MATH 108
FALL 2013
FINAL EXAM REVIEW

Definitions and theorems. The following definitions and theorems are fair game for you to have to state on the exam.

Definitions:

- Limit (precise δ-ϵ version; §2.4, Def. 2)
- Continuous at a number (§2.5, Def. 1) and on an interval (§2.5, Def. 3)
- Derivative (§2.7, Def. 4 and/or 5)
- Critical number (§4.1, Def. 6)
- Inflection point (§4.3, p. 294)
- Antiderivative (§4.9, p. 344)

Theorems:

- Squeeze Theorem (§2.3, Th. 3)
- Intermediate Value Theorem (§2.5, Th. 10)
- Differentiability implies continuity (§2.8, Th. 4)
- Product Rule (§3.2, p. 185)
- Quotient Rule (§3.2, p. 187)
- Chain Rule (§3.3, p. 199)
- Extreme Value Theorem (§4.1, Th. 3)
- Rolle's Theorem (§4.2, p. 284)
- Mean Value Theorem (§4.2, p. 285)
- First Derivative Test (§4.3, p. 291)
- Second Derivative Test (§4.3, p. 295)
- Fundamental Theorem of Calculus Part 1 (§5.3, p. 388)
- Fundamental Theorem of Calculus Part 2 (§5.3, p. 391)
- Substitution Rule, a.k.a. u-substitution (§5.5, Th. 4)

Practice problems.

(1) For what values of x are the following functions defined?
  (a) $\ln(2x+1)$
  (b) $\ln(-\sqrt{x^2-1})$
  (c) $\ln|x|$
  (d) $\ln(\ln x)$

(2) What is the value of the expression $\ln\left(e^{\ln e^{\pi}}\right)$?

(3) Solve the equation
$$(2-\ln x)(\ln x)=0.$$

(4) Solve the equation
$$e^{x^2-3}=1.$$

(5) Evaluate the following limits, or state that they do not exist
  (a) $\displaystyle\lim_{x\to 2}\frac{1-\frac{2}{x}}{x^2-4}$

(b) $\lim_{x\to 4} \dfrac{x-4}{\sqrt{x}-2}$
(c) $\lim_{x\to 0^-} \dfrac{|x|}{x}$
(d) $\lim_{x\to 0^+} \dfrac{1}{1+\frac{1}{x}}$
(e) $\lim_{x\to 0} \dfrac{1}{1+\frac{1}{x}}$
(f) $\lim_{x\to 0} \dfrac{x+\sin x}{x+\cos x}$

(6) Find $\lim_{x\to 1} \frac{3x^2+3x-6}{x-1}$ and give a δ-ϵ proof that you're right.

(7) Give an example of a function which is continuous everywhere but not differentiable everywhere. Can you give an example where "continuous" and "differentiable" are interchanged?

(8) Let

$$f(x) = \begin{cases} -x & x < 0; \\ 3x^2 - x & x \geq 0. \end{cases}$$

Is f differentiable at 0?

(9) Find $f'(t)$ if $f(t) = h\big(g(t)\big) + h(t)g^2(t)$, where g and h are differentiable.

(10) Suppose $F(\beta) = \Psi\big(\Phi(\beta)\big)$, $\Psi(\Phi) = \sqrt{3\Phi}$, and $\Phi(\beta) = \beta^3 - \beta$. Find $\frac{d}{d\beta}F(\beta)$.

(11) Let f, g, and h be differentiable functions on all of $\mathbb{R}$. Express $(f \circ g \circ h)'(x)$ in terms of f, g, h, and their derivatives.

(12) Suppose f is differentiable on $\mathbb{R}$, and let

$$g(x) = \frac{f(x) - f(10)}{x - 10}.$$

(a) At which points in $\mathbb{R}$ is g continuous? Why?
(b) How should g be defined at 10, in terms of f, so that g is continuous at 10?

(13) Prove that the derivative of a rational function is a rational function. (In doing the proof, you may take it as known that the derivative of a polynomial is a polynomial.)

(14) Find $\frac{dy}{dx}$ in each of the following.
(a) $(x-1)^2 + y^2 = 5$
(b) $xy^2 + yx^2 = 1$
(c) $x^3 + y^3 = x^3y^3$

(15) Compute $f'(x)$ for each of the following f.
(a) $f(x) = \displaystyle\int_1^{\cos x} \sqrt{1-t^2}\,dt$
(b) $f(x) = \displaystyle\int_{x^2}^{1} (t - \sin^2 t)\,dt$

(16) Let

$$G(x) = \int_{-x}^{x} x \frac{\sin \frac{\pi}{t+4}}{(t+4)^2}\,dt$$

for $x \in (-4, 4)$.

(a) Find $G(-2)$.
(b) Find $G(x)$ as an expression in x. (HINT: Part (a) is designed to help you think about what to do with the x in the integrand.)
(c) Find $G'(x)$ in two ways: by applying FTCI, and by differentiating your expression from (b).

(17) Compute each of the following.
(a) $\displaystyle\int_0^1 x^2(2x-1)(x+2)\,dx$
(b) $\displaystyle\int \frac{6x^2+2}{x^3+x+1}\,dx$
(c) $\displaystyle\int_{1/2}^1 \frac{1}{\sqrt{1-v^2}}\,dv$
(d) $\displaystyle\int_e^{e^2} \frac{\ln x}{x}\,dx$
(e) $\displaystyle\int_a^b \frac{f'(x)}{f(x)}\,dx$ (HINT: Do a u-substitution)
(f) $\displaystyle\int_0^{\pi/4} \tan x\,dx$ (HINT: Use the previous integral)
(g) $\displaystyle\int \frac{1}{3x+2}\,dx$
(h) $\displaystyle\frac{d}{dx}\left(e^{x^2}\right)$
(i) $\displaystyle\int e^{ax+b}\,dx$
(j) $\displaystyle\int \frac{\sin(e^{-2x})}{e^{2x}}\,dx$
(k) $\displaystyle\int_{-1}^2 xe^{-x^2}\,dx$
(l) $\displaystyle\int e^x\sec^2(e^x)\,dx$
(m) $\displaystyle\int_{-2}^2 \frac{1}{4+t^2}\,dt$

(18) Find $\int_{-\pi}^{\pi}(\sin x - x^3\cos x)^5\,dx$.

(19) Find the area of the region bounded by $y=x^2$ and $y=3-2x^2$.

(20) (a) Find the intersection points of the curves $y=x^2-1$ and $y=x^3-1$. Which curve lies above the other, and where? Make a quick sketch of the two graphs. Let R be the region in the plane bounded by the curves.
(b) Write down an integral that gives the area of R. Evaluate it only if you want more integration practice.
(c) Find the volume of the solid obtained by rotating R about the y-axis. Write down the volume integral we'd have gotten by revolving R about the line $y=-1$ instead. (Evaluate it only if you want more practice.)

(21) Problems on exponential growth and decay in the text: 3.8.1, 3.8.7, 3.8.11, 3.8.19

(22) A colony of 100 genetically engineered *mutant attack monkeys* has just escaped from a renegade professor's lab deep in the bowels of JHU. With no natural predators, and an essentially unlimited supply of bananas from Baltimore's grocery stores, the population grows exponentially. In three year's time the city is overrun with roughly $50,000$ monkeys.

(a) Assuming the monkeys continue their conquest of the city unchecked, estimate the population of the colony 6 years after the initial escape. By all means, leave your answer in terms of e and ln.

(b) Estimate the amount of time it takes for the population to triple from its original amount.

(23) Compute $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{\sin^{-1} 2x}$.

(24) Compute $\displaystyle\lim_{x\to\pi/3}\left(\frac{1}{x-\pi/3}\int_{\pi/3}^{x}\frac{dt}{\cos^4 t}\right)$.

(25) If $g(1) = 2$ and $g'(1) = 5$, compute $\displaystyle\lim_{h\to 0}\frac{g\big((h-1)^2\big)-2}{h}$.

(26) Compute $\displaystyle\frac{d}{dx}\left((\cos x)^{\cos^{-1} x}\right)$.

(27) Let $f(x) = \displaystyle\int_{-x^2}^{x^4}(\ln t)^2\,dt$. Compute $f'(x)$.

(28) Let f be defined on an open interval I.

(a) Define "f is decreasing on I."

(b) Suppose that $f'(x) < 0$ for all $x \in I$. Prove that f is decreasing on I. (HINT: Let x_1, $x_2 \in I$ with $x_1 < x_2$. Apply the Mean Value Theorem to f on $[x_1, x_2]$.)

(29) If $\lim_{x\to a} f(x)$ exists, is it necessarily true that

$$\lim_{x\to a}|f(x)| = \left|\lim_{x\to a} f(x)\right|?$$

If so, explain why ("explain" in a math class usually means to cite a theorem or give a proof). If not, give a counterexample.

(30) Find an equation for the tangent line to the curve $y\sin 2x = x\cos 2y$ at the point $(\frac{\pi}{2}, \frac{\pi}{4})$.

(31) The volume of a cylinder of radius r and height h is $\pi r^2 h$. By the Pythagorean theorem, the diagonal L of the cylinder satisfies $L^2 = (2r)^2 + h^2$. Of all cylinders of diagonal length L, determine the height and radius of the one with maximum volume.

(32) The minute hand of a clock is 6 inches long. Starting from noon, how fast is the area of the sector swept out by the minute hand increasing in in^2/min at any instant? (HINT: The area of a sector with angle θ and radius r is $r^2\theta/2$.)

(33) In this problem we are going to determine whether e^π or π^e is bigger.

(a) To begin, define $f(x) = \frac{\ln x}{x}$ for $x > 0$. Find the critical points of f.

(b) At each critical point you found in (a), determine whether f has an absolute max, an absolute min, or neither.

(c) What can you say about $f(e)$ relative to $f(\pi)$? (Smaller, larger, the same, can't tell....)

(d) Use (c) and some algebra to determine which is larger, e^π or π^e.

(34) (a) Let R be the region bounded inside the curve

$$|x|^{1/2} + |y|^{1/2} = r^{1/2},$$

where r is a positive constant. (To give you an idea of the picture, the curve shown in §3.5, Exercise 30 looks very similar.) Show that R has area $2r^2/3$.

(b) Let S be the right cone of height h whose base is the region R in part (a). Show that S has volume $2r^2h/9$. (HINT: By a "right cone," I mean that the base R scales linearly down to a point as we imagine moving along the height of S; more familiar examples of right cones are right circular cones, where the base is a circle, and pyramids, where the base is a square. In fact, if you understand the calculation of the volume of a pyramid that we did in class, then you know everything you need to do this problem.)

(35) Suppose that S is a solid obtained by rotating a region R about a horizontal line. If you want to compute the volume of S by integrating along the x-axis, will you use the disk/washer method or the shell method? What about if you want to integrate along the y-axis? What happens if S is obtained by rotating about a vertical line instead?