Final Exam Review
Practice Problem

(1)

(a) $2x+1>0 \iff x>-\frac{1}{2}$

(b) $-\sqrt{x^2-1} > 0 \iff \sqrt{x^2-1} < 0$

$\iff$ No such x

(c) $\ln|x|$, $|x|>0 \iff x \neq 0$

(d) $\ln(\ln x)$, thus, $\ln x > 0$

$\iff x > 1$

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(2) $\ln(e^{\ln e^{\pi}}) = \ln(e^{\pi \ln e})$
$= \ln(e^{\pi}) = \pi \ln(e)$
$= \pi$

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(3) $(2-\ln x)(\ln x) = 0$

b/c $\ln x \neq 0$ for all $x \in \mathbb{R}_{>0}$

$\iff 2-\ln x = 0 \iff \ln x = 2$

$\iff x = e^2$

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(4) Take ln both side

$\ln(e^{x^2-3}) = \ln(1) = 0$

$x^2-3 \iff x^2 = \pm\sqrt{3}$

(5)

(a) $\lim_{x\to 2} \frac{1-\frac{2}{x}}{x^2-4}$

$= \lim_{x\to 2} \frac{\frac{1}{x}(x-2)}{(x+2)(x-2)}$

$= \lim_{x\to 2} \frac{1}{x(x+2)} = \frac{1}{8}$

(b) $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4} \frac{(\sqrt{x}+2)(\sqrt{x}-2)}{\sqrt{x}-2}$

$= \lim_{x\to 4} (\sqrt{x}+2) = 4$

(c) $\lim_{x\to 0^-} \frac{|x|}{x}$, b/c $x<0 \Rightarrow |x| = -x$

$= \lim_{x\to 0^-} \frac{-x}{x} = -1$

(d) $\lim_{x\to 0^+} \frac{1}{1+\frac{1}{x}} = \lim_{x\to 0^+} \frac{x}{x+1}$

$= 0$

(e) Same as (d)

(f) $\cos x = 1$ & $\sin x = 0$ for $x = 0$

thus $\lim_{x\to 0} \frac{x+\sin x}{x+\cos x} = 0$

(g) $\lim_{x\to 0} \frac{\sin^{-1} x}{x}$, it's $\frac{0}{0}$, use L'Hôpital

$= \lim_{x\to 0} \frac{(\sin^{-1}x)'}{1} = \lim_{x\to 0} \frac{1}{\sqrt{1-x^2}}$

$= 1$

(h) $\lim_{x\to 0^+} \left(\frac{e^x - 1 - x}{x(e^x-1)} \right)$, it's $\frac{0}{0}$ use L'Hop

$= \lim_{x\to 0^+} \frac{e^x - 1}{(e^x-1) + xe^x}$ again

$= \lim_{x\to 0^+} \frac{e^x}{e^x + (e^x + xe^x)}$

$= \frac{1}{2}$

(i) $y = x^{\frac{1}{(1-x)}}$, $\ln y = \frac{\ln x}{(1-x)}$

$\lim_{x\to 1^+} \ln y = \lim_{x\to 1^+} \frac{\ln x}{(1-x)}$ ← L'H

$= \lim_{x\to 1^+} \frac{\frac{1}{x}}{-1}$

$= -1$

$\Rightarrow \lim_{x\to 1^+} y = \frac{1}{e}$

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(6)
① $\lim_{x\to 1} \frac{3x^2+3x-6}{x-1}$

$= \lim_{x\to 1} \frac{3(x+2)(x-1)}{(x-1)}$

$= \lim_{x\to 1} 3(x+2) = 9$

② let $f(x) = \frac{3x^2+3x-6}{(x-1)}$

then

$|f(x) - 9| = \left| \frac{3(x+2)(x-1) - 9(x-1)}{(x-1)} \right|$

$= \left| \frac{(3x+6-9)(x-1)}{(x-1)} \right|$

thus, for any $\varepsilon > 0$

$|f(x) - 9| < \varepsilon$

$\Leftrightarrow |(3x-3)(x-1)| < \varepsilon |x-1|$

$\Leftrightarrow 3|(x-1)||(x-1)| < \varepsilon |x-1|$

$\Leftrightarrow |(x-1)||(x-1)| < \frac{\varepsilon}{3} |(x-1)|$

Take $\delta \le \frac{\varepsilon}{3}$, then it works.

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(7) (e.g.) $y = |x|$

(conti but not diff at $x=0$)

But diff implies conti, thus no example if we switch conti & diff

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(8)
① $\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h\to 0^-} \frac{f(h)}{h}$ b/c $h<0$

$= \lim_{h\to 0^-} \frac{-h}{h}$

$= -1$

thus, f is diff at 0

② $\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h\to 0^+} \frac{f(h)}{h} = \lim_{h\to 0^+} \frac{3h^2 - h}{h}$

$= -1$

(9)

$f(t) = h(g(t)) + h(t)\, g^2(t)$

$\Rightarrow f'(t) = h'(g(t))\, g'(t)$

$+ h'(t)\, g^2(t)$

$+ 2h(t)\, g(t)\, g'(t)$

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(10)

$\frac{d}{d\beta} F(\beta) = \frac{d}{d\beta}\left(\Psi(\Phi(\beta))\right)$

$= \frac{1}{2}(3\Phi)^{-\frac{1}{2}} \cdot \frac{d}{d\beta}(3\Phi(\beta))$

$= \frac{3}{2}(3\Phi)^{-\frac{1}{2}} \cdot \Phi'(\beta)$

$= \frac{3}{2}(3\Phi)^{-\frac{1}{2}}(3\beta^2 - 1)$

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(11)

$(f \circ g \circ h)'(x)$

$= f'(g \circ h)(x) \cdot g'(h(x)) \cdot h'(x)$

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(12)

(a) It's conti as long as $x \neq 10$

b/c $f(x)$ is diff

(b)

Further,

$\lim_{x \to 10^+} g(x) = \lim_{x \to 10^-} g(x) = f'(10)$

thus, in fact

$g(x)$ is conti for all x

if $g(10) = f'(10)$

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(13)

If $f(x), g(x)$ are polynomials

& $h(x) = \frac{g(x)}{f(x)}$

$h'(x) = \frac{g'(x)f(x) - f'(x)g(x)}{f'(x)}$ ⇐ rat'l ft

b/c polynomials

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(14)

(a) $2(x-1) + 2y\frac{dy}{dx} = 0$

$\Leftrightarrow \frac{dy}{dx} = -\frac{(x-1)}{y}$

(b) $y^2 + 2y\frac{dy}{dx} + x^2\frac{dy}{dx} + 2xy = 0$

$\frac{dy}{dx}(2y + x^2) = -y^2 - 2xy$

$\Rightarrow \frac{dy}{dx} = \frac{-y^2 - 2xy}{2y + x^2}$

(c) $3x^2 + 3y^2\frac{dy}{dx} = 3x^2y^3 + 3y^2x^3\frac{dy}{dx}$

$\Leftrightarrow \frac{dy}{dx}(y^2 - y^2x^3) = x^2y^3 - x^2$

$\Leftrightarrow \frac{dy}{dx} = \frac{x^2(y^3 - 1)}{y^2(1 - x^3)}$

(15)

(a)

$$f'(x) = \sqrt{1-(\cos x)^2}\cdot(\cos x)'$$

$$= -\sqrt{1-(\cos x)^2}\,\sin x$$

(b)

$$f'(x) = -\left(x^2 - \sin^2 x^2\right)\cdot(2x)$$

(16)

(a)

$$G(-2) = -2\int_{-2}^{2}\frac{\sin\frac{\pi}{t+4}}{(t+4)^2}\,dt$$

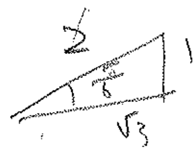
let $\frac{\pi}{t+4} = u$ ⇝ $t=-2 \Rightarrow u=\frac{\pi}{2}$; ⇝ $t=2 \Rightarrow u = \frac{\pi}{6}$

$$\Rightarrow -\pi\frac{1}{(t+4)^2}\,dt = du$$

$$\Rightarrow \frac{1}{(t+4)^2}\,dt = -\frac{1}{\pi}\,du$$

$$\Rightarrow G(-2) = (-2)\int_{\frac{\pi}{2}}^{\frac{\pi}{6}} -\frac{1}{\pi}\sin u\,du$$

$$= -\frac{1}{\pi}(-2)\int_{\frac{\pi}{2}}^{\frac{\pi}{6}}\sin u\,du$$



$$= \frac{2}{\pi}\Big[-\cos u\Big]_{\frac{\pi}{2}}^{\frac{\pi}{6}}$$

$$= \frac{2}{\pi}\left[-\frac{\sqrt{3}}{2}\right] = -\frac{\sqrt{3}}{\pi}$$

(b)

$$G(x) = x\int_{-x}^{x}\frac{\sin\frac{\pi}{t+4}}{(t+4)^2}\,dt$$

As in (a) $\frac{\pi}{t+4} = u$ ⇝ $t=-x \Rightarrow u = \frac{\pi}{4-x}$; ⇝ $t=x \Rightarrow u=\frac{\pi}{x+4}$

$$\Rightarrow G(x) = \left(-\frac{1}{\pi}\right)x\Big[-\cos u\Big]_{\frac{\pi}{4-x}}^{\frac{\pi}{x+4}}$$

$$= -\frac{x}{\pi}\left(-\cos\frac{\pi}{x+4} + \cos\frac{\pi}{4-x}\right)$$

(c)

(i) From def of $G(x)$

$$G(x) = x\int_{-x}^{x}\frac{\sin\frac{\pi}{t+4}}{(t+4)^2}\,dt$$

$$G'(x) = \int_{-x}^{x}\frac{\sin\frac{\pi}{t+4}}{(t+4)^2}\,dt$$

$$+\,x\left(\frac{\sin\frac{\pi}{x+4}}{(x+4)^2} - \frac{\sin\frac{\pi}{4-x}}{(4-x)^2}\right)$$

or you can differentiate (b)

(17)

(a) Extend, polynomial

(b) let $u = x^3 + x + 1$

$\Rightarrow du = (3x^2+1)dx$

$$\Rightarrow \int \frac{2(3x^2+1)}{x^3+1+1}\,dx$$

$$= \int \frac{2}{u}\,du$$

$$= 2\ln|u| + C$$

(c) b/c $(\sin^{-1} v)' = \frac{1}{\sqrt{1-v^2}}$

$$\Rightarrow \int_{1/2}^{1} \frac{1}{\sqrt{1-v^2}}\,dv = \left[\sin^{-1} v\right]_{1/2}^{1}$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

(d) let $\ln x = u \Rightarrow \frac{1}{x}dx = du$

$$\Rightarrow \int_1^2 u\,du = \left[\frac{1}{2}u^2\right]_1^2 = \frac{3}{2}$$

(e) let $f(x) = u \Rightarrow f'(x)dx = du$

$$\Rightarrow \int_{f(a)}^{f(b)} \frac{1}{u}\,du = \ln\left|\frac{f(b)}{f(a)}\right|$$

(f) Use $\tan x = \frac{\sin x}{\cos x}$

& $\sin x = (\cos x)'$ & (e)

(g) $u = 3x+2 \Rightarrow du = 3dx$

$$\Rightarrow \int \frac{1}{u}\left(\frac{1}{3}du\right) = \frac{1}{3}\ln|3x+2| + C$$

(h) $e^{x^2} \cdot 2x$

(i) $\int e^{ax+b}dx = \int e^b \cdot e^{ax}dx$

$$= e^b \int e^{ax}dx$$

$$= e^b\left(\frac{1}{a}e^{ax} + C\right)$$

(j) let $e^{-2x} = u$

$\Rightarrow -2e^{-2x}dx = du$

$$\Rightarrow \int -\frac{1}{2}\sin(u)\,du = -\frac{1}{2}\int \sin u\,du$$

$$= \frac{1}{2}\cos u + C$$

$$= \frac{1}{2}\cos(e^{-2x}) + C$$

(k) $u = -x^2$

$\Rightarrow du = -2x\,dx$

$$\int_{-1}^{-4} -\frac{1}{2}e^u\,du = -\frac{1}{2}\left[e^u\right]_{-1}^{-4}$$

$$= -\frac{1}{2}\left(\frac{1}{e^4} - \frac{1}{e}\right)$$

(l) $u = e^x \Rightarrow du = e^x\,dx$

$$\Rightarrow \int \sec^2(u)\,du = \tan u + C$$
$$= \tan(e^x) + C$$

(m) $$\int_{-2}^{2} \frac{1}{4+t^2}\,dt$$

$t = 2u \qquad dt = 2\,du$

$$\Rightarrow \frac{1}{2}\int_{-1}^{1} \frac{2}{4+4u^2}\,du$$

$$\Rightarrow \int_{-1}^{1} \frac{2}{4(1+u^2)}\,du$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{1}{(1+u^2)}\,du$$

But we know $(\tan^{-1}u)' = \frac{1}{1+u^2}$

$$\Rightarrow \frac{1}{2}\left[\tan^{-1}u\right]_{-1}^{1} = \frac{\pi}{4}$$

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(18) $\sin x$ = odd ftn

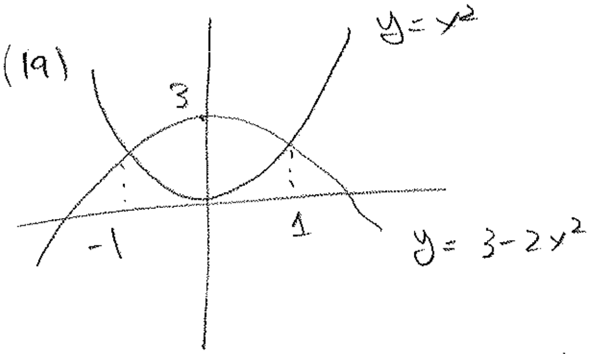
$x^3\cos x$ = odd "

$\Rightarrow \sin x - x^3\cos x =$ "

& $(\sin x - x^3\cos x)^5$ = odd ftn (check)

thus $\int_{-\pi}^{\pi} (\sin x - x^3\cos x)^5\,dx = 0$

(19)



$(x^2 = 3 - x^2 \Leftrightarrow x = \pm 1)$

thus
$$A = \int_{-1}^{1} (3-2x^2) - x^2\,dx$$
$$= \int_{-1}^{1} 3 - 3x^2\,dx$$
$$= \left[3x - x^3\right]_{-1}^{1} = (2+2) = 4$$

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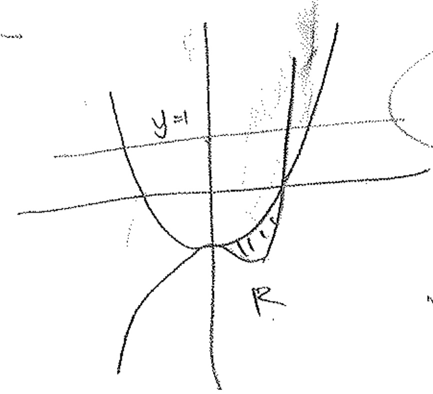
(20)

(a) Intersection of $\begin{cases} y = x^2-1 \leftarrow C_1 \\ y = x^3-1 \leftarrow C_2 \end{cases}$

$x^2 - 1 = x^3 - 1 \Leftrightarrow x^2 = x^3 \Leftrightarrow x^2(1-x) = 0$

$\Leftrightarrow x = 0, 1$

thus $(0,-1), (1,0)$



C_1 lies above

check for example by plugging in $x = \frac{1}{2}$