Quillen's Theorem

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1 Homology of MU

Recall that the spectrum MU has a canonical complex orientation $t_{MU}\colon \mathbb{C}P^\infty \to MU$. Recall also that $H_*(\mathbb{C}P^\infty) = \mathbb{Z}\{z_0, z_1, z_2, \dots\}$ where each z_i is the dual of the generator $t^i \in H^{2i}(\mathbb{C}P^\infty)$

Fact 1.1 Let $\beta_i \in H^{2i}(BU)$ is the image of $x_i \in H_{2i}(\mathbb{C}P^\infty)$ via the natural map $\mathbb{C}P^\infty \to BU$. Then,

$$H_*(BU) = \mathbb{Z}[\beta_1, \beta_2, \dots]$$

By Thom isomorphism for homology, we have a ring isomorphism $H_*(MU) \cong H_*(BU)$. In particular, $H_*(MU) \cong \mathbb{Z}[b_1, b_2, \dots]$ where each b_i is the image of $z^{i+1} \in H_*(\mathbb{C}P^\infty)$ via the complex orientation $t_{MU}\colon \mathbb{C}P^\infty \to MU$.

Recall that for a complex oriented spectrum E, we have $E_*(\mathbb{C}P^\infty) = \pi_*(E)\{z_0^E, z_1^E, z_2^E, \dots\}$. In fact, the Atiyah-Hirzebruch spectral sequence $H_*(\mathbb{C}P^\infty, \pi_*(E)) \Rightarrow E_*(\mathbb{C}P^\infty)$ collapses.

Lemma 1.2 If E is a complex oriented spectrum, then

$$E_*(MU) = \pi_*(E)[b_1^E, b_2^E, \dots]$$

where each b_i^E is the image of $z_{i+1}^E \in E_{2i+2}(\mathbb{C}P^\infty)$.

Proof By naturality of the Atiyah-Hirzebruch spectral sequence, each $b_i \in H_{2i}(MU, \pi_0(E)) = E_2^{2i,0}$ is a permanent cycle. Since the spectral sequence is multiplicative, this spectral sequence collapses. □

2 Main Theorem

If we have a complex orientation $t_E \in E^2(\mathbb{C}P^\infty)$ for a ring spectrum E, we have a corresponding formal group law $F_E(x, y)$. Then we have a ring homomorphism from a Lazard ring $\theta_E\colon L \to \pi_* E$ classifying the formal group law F_E.

Theorem 2.1（Quillen） The map $\theta_{MU}\colon L \to MU$ is isomorphism.

The strategy to prove this theorem is as follows.

(i) $\pi_*(MU) = \mathbb{Z}[x_1, x_2, \dots]$ with $|x_i| \in \pi_{2i}(MU)$.

(ii) Let h be the Hurewicz map $h\colon \pi_*(MU) \to H_*(MU)$. Then modulo decomposables,

$$h(x_i) = \begin{cases} pb_i & \text{if } i = p^k - 1 \text{ for some prime } p \\ b_i & \text{otherwise.} \end{cases}$$

(iii) Let $g_H(x) \in H_*(MU)[[x]]$ be $g_H(x) = x + b_0x^2 + b_1x^3 + \cdots$. Then show that the formal group law induced by the map $h\theta_{MU}$ is $g_H(g_H^{-1}(x) + g_H^{-1}(y))$.

At this point we can deduce the Quillen's theorem.

Proof of Quillen's theorem. We will show that θ_{MU} is bijection. Recall that $L \cong \mathbb{Z}[t_1, t_2, \dots]$ and modulo decomposables

$$h\theta_{MU}(t_i) = \begin{cases} pb_i & \text{if } i = p^k - 1 \text{ for some prime } p \\ b_i & \text{otherwise.} \end{cases}$$

In particular $h\theta_{MU}$ is injective because L and $H_*(MU)$ are torsion-free and $h\theta_{MU}$ induces isomorphism after tensoring with $\mathbb{Q}$. Therefore θ_{MU} is injective. Now $\theta_{MU}(t_i) = x_i$ modulo decomposables since we know (ii). Since θ_{MU} is surjective on degree 0 and 2, we can inductively show that θ_{MU} is surjective on every degree. $\square$

3 Proof of (iii)

First note that for a complex oriented ring spectrum E, $E \wedge MU$ becomes a complex oriented ring spectrum. In fact, both $\hat{t}_E\colon \mathbb{C}P^\infty \to E \to E \wedge MU$ and $\hat{t}_{MU}\colon \mathbb{C}P^\infty \to MU \to E \wedge MU$ are complex orientations.

Lemma 3.1 $\hat{t}_{MU} = \hat{t}_E + b_1^E \hat{t}_E^2 + b_2^E \hat{t}_E^3 \cdots$ in $(E \wedge MU)^*(\mathbb{C}P^\infty) = E_*(MU)[[\hat{t}_E]]$.

Proof It suffices to show that $\langle \hat{t}_{MU}, z_{i+1}^{E\wedge MU} \rangle = b_i^E$. Recall that

$$\hat{t}_{MU} : \mathbb{C}P^\infty \xrightarrow{t_{MU}} MU \longrightarrow E \wedge MU$$

and

$$z_{i+1}^{E\wedge MU} : S \xrightarrow{z_{i+1}^E} \mathbb{C}P^\infty \wedge E \longrightarrow \mathbb{C}P^\infty \wedge E \wedge MU.$$

Then we can easily compute that

$$\langle \hat{x}_{MU}, z_{i+1}^{E\wedge MU} \rangle = S \xrightarrow{z_{i+1}^E} \mathbb{C}P^\infty \wedge E \xrightarrow{t_{MU}\wedge 1} MU \wedge E = E \wedge MU.$$

This means that $\langle \hat{x}_{MU}, z_{i+1}^{E\wedge MU} \rangle$ is the image of z_{i+1}^E via the complex orientation t_{MU}. Therefore $\langle \hat{x}_{MU}, z_{i+1}^{E\wedge MU} \rangle = b_i^E$. $\square$

Corollary 3.2 Let $g_E(x) \in \pi_*(E \wedge MU)[[x]]$ be $g_E(x) = x + b_1x^2 + b_2x^3 \cdots$. In $\pi_*(E \wedge MU)[[x, y]]$,

$$F_{MU}(x, y) = g_E(F_E(g_E^{-1}(x), g_E^{-1}(y))).$$

Proof

$$
\begin{aligned}
F_{MU}(\hat{x}_{MU} \otimes 1, 1 \otimes \hat{x}_{MU}) &= \alpha^*(\hat{t}_{MU}) \\
&= \alpha^*(g_E(\hat{t}_E)) \\
&= g_E(\alpha^*(\hat{t}_{MU})) \\
&= g_E(F_E(\hat{x}_E \otimes 1, 1 \otimes \hat{x}_E)) \\
&= g_E(F_E(g_E(\hat{x}_{MU} \otimes 1), g_E(1 \otimes \hat{x}_{MU}))).
\end{aligned}
$$

□

By taking $E = H\mathbb{Z}$, we have (iii).

4 Sketch of a proof of (i) and (ii)

Fact 4.1 There is a spectral sequence satisfying the following properties.

(a) $E_2^{*,*} = \mathrm{Ext}_{A_*}^{*,*}(\mathbb{Z}_p, H_*(X, \mathbb{Z}_p))$ where A_* is the dual of Steenrod algebra.

(b) If $H_*(X, \mathbb{Z}_p)$ is of finite type, this spectral sequence converges to the p-primary part of $\pi_*(X)$. In other words, there is a filtration of p-primary part

$$
{}_p\pi_n(X) = F^{0,n} \supset F^{1,n-1} \supset F^{2,n-2} \supset \cdots
$$

and isomorphism $F^{s,t}/F^{s+1,t-1} \cong E_\infty^{s,t}$ where $E_\infty^{s,t} = \bigcap E_r^{s,t}$.

(c) The edge homomorphism

$$
\pi_n(X) \to E_\infty^{0,n} \to E_2^{0,n} \to \mathrm{Hom}_{A_*}^n(\mathbb{Z}_p, H_*(X, \mathbb{Z}_p)) \cong H_n(X, \mathbb{Z}_p)
$$

is the Hurewicz homomorphism.

In particular, we apply $X = MU$ in this spectral sequence. In fact, this spectral sequence collapses, and you can show that every p-primary component is isomorphic to the polynomial ring as a group. With a bit of algebraic argument, you can show that $\pi_*(MU)$ is isomorphic to the polynomial ring. By looking at the argument of taking generators of $\pi_*(MU)$ and using the property (c), you can show (ii). For more details, you can see Switzer's textbook.

References

[1] Robert M. Switzer, “Algebraic Topology - Homology and Homotopy”, Springer-Verlag, 1975.

[2] Douglas C. Ravenel, “Complex Cobordism and Stable Homotopy Groups of Spheres”, AMS Chelsea Publishing, 2003.