2}{2}\right]\Big|_0^h$$

$$= \boxed{\pi\left(rh^2 - \frac{h^3}{3}\right)}$$

(b) If $r = 9$ inches and Morris pours the eggnog in the bowl at a constant rate of 2 cubic inches per second, what's the rate of change of h when $h = 3$ inches? (You may use the result of (a) even if you haven't actually done (a).)

By (a), applied w/ $r=9$,

$$V = \pi\left(9h^2 - \frac{h^3}{3}\right)$$

$$\Rightarrow \frac{dV}{dt} = \pi\left(18h\frac{dh}{dt} - h^2\frac{dh}{dt}\right)$$

$$\Rightarrow 2 = \pi(18\cdot 3 - 9)\left.\frac{dh}{dt}\right|_{h=3}$$

$$\Rightarrow \boxed{\left.\frac{dh}{dt}\right|_{h=3} = \frac{2}{45\pi}\ \text{in/sec}}$$

Problem 6. [10 points] Let $I = [0, \pi]$, and let $f(x) = \int_1^{x+\cos x} e^{-t^2}\, dt$ for $x \in I$.

(a) Compute $f'(x)$.

By FTC I, $f'(x) = e^{-(x+\cos x)^2}(1-\sin x)$

[Reason: let $g(x) = \int_1^x e^{-t^2} dt$, $h(x) = x + \cos x$.

Then $f(x) = g(h(x))$, so

$f'(x) = g'(h(x))h'(x) = e^{-(x+\cos x)^2}(1-\sin x)$]

(b) Find the critical points of f on I, and at each critical point, determine whether f has a local max, a local min, or neither. What is the absolute minimum value of f on I?

f is differentiable everywhere, so its critical pts are pts where $f'(x) = 0$.

We solve $e^{-(x+\cos x)^2}(1-\sin x) = 0 \Rightarrow 1 - \sin x = 0 \Rightarrow \sin x = 1$

The only solution to this in $[0, \pi]$ is $\boxed{x = \frac{\pi}{2}}$.

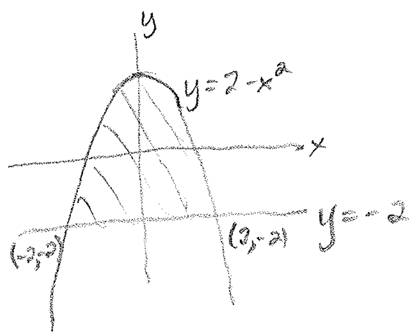
Since $e^{-(x+\cos x)^2}$ is always positive, we see right away that $f'(x) > 0$ for $x < \frac{\pi}{2}$ & for $x > \frac{\pi}{2}$ $\Rightarrow$ f has neither a local max nor local min at $\frac{\pi}{2}$. Moreover, since $f'(x) > 0$ for $x \neq \frac{\pi}{2}$, we conclude that f is increasing on $[0, \frac{\pi}{2}]$ & on $[\frac{\pi}{2}, \pi]$ $\Rightarrow$ f is increasing on all of I

$\Rightarrow$ the abs min of f occurs at the left endpt 0, where f has value

$f(0) = \int_1^{\cos(0)} e^{-t^2} dt = \int_1^1 e^{-t^2} dt = \boxed{0}$

Problem 7. [12 points] Let R be the region in the plane bounded by the curves $y = 2 - x^2$ and $y = -2$. Set up, but *do not evaluate,* integrals to compute the following quantities.

(a) The area of R.



$$A = \int_{-2}^{2} (2-x^2-(-2))\,dx = \int_{-2}^{2} (4-x^2)\,dx$$

(b) The volume of the solid obtained by rotating R about the y-axis.

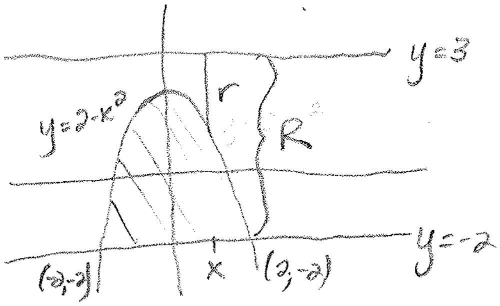
Shell method:

$$V = \int_0^2 2\pi x(4-x^2)\,dx$$ (note if you integrate from -2 to 2, you double-count the volume)

Washer method:

$$V = \int_{-2}^{2} \pi x^2\,dy = \int_{-2}^{2} \pi(2-y)\,dy$$ (note that $y=2-x^2 \Rightarrow x^2 = 2-y$!)

(c) The volume of the solid obtained by rotating R about the horizontal line $y = 3$.



Washer method:

$$V = \int_{-2}^{2} (\pi R^2 - \pi r^2)\,dx = \int_{-2}^{2} \left[\pi(3-(-2))^2 - \pi(3-(2-x^2))^2\right] dx$$

$$= \pi \int_{-2}^{2} \left[25 - (1+x^2)^2\right] dx$$

Shell method: $V = \displaystyle\int_{-2}^{2} 2\pi(3-y)\overbrace{\left(\sqrt{2-y} - (-\sqrt{2-y})\right)}^{\text{height of shell.}}\,dy$

$$= 4\pi \int_{-2}^{2} (3-y)\sqrt{2-y}\;dy$$

(d) The arc length of the curve $y = 2 - x^2$ between its points of intersection with $y = -2$.

$$\frac{dy}{dx} = -2x \quad \Rightarrow L = \int_{-2}^{2} \sqrt{1+\left(\tfrac{dy}{dx}\right)^2}\,dx = \int_{-2}^{2} \sqrt{1+4x^2}\,dx$$

Problem 8. [10 points] In this problem we're going to prove the main theorem used in exponential growth and decay problems: If f is a differentiable function defined on an open interval I and there exists a constant k such that $f'(x) = kf(x)$ for all $x \in I$, then there exists a constant C such that $f(x) = Ce^{kx}$ for all $x \in I$.

(a) Suppose that f satisfies the hypotheses of the theorem, and define $g(x) = f(x)e^{-kx}$. Show that $g'(x) = 0$ for all $x \in I$.

By the product rule,

$$\begin{aligned} g'(x) &= f'(x)e^{-kx} + f(x)(-ke^{-kx}) \\ &= kf(x)e^{-kx} - kf(x)e^{-kx} \quad \text{(by hypothesis on } f\text{)} \\ &= 0 \end{aligned}$$

Remark Many students did this problem by assuming the conclusion of the thm, namely that $f(x) = Ce^{kx}$, which was not allowed!

(b) What does the result of (a) tell you about g? Use this to deduce the conclusion of the theorem. (To be clear, you may use the result of (a) even if you haven't actually done (a).)

$$g'(x) = 0 \ \forall\, x \in I \xrightarrow[\text{from class}]{\text{Thm}} g(x) = C \quad \text{some const } C$$

$$\Rightarrow f(x)e^{-kx} = C \Rightarrow f(x) = Ce^{kx}, \text{ as desired}$$