Practize

1 (40 pts.) Evaluate the following limits.

a) $\lim_{x\to 0}(\cos x - (x+2)^3)$

$$= \lim_{x\to 0} \cos x - \lim_{x\to 0}(x+2)^3$$

$$= \cos 0 - (0+2)^3$$

$$= 1 - 8 = -7$$

b) $\lim_{x\to\infty} \frac{x^3+100x^2+x}{5x^5+2x^2+2}$

$$= \lim_{x\to\infty} \frac{x^{-2}+100x^{-3}+x^{-4}}{5+2x^{-3}+2x^{-5}} \qquad \left(\lim_{x\to\infty} \frac{1}{x^r} = 0 \quad r>0\right)$$

$$= \frac{0+0+0}{5+0+0} = 0$$

c) $\lim_{x\to 1} \arcsin(\frac{\sqrt{x}-1}{x-1})$

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \frac{d}{dx}\sqrt{x}\Big|_{x=1} = \frac{1}{2\sqrt{x}}\Big|_{x=1} = \frac{1}{2}$$

$$\therefore \lim_{x\to 1} \arcsin\left(\frac{\sqrt{x}-1}{x-1}\right) = \arcsin\left(\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}\right) = \arcsin\frac{1}{2} = \frac{\pi}{6}$$

d) $\lim_{x\to\infty}(\sqrt{x^2-1} - x)$

$$\because \sqrt{x^2-1} - x = \frac{(\sqrt{x^2-1}-x)(\sqrt{x^2-1}+x)}{\sqrt{x^2-1}+x} = \frac{-1}{\sqrt{x^2-1}+x}$$

$$\therefore \lim_{x\to\infty}(\sqrt{x^2-1}-x) = \lim_{x\to\infty}\frac{-1}{\sqrt{x^2-1}+x} = \lim_{x\to\infty}\frac{-1/x}{\sqrt{1-\frac{1}{x^2}}+1} = \frac{0}{1+1} = 0$$

2 (20 pts.) Find asymptotes (horizontal and vertical) for the function

$$f(x) = \begin{cases} 1 - \frac{1}{1+x} & x < -1 \\ \tan \frac{\pi x}{2} & -1 < x < 2 \\ \frac{1}{x^2} & x > 2 \end{cases}$$

horizontal, if exist,

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x^2} = 0$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(1 - \frac{1}{1+x}\right) = 1 - \lim_{x\to-\infty} \frac{1/x}{1+\frac{1}{x}} = 1 - \frac{0}{1+0} = 1$

∴ two horizontal asymptotes: $y=0$ and $y=1$

vertical.

i) $1 - \frac{1}{1+x}$ defined for any $x<-1$

$\Rightarrow$ only possible vertical: $\lim_{x\to-1^-} \left(1 - \frac{1}{1+x}\right) = \lim_{t\to0^+} \left(1 + \frac{1}{t}\right) = \infty$

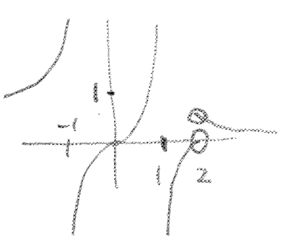
$-t = x+1 < 0$ for $x<-1$

$\Rightarrow$ $\underline{x=-1}$ vertical

ii) $-1 \le x \le 2$ since $\tan\frac{\pi x}{2} = \frac{\sin\frac{\pi}{2}x}{\cos\frac{\pi}{2}x}$

the only possible vertical: $\cos\frac{\pi x}{2} = 0 \Rightarrow x = \pm 1$

for $x=1$, $\cos\frac{\pi}{2}x = 0$ $\sin\frac{\pi}{2}x = 1$ $\Rightarrow \lim_{x\to1^-} \tan\frac{\pi}{2}x = \infty$



$\Rightarrow$ $x=1$ vertical

iii) $x>2$ $\frac{1}{x^2}$ defined for any $x>0$ no vertical

∴ $\boxed{\begin{array}{ll} y=0,\ 1 & \text{horizontal} \\ x=\pm1 & \text{vertical.} \end{array}}$

3 (20 pts.)

For the following functions $y = f(x)$, determine if it is invertible. If not, explain the reason, if yes, find the inverse function $y = f^{-1}(x)$.

a) $f(x) = \frac{1}{\sqrt{x^2-1}}$

In domain, the function is even : $f(-x) = f(x)$
$(|x|>1)$

$\Rightarrow$ not invertible

e.g. $f(-2) = \frac{1}{\sqrt{2^2-1}} = \frac{1}{\sqrt{3}} = f(2)$

b) $f(x) = \frac{1}{\sqrt{x-1}}$

Yes. $y = f(x) = \frac{1}{\sqrt{x-1}}$ Domain of $f = \{x \mid x>1\} = (1,\infty)$

Range of $f = \{y \mid y>0\} = (0,\infty)$

i) $y = f(x) \Leftrightarrow x = f^{-1}(y)$
$\Downarrow$
$y^2 = \frac{1}{x-1} \Rightarrow x-1 = \frac{1}{y^2} \Rightarrow x = 1 + \frac{1}{y^2}$

ii) ~~change~~ $x = f^{-1}(y) = 1 + \frac{1}{y^2}$

iii) change $x, y \Rightarrow y = f^{-1}(x) = 1 + \frac{1}{x^2}$

Domain of $f^{-1} = (0,\infty)$

4 (20 pts.) Determine (the equation of) the tangent line to the curve $y = x\sqrt{x}$, that parallel to the line $2y = 3x + 10$. At which point does this tangent line touch the curve? Write out the equation of the normal line through this point.

The line $2y = 3x+10$

$\Leftrightarrow y = \frac{3}{2}x + 5$ it's slope is $\frac{3}{2}$

tangent line. parallel to this line. $\Rightarrow$ slope $m = \frac{3}{2}$.

$y = f(x) = x\sqrt{x} \Rightarrow f'(x) = \frac{d}{dx}(x\sqrt{x}) = \frac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}x^{\frac{1}{2}}$

$\Rightarrow$ points s.t. tangent line parallel to that line

$\Leftrightarrow \frac{3}{2}x^{\frac{1}{2}} = f'(x) = \frac{3}{2}$

$\Leftrightarrow x = 1 \qquad y = f(x) = 1$

$\therefore$ Point $P(x, f(x)) = (1,1)$, slope $\frac{3}{2}$

$\Rightarrow$ $y - 1 = \frac{3}{2}(x-1)$ is the equation of tangent

normal line is the line through P and perpendicular with tangent line

$\Rightarrow$ slope $= -\frac{1}{m} = -\frac{2}{3}$

$\Rightarrow$ equation. $y - 1 = -\frac{2}{3}(x-1)$ normal line