NOTES ON J-HOMOMORPHISM

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1. Introduction

To begin with, we generalize the Hopf construction a bit.

Definition 1.1. Let $f\colon S^i \to O(n) \in \pi_i(O(n))$. We define

$$Jf\colon S^{n+i} = S^i \times D^n \cup D^{i+1} \times S^{n-1} \to S^n$$

sending $(x, y) \in S^i \times D^n$ to $f(x)(y) \in D^n/\partial D^n$ and $D^{i+1} \times S^{n-1}$ to a basepoint.

Jf is homotopy equivalent to the map $S^{n+i} = S^i \wedge S^n \to S^n$ induced by $(x, y) \mapsto f(x)(y)$. In other words,

Proposition 1.2. *J is equal to the map induced by*

$$f\colon S^i \to \mathrm{Map}_*(S^n, S^n).$$

From this description, we can prove the following properties.

Proposition 1.3. *$J\colon \pi_i(O(n)) \to \pi_{n+i}(S^n)$ is a group homomorphism.*

Proposition 1.4. *The following diagram commute*

$$\begin{tikzcd}
\pi_i(O(n)) \arrow[r, "J"] \arrow[d, "i_*"'] & \pi_{n+i}(S^n) \arrow[d, "\Sigma"] \\
\pi_i(O(n+1)) \arrow[r, "J"] & \pi_{n+i+1}(S^{n+1})
\end{tikzcd}$$

where $i\colon O(n) \to O(n+1)$ is an inclusion and Σ is a suspension homomorphism.

Therefore we obtain a homomorphism $J\colon \pi_i(O) \to \pi_i^S$. Recall that, by Bott periodicity, $\pi_{4s-1}(O) \cong \mathbb{Z}$. Our goal is to determine the image of J. Let $B_{2s} \in \mathbb{Q}$ be a Bernoulli number.

Theorem 1.5. *The order of $J\colon \pi_{4s-1}(O) \to \pi_{4s-1}^S$ is $m(2s)$, where $m(2s)$ is the denominator of $B_{2s}/4s$.*

In this note, our main goal is to provide the lower bound of the order of $\mathrm{Im}J$, i.e., to show that the order of $\mathrm{Im}J$ is a multiple of $m(2s)$. Giving the upper bound concerns with the Adams conjecture. To show that there are a number of non-trivial elements in π_i^S, we need an invariant to distinguish these elements.

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2. e-invariant

For $f\colon X \to Y$, take $d_\Lambda(f) \coloneqq f^*\colon \tilde{K}_\Lambda(Y) \to \tilde{K}_\Lambda(X) \in \mathrm{Hom}_A(\tilde{K}_\Lambda(Y), \tilde{K}_\Lambda(X)) = \mathrm{Ext}^0_A(\tilde{K}_\Lambda(Y), \tilde{K}_\Lambda(X))$ where $\Lambda = \mathbb{R}$ or $\mathbb{C}$. (A is an abelian category of finitely generated abelian groups with Adams operations.)[1] The $d_\Lambda(f)$ is called the d-invariant of f. If $d_\Lambda(f) = 0$ and $d_\Lambda(\Sigma f) = 0$, then we have the following short exact sequence

$$0 \longrightarrow \tilde{K}_\Lambda(\Sigma X) \longrightarrow \tilde{K}_\Lambda(C_f) \longrightarrow \tilde{K}_\Lambda(Y) \longrightarrow 0$$

where C_f is the cofiber of f. In other words, we obtain an element $e_\Lambda(f) \in \mathrm{Ext}^1_A(\tilde{K}_\Lambda(Y), \tilde{K}_\Lambda(\Sigma X))$. $e_\Lambda(f)$ is called the e-invariant of f. We want to compute this invariant, at least for $f \in \pi_m(S^n)$. Since it is hard to distinguish two different elements directly from definition, we attach a number to $e_\Lambda(f)$ as we did in the Hopf invariant.

Let $f\colon S^{n+q} \to S^q \in \pi^S_n$. Suppose that $q \equiv 0 \bmod 2$ when $\Lambda = \mathbb{C}$ or $q \equiv 0 \bmod 8$ when $\Lambda = \mathbb{R}$. Then $\tilde{K}_\Lambda(S^q) \cong \mathbb{Z}$ with $\psi^k(x) = k^q x$. It can be checked that we can define $e_\Lambda(f)$ when $n = 4s-1$. Now we have $\tilde{K}_\Lambda(C_f) \cong \mathbb{Z} \oplus \mathbb{Z}$ as an abelian group. Let $\xi \in \tilde{K}_\Lambda(C_f)$ be an element which projects to a generator in $\tilde{K}_\Lambda(S^n)$ and $\eta \in \tilde{K}_\Lambda(C_f)$ be a image of a generator of $\tilde{K}_\Lambda(S^{n+q+1})$. Then we must have

$$\begin{aligned} \psi^k \xi &= k^q \xi + \lambda(k^{n+q+1} - k^q)\eta \\ \psi^k \eta &= k^{n+q+1}\eta \end{aligned}$$

for some $\lambda \in \mathbb{Q}$. We write $e_\Lambda(f) \coloneqq \lambda \in \mathbb{Q}/\mathbb{Z}$ by abuse of notation. Then it is easy to see that $\lambda \in \mathbb{Q}/\mathbb{Z}$ is independent of the choice of ξ. Also, we can see that λ is independent of k by computing $\psi^k\psi^l = \psi^l\psi^k$.

Now we have the following diagram.

$$\begin{tikzcd} & \pi^S_{4s-1} \arrow[dr, "e"] & \\ \pi_{4s-1}(O) \arrow[ur, "J"] \arrow[rr] & & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

Our next goal is to compute the value of e for the image of the generator of $\pi_{4s-1}(O)$.

3. value of e-invariant

Note that we have following two maps

$$\pi_{4s-1}(U) \xrightarrow{r} \pi_{4s-1}(O) \xrightarrow[e_{\mathbb{C}}]{e_{\mathbb{R}}} \mathbb{Q}/\mathbb{Z}$$

where r is induced by canonical map $U(n) \to O(2n)$.

Proposition 3.1. $e_{\mathbb{C}} \circ r = 2e_{\mathbb{R}} \circ r$.

Proof. If $4s-1 \equiv 3 \bmod 8$, then $r = 1$ and $e_{\mathbb{C}} = 2e_{\mathbb{R}}$. If $4s-1 \equiv 7 \bmod 8$, then $r = 2$ and $e_{\mathbb{C}} = e_{\mathbb{R}}$. [1] □

[1] The precice definition of A is in [1].

Hence it suffices to compute the $e_{\mathbb{C}}$ of the generator of $\pi_{4s-1}(U)$. Now we can inject $\tilde{K}_{\mathbb{C}}(C_f)$ to $\tilde{H}^*(C_f;\mathbb{Q})$ by the Chern character. In fact, we have

$$\mathrm{ch}\xi = h^q + \lambda h^{4s+q}$$
$$\mathrm{ch}\eta = h^{4s+q}$$

where $h \in H^q(C_f;\mathbb{Q})$ is the generator.

Proof. Use the property that $\mathrm{ch}_{2q} \circ \psi^k = k^q \circ \mathrm{ch}_{2q}$ where ch_{2q} is the degree $2q$ component of ch. [2] □

We need a following lemma.

Lemma 3.2. *Let $f\colon S^{4s-1} \to U(n)$. Then C_{Jf} is homotopy equivalent to the Thom space of the complex vector bundle $E_f \to S^{4s}$ determined by f.*

Proof. Note that E_f is constructed from $D^{4s} \times \mathbb{C}^n \sqcup \mathbb{C}^n$ by identifying (x, v) and $f(x)(v)$. Therefore $T(E_f)$ is constructed from $D^{4s} \times D^{2n} \sqcup D^{2n}$ by identifying $(x, v) \sim f(x)$ and collapsing $D^{4s} \times S^{2n-1} \cup S^{2n-1}$ to a point. In other words, $T(E_f)$ is constructed from attaching a $4s + 2n$-cell to S^{2n} by the attaching map $Jf\colon D^{4s} \times S^{2n-1} \cup S^{4s-1} \times D^{2n} \to S^{2n}$. □

With this identification, we can see that $\xi \in \tilde{K}_{\mathbb{C}}(T(E_f))$ restricts to the generator of $\tilde{K}_{\mathbb{C}}(S^{2n})$. In other words, ξ is the Thom class. We can compute the Chern character of the Thom class. Indeed, we have the following fact.

Theorem 3.3. [3] *Let $\Phi\colon H^*(X;\mathbb{Q}) \to \tilde{H}^*(T(E);\mathbb{Q})$ be a Thom isomorphism and $\xi \in \tilde{K}(T(E))$ be a Thom class. Then we have*

$$\log \Phi^{-1}\mathrm{ch}(\xi) = \sum_j \alpha_j \mathrm{ch}_{2j}(E)$$

where $\log((e^y - 1)/y) = \sum_j \alpha_j y^j/j!$ is a power series expansion.

computation of $e(Jf)$. Let $f\colon S^{4s-1} \to U(n)$ be a generator of $\pi_{4s-1}(U)$. We will compute $\mathrm{ch}(\xi) \in \tilde{H}(C_{Jf})$ where $\xi \in \tilde{K}(C_{Jf}) = \tilde{K}(T(E_f))$ is a Thom class. Since $\mathrm{ch}(\xi) = h^{2n} + \lambda h^{4s+2n}$, we have $\Phi^{-1}\mathrm{ch}(\xi) = 1 + \lambda h^{4s}$. Hence $\log \Phi^{-1}\mathrm{ch}(\xi) = \lambda h^{4s}$. (Note that $\log(1 + z) = z - z^2/2 \ldots.$) On the other hand, $\log \Phi^{-1}\mathrm{ch}(\xi) = \sum_j \alpha_j \mathrm{ch}_j(E_f) = \alpha_{2s}\mathrm{ch}_{4s}(E_f)$ holds by the general formula above. Since f is a generator of $\pi_{4s-1}(U) = \tilde{K}(S^{4s})$, we must have $\mathrm{ch}_{4s}(E_f) = h^{4s}$. Therefore we have $\lambda = \alpha_{2s}$. □

However, we have $\alpha_k = B_k/k$ for $k > 1$ [3]. This is what we wanted.

References

[1] J.F. Adams, On the groups of $J(X)$-IV, Topology, Volume 5, Issue 1, March 1966, pp. 21-71.
[2] J.F. Adams, Vector field on spheres, Ann. Math., 75 (1962), pp. 603-632.
[3] Allen Hatcher, Vector bundles and K-Theory.