The Axiom of Steenrod Algebra and proof of Adem Relation

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1 The Axiom of Steenrod Square

In this note, we take the coefficient of ordinary cohomology to be $\mathbb{F}_2$.

Theorem 1.1 There exists a natural transformation between cohomology functor $Sq^i\colon H^n(-) \to H^{n+i}(-)$ satisfying the following axioms. These natural transformations are called Steenrod Squares.

1. Sq^i is a homomorphism between $\mathbb{F}_2$-module.
2. $Sq^0 = id$.
3. If $x \in H^n(X)$, then $Sq^n(x) = x^2$, and $Sq^i(x) = 0$ if $i > n$.
4. (Cartan Formula) The relation
$$Sq^n(xy) = \sum_{i+j=n} Sq^i(x)Sq^j(y)$$
holds.
5. Sq^i is a stable operation. In other words, for a coboundary map of long exact sequence $\delta\colon H^n(X) \to H^{n+1}(X, A)$, $Sq^i \circ \delta = \delta \circ Sq^i$ holds.

For a tuple of non-negative integers $I = (i_1, i_2, \cdots i_k)$, we write the composition $Sq^{i_1}Sq^{i_2}\cdots Sq^{i_k}$ as Sq^I. We call k to be the length of I and we write it as $l(I)$. Define the degree of Sq^I to be $i_1 + i_2 + \cdots + i_k$. If $i_1 > 2i_2, i_2 > 2i_3, \ldots, i_{k-1} > 2i_k$ holds, we call I or Sq^I to be admissible. If I is admissible, we define the excess $e(I)$ of I or Sq^I to be $(i_1 - 2i_2) + (i_2 - 2i_3) + \cdots (i_{k-1} - 2i_k)$.

Theorem 1.2 Steenrod Squares satisfy the following properties.

1. (Adem relation) If $a < b$, then
$$Sq^aSq^b = \sum_j \binom{b-1-j}{a-2j} Sq^{a+b-j}Sq^j$$

holds.

2. Sq^1 coincides with the Bockstein homomorphism corresponding the short exact sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$.

Actually, Sq^i can be characterized by the properties in Theorem 1.1. Therefore it is natural to try to deduce Theorem 1.2 from Theorem 1.1.

2 Proof of Adem Relation

I'll introduce the proof of Adem Relation in [1]. Fix a topological space X. Define a ring homomorphism $P(t)\colon H^*(X) \to H^*(X)[t]$ by $x \in H^*(X) \mapsto \sum_i t^i Sq^i(x)$. By abuse of notation, we also use the same symbol $P(s)$ for the ring homomorphism $P(s)\colon H^*(X)[t] \to H^*(X)[t,s]$ defined by $x \in H^*(X) \mapsto \sum_i t^i Sq^i(x)$ and $t \mapsto t$. (Note that these maps are ring homomorphism because of Cartan formula.)

Lemma 2.1 (Bullett-Macdonald) The equation

$$P(t+t^2) \circ P(1) = P(1+t) \circ P(t^2)$$

holds.

In order to prove this lemma, we need the following. Let $x_i \in H^1((\mathbb{R}P^\infty)^{\times q})$ be the pullback of the generator $x \in H^1(\mathbb{R}P^\infty)$ by the projection to i-component. Let $\sigma = x_1 x_2 \cdots x_q \in H^q((\mathbb{R}P^\infty)^{\times q})$.

Lemma 2.2 Elements in $\{Sq^I(\sigma) \mid I : \text{admissible}, \deg(I) \le q\}$ are linearly independent over $\mathbb{F}_2$.

Proof Induction on q. This statement is clear for $q = 1$. Let $\sum_I a_I Sq^I(\sigma) = 0$. We show that $a_I = 0$ by induction on the length of I. Suppose that $a_I = 0$ for I with $l(I) > m$. Then $\sum_{l(I)=m} a_I Sq^I(\sigma) + \sum_{l(I)<m} a_I Sq^I(\sigma) = 0$ holds. Note that, by Künneth formula, $H^{q+r}((\mathbb{R}P^\infty)^{\times q}) = \oplus_s H^s(\mathbb{R}P^\infty) \otimes H^{q+r-s}((\mathbb{R}P^\infty)^{\times q-1})$ holds. Let p the projection to the $s = 2^m$ summand. Also, let $\sigma' = x_2 x_3 \cdot x_q$, so that $\sigma = x_1\sigma'$ holds. Define a tuple of integers J_m to be $J_m = \{2^m, 2^{m-1}, \ldots, 1\}$. Now we claim the following equation.

$$p(Sq^I(\sigma)) = \begin{cases} 0, & l(I) < m \\ x_1^{2^m} \cdot Sq^{I-J_m}(\sigma') & l(I) = m. \end{cases}$$

In fact, by the Cartan formula,

$$Sq^I(\sigma) = Sq^I(x_1 \cdot \sigma') = \sum_{J \le I} Sq^J(x_1) \cdot Sq^{I-J}(\sigma')$$

holds. Also, we can easily check the following equation.

$$Sq^J(x_1) = \begin{cases} x_1^{2^m} & J = J_m \\ 0 & \text{otherwise.} \end{cases}$$

Then we can easily deduce the second equation. In addition, if $l(I) < m$, then $l(J) < m$ holds, so that $p(Sq^I(\sigma)) = 0$ follows. Now we have

$$p(\sum_{l(I)=m} a_I Sq^I(\sigma) + \sum_{l(I)<m} a_I Sq^I(\sigma)) = x_1^{2^m} \cdot (\sum_{l(I)=m} a_I Sq^{I-J_m}(\sigma')) = 0.$$

Because of induction hypothesis on q, $a_I = 0$ for I with $l(I) = m$. Therefore, by induction on $l(I)$, we can deduce that $a_I = 0$ for all I. □

Lemma 2.3 Let C to be a natural transformation between cohomology defined by a linear sum of Steenrod Squares Sq^I with $\deg(I) < q$. If $C(\sigma) = C(x_1 x_2 \cdots x_q) = 0$, then $C = 0$.

Proof Note that all natural transformations between H^q and H^n as a set functor can be written uniquely as $P(Sq^{I_1}, Sq^{I_2}, \ldots, Sq^{I_k})$ where P is a polynomial over $\mathbb{F}_2$ and $I_1, \ldots I_k$ are admissible and excess $< n$. (This can be shown by calculating the cohomology $H^*(K(\mathbb{F}_2, n))$.) In particular, the degree of polynomial P cannot be equal or larger than 2 when $n \leq 2q$, so that the set $\{Sq^I \mid I : \text{admissible}, deg(I) = n - q\}$ becomes a $\mathbb{F}_2$-basis of natural transformations from H^q to H^n. Because that the image of this basis $\{Sq^I(\sigma) \mid I : \text{admissible}, \deg(I) = n - q\}$ is linearly independent according to previous lemma, we know that the map $C \mapsto C(\sigma)$ is injective. □

Therefore, in order to prove Lemma 2.1, it suffices to show that the image of $\sigma = x_1 \cdots x_q$ is equal for both terms. Because $P(t)$ is ring homomorphism, it suffices to check when $q = 1$. In this case, $P(t + t^2) \circ P(1)(x) = x + (1 + t + t^2)x^2 + t^2(1 + t)^2 x^4 = P(1 + t) \circ P(t^2)(x)$ holds. This proves Lemma 2.1.

Deduction of Adem Relations Take $x \in H^i(X)$. By doing calculation, we can check that

$$P(t + t^2) \circ P(1)(x) = \sum_{a,k} (t + t^2)^a Sq^a Sq^k(x),$$

$$P(1 + t) \circ P(t^2)(x) = \sum b, j(1 + t)^{a+b-j} t^{2j} Sq^{a+b-j} Sq^j(x).$$

Now, for a formal Laurent series $f(z) = \sum_{k=-N}^{\infty} a_k z^k$, we define the residue $\mathrm{Res}_{z=0} f$ to be

a_{-1}. Now we have

$$\begin{aligned}
\sum_k Sq^a Sq^k &= \mathrm{Res}_{t+t^2=0} \frac{P(t+t^2) \circ P(1)(x)}{(t+t^2)^{a+1}} \\
&= \mathrm{Res}_{t+t^2=0} \frac{P(1+t) \circ P(t^2)(x)}{(t+t^2)^{a+1}} \\
&= \mathrm{Res}_{t+t^2=0} \sum_{b,j} (1+t)^{b-j-1} t^{2j-a-1} Sq^{a+b-j} Sq^j(x) \\
&= \mathrm{Res}_{t=0} (\sum_{b,j} (1+t)^{b-j-1} t^{2j-a-1} Sq^{a+b-j} Sq^j(x))(t+t^2)' \\
&= \mathrm{Res}_{t=0} \sum_{b,j} (1+t)^{b-j-1} t^{2j-a-1} Sq^{a+b-j} Sq^j(x).
\end{aligned}$$

By looking at the elements with same degree, we see

$$\begin{aligned}
Sq^a Sq^b(x) &= \mathrm{Rez}_{t=0} \sum_j (1+t)^{b-j-1} t^{2j-a-1} Sq^{a+b-j} Sq^j(x) \\
&= \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j(x).
\end{aligned}$$

Note that we used the condition $a < 2b$ for the last equation. $\square$

References

[1] S. R. Bullett, I. G. Macdonald, "On the Adem Relations", Topology, Volume 21, Issue 3, 1982, 329-332.

[2] Larry Smith, "An algebraic introduction to Steenrod algebra", Geometry & Topology Monographs, 11 (2007), 327-348.

[3] J. P. Serre, "Cohomologie modulo 2 des complexes d'Eilenberg-MacLane", Comment. Math. Helv., 27 (1953), 198-232.

[4] N. E. Steenrod, D. B. Epstein, "Cohomology Operations", Princeton University Press, 1962.