Curvature Estimates for Constant Mean Curvature Surfaces in Three Manifolds

by

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Abstract

In this thesis, we give curvature estimates for strongly stable constant mean curvature surfaces in a complete three dimensional manifold. We use a key observation of Colding and Minicozzi to obtain area and small total curvature estimates of constant mean curvature surfaces. Then following Choi and Schoen we show that small total curvatures yield curvature estimates. By giving a much shorter proof, this thesis extends the work of Bérard and Hauswirth, where they gave curvature estimates for constant mean curvature surfaces in a space form.

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Contents

Introduction

Background

Surfaces in three manifolds have been actively studied since the eighteenth century. It is a beautiful topic in itself and it helps us to understand the ambient three manifold because of the interaction between the surface and the topology of the three manifold.

Let M be a three dimensional manifold with Riemannian metric g. For any two vector fields X and Y, we have a Riemannian connection ∇ which defines the directional derivative of Y along X by $\nabla_X Y$. For any immersed surface Σ in M, we denote by X^T and X^N the tangential and normal components of X respect to Σ. Now we define two very important symmetric bilinear forms on Σ.

First fundamental form

$$g_\Sigma(X, Y) = g(X, Y) \tag{1}$$

Here X, Y are vector fields on Σ(and also on M). This is the induced metric g on Σ, which describes distances on the surface.

Second fundamental form

$$A(X, Y) = (\nabla_X Y)^N \tag{2}$$

A is a vector-valued symmetric bilinear form. We can define a real-valued form b by $b(X, Y) = g(A(X, Y), \overrightarrow{n})$. $\overrightarrow{n}$ is the unit normal vector field on Σ.

The second fundamental form is more interesting to us because it describes how the surface curves in the manifold. Let us look at a neighborhood of a point. The second fundamental form b can be expressed locally as a 2×2 symmetric matrix. We find that the two eigenvalues k_1 and k_2 of the matrix b are the maximal and minimal values of the curvatures of all normal slices to the surface through the point. We call them the principal curvatures.

Moreover we define the mean curvature $h = \frac{k_1+k_2}{2}$ as half value of the trace of b and Gauss curvature $K = k_1 \cdot k_2$ as the determinant of b. Since the celebrated *Gauss Theorem egregium* says that the Gauss curvature are determined entirely by the first fundamental form, the mean curvature measures how a surface lies in the manifold.

Definition 0.1 *A constant mean curvature surface is a surface whose mean curvatures equal some constant at any point. We denote the constant h. We call the surface a CMC h-surface.*

When $h \equiv 0$, we call it a minimal surface.

History

Generally constant mean curvature surfaces are not as well understood as minimal surfaces. Until H.Wente's CMC torus was discovered in 1984[We], people knew very few examples of CMC surfaces(not minimal surfaces)(See §1.1) Since then, many CMC surfaces have been discovered by different techniques. However it is not very clear how h(not zero) is restricted by the curvature of the ambient manifold. CMC surfaces need more investigation.

Recently minimal surface theory has been rapidly developed from the perspective of PDEs[CM2]. Since we can locally express a minimal surface(and a CMC surface) as the solution of a second order elliptic PDE equation, it is natural to borrow some PDE ideas to study CMC surfaces. Curvature estimates have played a key role in minimal surface theory. In this work, we are going to obtain curvature estimates for CMC surfaces..

The study of curvature estimates for minimal surfaces goes back at least to E.Heinz's work [He] in 1952. He proved for a solution of a minimal surface equation over a disc $\{x \in \mathcal{R}^2 : |x - x_0| < R\}$, there is a absolute constant β such that

$$(k_1^2 + k_2^2)(x_0) \leq \beta / R^2 \tag{3}$$

where k_1 and k_2 are the principal curvatures of the graph of the solution. When the solution is on the entire $\mathcal{R}^2$, letting $R \to \infty$, we have $k_1 = k_2 = 0$, so the solution is a plane, i.e., the curvature estimate (3) implies Bernstein's theorem in $\mathcal{R}^3$. In 1975, R. Schoen, L. Simon and Yau showed curvature estimates of minimal hyper-surfaces in higher dimensions[ScSimY], which can give Bernstein's theorem for higher dimensions. Note that there are similar results under additional hypotheses by F. Almgren, R.Osserman, L. Simon, B. White etc. [1]

In R. Schoen's fundamental work [Sc](1983), he proved an estimate of the Gauss curvature for *stable* minimal surfaces in $\mathcal{R}^3$, which yielded the Bernstein theorem for complete stable minimal surfaces in $\mathcal{R}^3$. His key idea was to apply the stability inequality[See §1.2] to different well chosen functions.

[1] See [CM1] [CM2] for further reference.

He obtained the result by using Simons' inequality(See §2.1), the stability inequality and the de Giorgi-Moser iteration. Later in 2002 [CM1], T. Colding and W. Minicozzi II found that stability implied certain upper bounds on intrinsic balls (and general domains). Using this observation, they gave more general and useful estimates for stable parametric elliptic integrands, which were very useful in their important study of embedded minimal surfaces in three manifolds[CM3]. It also provides a key ingredient for our work.

Curvature estimates for CMC surfaces were studied by J. Spruck [Sp] for $\mathcal{R}^3$, Ecker and Huisken [EcHu] for higher dimensional Euclidean spaces.[2] The stability of CMC surfaces has been explored by J. Barbosa and M. do Carmo since 1984([BaCa],[BaCaE]). It turned out there were two kinds of stabilities(See §1.2), which is different from the minimal surface case. Then in 1999 P. Bérard and L. Hauswirth [BeHa] gave curvature estimates for (strongly) stable CMC surfaces in a space form. Their work was mostly inspired by the above work of R. Schoen[Sc], which made extensive use of the stability inequality and the de Giorgi-Moser iteration method. From their paper, we can see that, for CMC surfaces in a general three manifold, curvature estimates may become very complicated if we follow their method.

Result

Our work generalizes the results of [BeHa] to strongly stable CMC surfaces in a general three dimensional manifold. The main idea follows [CM1].

[2]See [BeHa] for further reference.

We use a key observation in [CM1] to obtain area and total curvature estimates for strongly stable CMC surfaces. Then we follow H. Choi and R. Schoen [ChSc] to show that small total curvature implies curvature estimates. In [ChSc] they use a different scaling method, which is shorter and more elementary than the de Giorgi-Moser iteration.

We state the main theorem below:

Theorem 0.1 *Let Σ^2 be an immersed (strongly) stable CMC h-surface with trivial normal bundle in a complete three dimensional manifold M, where $|K_M| \leq k^2$. There exists $0 < r_0 < \frac{\pi}{\sqrt{6(h^2+k^2)}}$, such that for any x in Σ with geodesic ball $B_{r_0}(x) \cap \partial\Sigma = \emptyset$, we have, for all $0 < \sigma \leq r_0$,*

$$\sup_{B_{r_0-\sigma}} |A^0(x)|^2 \leq \sigma^{-2} \tag{4}$$

$$\sup_{B_{r_0-\sigma}} |K_\Sigma(x)| \leq C_1\sigma^{-2} \tag{5}$$

Here r_0 depends on h, the curvature tensor of M and its covariant derivative. C_1 is a constant depending on h and k.

We present our work in three sections. Section 1 gives examples and known facts about CMC surfaces. In §1.1 we give some examples in $\mathcal{R}^3$. In §1.2 we induce the first and second variation formulas and the stability inequality. In §1.3 we describe P. Bérard and L. Hauswirth's results for CMC surfaces in a space form [BeHa].

Section 2 proves the tools(inequalities) of curvature estimates we need. In §2.1 we obtain the Simons' inequality for CMC surfaces in a general three manifold. In §2.2 we show a useful version of the mean value inequality for CMC surfaces. In §2.3 we show the estimates of elliptic integrands by T.Colding and W.Minicozzi II[CM1].

Section 3 gives the proof of our main theorem. In §3.1 we obtain area estimates and total curvature estimates which follow from §2.3. In §3.2 we give curvature estimates for a topological disk following [ChSc]. In §3.3 we prove the main theorem and give remarks.

Throughout this thesis, Σ denotes an immersed CMC h-surface with trivial normal bundle in a complete three manifold M. g is the Riemannian metric. We denote the connections ∇^T and ∇, the sectional curvatures K_Σ and K_M, the curvature tensors R and $\overline{R}$ respectively for the surface Σ and the manifold M. A(or A^0) is the (traceless) second fundamental form. Notice that $B_r(x)$ denotes an intrinsic geodesic ball centered at x with radius r.

1 Constant mean curvature surfaces

1.1 Examples in $\mathcal{R}^3$

Locally a surface in a three manifold is just a graph over its tangent plane. To imagine surfaces in a three manifold, it is often useful to look at the graphs in $\mathcal{R}^3$.

Let D be a domain in the (u, v) plane and X be a smooth map from D to $\mathcal{R}^3$. We write

$$X(u, v) = (x(u, v), y(u, v), z(u, v)),\ (u, v) \in D \tag{6}$$

It is called a local parametric surface. In the local frame by X_u, X_v and $\overrightarrow{n} = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}$, $\overrightarrow{n}$ is the unit normal vector field. We compute the fundamental forms to obtain:

$$I = Edu^2 + 2Fdudv + Gdv^2 \tag{7}$$

$$II = Ldu^2 + 2Mdudv + Ndv^2 \tag{8}$$

where

$$E = \langle X_u, X_u \rangle, F = \langle X_u, X_v \rangle, G = \langle X_v, X_v \rangle,$$
$$L = \langle X_{uu}, \overrightarrow{n} \rangle, M = \langle X_{uv}, \overrightarrow{n} \rangle, N = \langle X_{vv}, \overrightarrow{n} \rangle.$$

Moreover we have

$$h = \frac{1}{2} \frac{GL + EN - 2FM}{EG - F^2} \tag{9}$$

$$K = \frac{LN - M^2}{EG - F^2} \tag{10}$$

Now we can find some simple examples by direct computation.

Example 1 (Plane) $X(u, v) = (u, v, au + bv + c)$. The second fundamental form is zero, thus $h = 0$. Here all principal curvatures are zeros.

Example 2 (Helicoid) $X(u, v) = (u \cos v, u \sin v, av + b)$. Here $h \equiv 0$ but the second fundamental form is not identically 0. Helicoid is a minimal surface. For more examples of minimal surfaces please see [Ni].

Example 3 (Sphere) Let S^2 be a sphere centered at the origin with radius r in $\mathcal{R}^3$. Taking the inward normal vector field, the mean curvature $h = 1/r$. In fact, both of the two principal curvatures are $1/r$. Usually the two parts of the sphere cut by a plane are called spherical caps, which are also CMC surfaces.

Example 4 (Cylinder) $X(u, v) = (r \cos u, r \sin u, v)$. It is a rotational surface of a straight line. The two principal curvatures are 0 and $1/r$. So $h = \frac{1}{2r}$. Here the Gauss curvature $K \equiv 0$.

Except for the minimal surfaces, the above examples are very simple. However there is a class of CMC surfaces like the cylinder which are rotational surfaces. Delaunay determined all such surfaces in 1841. We call rotational CMC surfaces Delaunay surfaces.

Let $C : (x(s), y(s))$ be the smooth curve in the plane of $z = 0$ in $\mathcal{R}^3$ parametrized by the arc length s. The rotational surface X generated by C

around the x-axis is

$$X(s,\theta) = (x(s), y(s)\cos\theta, y(s)\sin\theta), 0 \le \theta \le 2\pi$$

We give Delaunay's result below (for a proof see [Ke]).

Theorem 1.1 *For any real numbers b and h, there exists one-parameter family of rotational surfaces $X(s,\theta;h,b)$ that are CMC h-surfaces. Here the generating curves C can be written as*

$$C(s;h,b) := \left(\int_0^s \frac{1+b\sin 2ht}{\sqrt{1+b^2+2b\sin 2ht}}dt, \frac{1}{2|h|}\sqrt{1+b^2+2b\sin 2hs}\right) \quad (11)$$

In 1984, H. Wente discovered an immersion CMC torus in $\mathcal{R}^3$. It has inspired lots of examples by using different techniques.(see [We] and [Ke] for more details.)

We conclude this section with an interesting fact[Ke].

Theorem 1.2 *Let $X(u,v) = (u,v,f(u,v))$ be a map on the open disk $B_R(0)$ on the uv-plane. If $|h| \ge a > 0$ on $B_R(0)$ for some constant a, then*

$$R \le \frac{1}{a}$$

1.2 Stability and variation formulas

From the point of view of calculus of variations, constant mean curvature surfaces are critical points of certain area functionals. R.Schoen's work[Sc] established curvature estimates for stable minimal surfaces. But stability of CMC surfaces is a little different from the minimal surface case. We investigate the stability following Barbosa and do Carmo([BaCa]).

Let Σ be an immersed surface in a three manifold M with trivial normal bundle.

Definition 1.1 *A variation of* Σ *is a differential map* $F : \Sigma \times (-\varepsilon, \varepsilon) \to M$ *such that*

$\Sigma_t(\cdot) = F(\cdot, t)$ *is an immersion for each* t, $\Sigma_0 = \Sigma$, *and* $\partial(\Sigma_t) = \partial\Sigma$.

We define the area functional by $Area(t) = \int_\Sigma d\Sigma_t$ where $d\Sigma_t$ is the area element of Σ in the metric induced by Σ_t.

The vector field $\frac{\partial F}{\partial t}$ restricted on Σ is often called the variation vector field. Let $\overrightarrow{n}$ be the normal vector field, we denote the normal component of $\frac{\partial F}{\partial t}$ by $f = \langle \frac{\partial F}{\partial t}, \overrightarrow{n} \rangle$. We obtain the two variation formulas by direct computation.

Proposition 1.1 (First variation formula) $\frac{dArea(t)}{dt}(0) = -\int_\Sigma 2hf d\Sigma$.

Proposition 1.2 (Second variation formula) *For the critical points,*

$$\frac{d^2 Area(t)}{dt^2}(0) = -\int_\Sigma (f \triangle f + (|A|^2 + Ric(\overrightarrow{n}, \overrightarrow{n}))f^2) d\Sigma,$$

where $\triangle_\Sigma$ *is the Laplacian,* $|A|^2$ *is the square norm of the second fundamental form and* Ric *is the Ricci tensor of* M.

When $h \equiv 0$, the minimal surfaces are the critical points of the area functional. Stable minimal surfaces are the points such that $Area''(t) \geq 0$. These include area-minimizing surfaces. In particular, minimal graphs are area-minimizing.

When $h \neq 0$, CMC surfaces are not critical points for all variations. But they are critical points for all *volume-preserving variations*, which are variations such that $\int_\Sigma f d\Sigma = 0$. We denote the set of such variations by $\mathcal{F}$.

Now we could define that (weakly) stable CMC surfaces are the points such that $Area''(t) \geq 0$ for all *volume-preserving variations.*

However, when applying the stability condition, general variations are more easy to use. So we define that *strongly stable CMC surfaces* are the points such that $Area''(t) \geq 0$ for all variations. Throughout this thesis, we work on strongly stable CMC surfaces.

For (strongly) stable CMC surfaces, we define the stability operator

$$L = \triangle + |A|^2 + Ric(\overrightarrow{n}, \overrightarrow{n}). \tag{12}$$

We state two useful propositions.

Proposition 1.3 *Σ is strongly stable if and only if -$\int_\Sigma f \cdot Lf \geq 0$ for all $f \in C_0^\infty(\Sigma)$.*

Proposition 1.4 *(Stability inequality) for all $f \in C_0^\infty(\Sigma)$,*

$$\int_\Sigma (\inf_M Ric_M + |A|^2) f^2 \leq \int_\Sigma |\nabla f|^2. \tag{13}$$

The relationship between weak stability and strong stability can be explained by the Morse index.

Definition 1.2 *Morse index is the number of negative eigenvalues of the stability operator L acting on smooth functions.*

Strong stability implies that the Morse index is 0. By the study of Barbosa and Bérard[BaBe], weak stability implies that the Morse index is 1 or 0.

At the end of the section, we shall give a very useful argument for stability from Fischer-Colbrie and Schoen[FiSc].

Theorem 1.3 *Let L be the stability operator. $-L \geq 0$ on $C_0^\infty(\Sigma)$ if and only if there exists a function u such that $u > 0$ and $Lu = 0$ on Σ.*

1.3 Surfaces in space forms

In this section we describe the curvature estimate for CMC surfaces in a space form. J.Spruck[Sp] established this for graphs when the ambient space is $\mathcal{R}^3$ by using PDE methods. When the ambient three manifold is a space form, i.e. the constant curvature space, the curvature estimate was given by P. Bérard and L. Hauswirth [BeHa].

Their work was inspired by R.Schoen[Sc]. Because of the length of their proof, we only give a short description here.

Sobolev inequality

$$\left(\int_\Sigma f^2\right)^{1/2} \leq A_\Sigma \left(\int_\Sigma |\nabla f|^2 \ + \int_\Sigma B_\Sigma f\right). \tag{14}$$

Here the constant A_Σ and the non-negative function B_Σ are depending on the geometry of the surface. The Sobolev inequality holds in many situations. For stable surfaces, we have the stability inequality (13).

Simons' inequality $\triangle(|A^0|^2 + v) \geq -f(|A^0|^2 + v)$.

It holds for CMC surfaces in space forms. Here v and f are some functions.

de Giorgi-Moser iteration :

Suppose that the surface satisfies some Sobolev inequality and the area estimate $Area(B_R) \leq c_1 R^2$, and also there exists a nonnegative function u such that $\triangle u \geq -fu$. Moreover in the ball $B_{3R/4}$, u and f satisfied integral estimates:

- there exists some $q \geq 6$,

$$\left(\int_{B(3R/4)} u^{2q}\right)^{1/q} \leq c_2 R^{-2+2/q};$$

- for all $0 \leq a \leq 1/2$,

$$\int_{B(3R/4)} (f + B_\Sigma^2)_+^{1+a} \leq c_3 R^{-2a}.$$

Then we have the point-wise estimate on $B_{R/2}$,

$$\sup_{B_{R/2}} u^2 \leq cq^2 R^{-2}.$$

The idea is that we choose a series of the indices(q) to get a series of integral inequalities, then take the limit to obtain the point-wise estimate. To show the existence of the limit, we use the Sobolev inequality and some nice cutoff functions.

Now we can state their main result.

Theorem 1.4 *(Theorem 4.1, [BeHa]) Let Σ be an oriented Riemannian surface. Let $i : \Sigma \to M(c)$ be an immersed CMC h-surface into a simply-connected three manifold M with constant curvature c. For any geodesic ball $B_R(x_0)$ where stability operator L is non-positive, given $\Lambda > 0$, there exists a constant $C(\Lambda)$, which depends on Λ, such that*

$$|A^0|^2(x_0) \leq C(\Lambda)R^{-2} \quad and \quad |K(x_0)| \leq C(\Lambda)R^{-2}, \tag{15}$$

under one of the following conditions:

(A) $c + H^2 \leq 0$ *and* $4R^2(c + H^2)_- \leq \Lambda$

or

(B) $c + H^2 > 0$ *and* $4R^2(c + H^2) \leq \pi^2$

In their paper[BeHa], since the Simons' inequality and the Sobolev inequality hold, they only need to prove estimates for $\int |A|^2$(total curvature) and the area. But even in space forms, they have to prove estimates in two cases, which depend on the curvature c of the space and the mean curvature h. For each case it is a long way before some nice-chosen cutoff functions yield estimates. If we study the surface in a general manifold. There is no way we could decide all cases. Moreover the work to plug in different cutoff functions is considerable.

From the next section, we shall give our approach to this problem.

2 Tools of curvature estimates

2.1 Simons' inequality

J. Simons [Si] in 1968 obtained an identity for the Laplacian of the second fundamental form of a minimal hyper-surface, which has led to a number of inequalities in different settings. Here we shall give the Simons' inequality for CMC surfaces in a general three manifold.

Lemma 2.1

$$\triangle_\Sigma |A^0|^2 \geq -2|A^0|^4 - 8|h||A^0|^3 - 2C_2|A^0|^2 - 2C_3|A^0|, \tag{16}$$

where C_2 depends on h and the curvature of M, C_3 depends on h, the curvature of M and its covariant derivative.

Proof. First, we choose $\{E_i, i = 1,2,3\}$ as a locally defined orthonormal frame in a neighborhood of some x in Σ, such that E_3 is normal to Σ. For the second fundamental form A , we define the symmetric two-tensor b on Σ by

$$b(X,Y) = g(A(X,Y), E_3) = -g(\nabla_X E_3, Y), \tag{17}$$

and set $b_{i,j} = -g(\nabla_{E_i} E_3, E_j)$. Moreover we need to define another symmetric two-tensor, $a(X,Y) = b(X,Y) - hg(X,Y)$. It is easy to see it corresponds to the traceless second fundamental form A^0 and

$$a_{ij} = b_{ij} - h\delta_i^j, \tag{18}$$

where $\delta_i^j = 1$ if $i = j$, $\delta_i^j = 0$ if $i \neq j$.

Then we compute covariant derivatives of the tensor a to give $a_{ij,k}$ and $a_{ij,kl}$. Following the computation in [CM2] (Lemma 2.1, page 26), we obtain:

Proposition 2.1 $a_{ij,k} = a_{ik,j} + \overline{R_{3ijk}}$,

Proposition 2.2 $a_{ij,kl} = a_{ij,lk} + \sum_{m=1}^{m=2} R_{klim} a_{mj} + \sum_{m=1}^{m=2} R_{kljm} a_{mi}$.

Also by the Gauss equation, we have

$$R_{ijkl} = \overline{R_{ijkl}} + b_{jk} b_{il} - b_{ik} b_{jl}. \tag{19}$$

In particular, we plug (18) and (19) into Proposition 2.2 to obtain

$$\begin{aligned} a_{ik,jk} &= a_{ik,kj} + \sum_m \overline{R_{jkim}} a_{mk} + \sum_m \overline{R_{jkkm}} a_{mi} \\ &\quad + \sum_m (a_{ki} a_{jm} - a_{ji} a_{km} + P1) a_{mk} \\ &\quad + \sum_m (a_{kk} a_{jm} - a_{jk} a_{km} + P2) a_{mi}, \end{aligned} \tag{20}$$

where

$$P1 = h^2(\delta_k^i \delta_j^m - \delta_j^i \delta_k^m) + h(\delta_k^i a_{jm} + \delta_j^m a_{ki} - \delta_j^i a_{km} - \delta_k^m a_{ij}),$$

and

$$P2 = h^2(\delta_k^k \delta_j^m - \delta_j^k \delta_k^m) + h(\delta_k^k a_{jm} + \delta_j^m a_{kk} - \delta_j^k a_{km} - \delta_k^m a_{jk}).$$

Now we can compute

$$\begin{aligned} \triangle_\Sigma |A^0|^2 &= \triangle_\Sigma \sum_{i,j} a_{ij}^2 \\ &= 2 \sum_{i,j} a_{ij} \triangle_\Sigma a_{ij} + 2 \sum_{i,j} |\nabla_\Sigma a_{ij}|^2 \end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{i,j,k} a_{ij}a_{ij,kk} + 2\sum_{i,j,k} a_{ij,k}^2 \\
&= 2\sum_{i,j,k} a_{ij}\left\{a_{ik,jk} + (\overline{R_{3ijk}})_k\right\} + 2\sum_{i,j,k} a_{ij,k}^2 \text{ (by Prop. 2.1)} \\
&= 2\sum_{i,j,k} a_{ij}\left\{a_{ik,kj} + \sum_m \overline{R_{jkim}}a_{mk} + \sum_m \overline{R_{jkkm}}a_{mi}\right. \\
&\quad + \sum_m (a_{ki}a_{jm} - a_{ji}a_{km} + P1)a_{mk} \\
&\quad \left. + \sum_m (a_{kk}a_{jm} - a_{jk}a_{km} + P2)a_{mi}\right\} \\
&\quad + 2\sum_{i,j,k} a_{ij}(\overline{R_{3ijk}})_k + 2\sum_{i,j,k} a_{ij,k}^2 \text{ (by (20))} \\
&= 2\sum_{i,j,k} a_{ij}\left\{a_{kk,ij} + (\overline{R_{3kik}})_j + \sum_m \overline{R_{jkim}}a_{mk} + \sum_m \overline{R_{jkkm}}a_{mi}\right. \\
&\quad + \sum_m (a_{ki}a_{jm} - a_{ji}a_{km} + P1)a_{mk} \\
&\quad \left. + \sum_m (a_{kk}a_{jm} - a_{jk}a_{km} + P2)a_{mi}\right\} \\
&\quad + 2\sum_{i,j,k} a_{ij}(\overline{R_{3ijk}})_k + 2\sum_{i,j,k} a_{ij,k}^2 \text{ (by Prop. 2.1).} \qquad (21)
\end{aligned}
$$

From here we are going to estimate the righthand side of (21).

First we estimate the terms of the form $(\overline{R_{3ijk}})_l$. We denote the covariant derivative of $\overline{R_{ijkm}}$, as a curvature tensor in M, by $\overline{R_{ijkm:l}}$. Then by restricting to Σ, we obtain (see [ScSimY],page 278)

Proposition 2.3 $\overline{R_{3ijk:l}} = (\overline{R_{3ijk}})_l - \overline{R_{3i3k}}b_{jl} - \overline{R_{3ij3}}b_{kl} + \sum_m b_{ml}\overline{R_{mijk}}$.

Note that all computations occur in the neighborhood of x on the CMC h-surface. We have

$$
\left|\overline{R_{ijkl}}\right| < c_1 \text{ and } \left|\nabla\overline{R_{ijkl}}^2\right| = \sum_{i,j,k,m,l} \overline{R_{ijkm:l}}^2 < c_2^2. \qquad (22)
$$

Also on the constant mean curvature surface, we have

$$\sum_i a_{ii} = \sum_i b_{ii} - 2h = 0. \tag{23}$$

By using Proposition 2.3, (22),(23) and the Cauchy inequality, after a short computation, we obtain the estimate:

$$\begin{aligned}
& 2\sum_{i,j,k} a_{ij}\left\{(\overline{R_{3kik}})_j + (\overline{R_{3ijk}})_k\right\} \\
= \; & 2\sum_{i,j,k} a_{ij}\left\{(\overline{R_{3kik:j}} + \overline{R_{3ijk:k}}) + \overline{R_{3k3k}}b_{ij} + \overline{R_{3ij3}}b_{kk}\right\} \\
& -2\sum_{i,j,k,m} \overline{R_{mkik}}a_{ij}b_{mj} - 2\sum_{i,j,k,m} \overline{R_{mijk}}a_{ij}b_{mk} \\
\geq \; & -4c_2|A^0| - 28|h|c_1|A^0| - 16c_1|A^0|^2.
\end{aligned} \tag{24}$$

We plug (24) into (21) and use elementary inequalities to obtain

$$\begin{aligned}
& \triangle_\Sigma |A^0|^2 \\
\geq \; & 0 - 2\sum_{i,j,k} a_{i,j}\left\{(\overline{R_{3kik}})_j + (\overline{R_{3ijk}})_k\right\} \\
& -2\sum_{i,j,k,m} c_1|a_{ij}a_{mk} + a_{ij}a_{mi}| \\
& -2\sum_{i,j,k,m}\left(a_{ij}^2 a_{km}^2 + |a_{ij}a_{mk}P1 + a_{ij}a_{mi}P2|\right) + 2\sum_{i,j,k} a_{ij,k}^2 \\
\geq \; & 0 - 4c_2|A^0| - 28|h|c_1|A^0| - 16c_1|A^0|^2 \\
& -4c_1|A^0|^2 - 2|A^0|^4 - 8h^2|A^0|^2 - 8|h||A^0|^3 \\
\geq \; & -2|A^0|^4 - 8|h||A^0|^3 - 2(10c_1 + 4h^2)|A^0|^2 - 2(2c_2 + 14|h|c_1)|A^0| \\
\geq \; & -2|A^0|^4 - 8|h||A^0|^3 - 2C_2|A^0|^2 - 2C_3|A^0|,
\end{aligned}$$

where $C_2 = 10c_1 + 4h^2$ and $C_3 = 2c_2 + 14|h|c_1$. Q.E.D.

Remark 2.1 *When Σ is a minimal surface in $\mathcal{R}^3$ (or a CMC surface in a space form), Simon's inequalities have been obtained in [Si](or [BeHa]).*

2.2 Mean value inequality

In this section, we give a local version of the mean value inequality obtained by Schoen and Yau.

Theorem 2.1 *(Theorem 6.2 in [ScY],p77) Let* Σ *is a complete Riemannian manifold with* $Ric(\Sigma) \geq -K$*. Let* u *be a non-negative subharmonic function on* Σ*. Then for any* $\tau \in (0, 1/2)$ *and* $R > 0$ *we have*

$$\sup_{B((1-\tau)R)} u^2 \leq c_1 \tau^{-c_2(1+\sqrt{K}R)} \frac{\int_{B_R} u^2}{Vol(B_R)}. \tag{25}$$

For a CMC h-surface Σ in a complete three dimensional manifold M with $|K_M| \leq k^2$, we have a mean value inequality in a disk.

Proposition 2.4 *Let* $x_0 \in \Sigma, 0 < R < \frac{\pi}{|h|}$ *and the geodesic ball* $B_R(x_0) \cap \partial\Sigma = \emptyset$*. If* f *is a nonnegative function on* Σ *with* $\triangle_\Sigma f \geq -R^{-2} f$*, then*

$$f^2(x_0) \leq \frac{C_4}{Vol(B_R(x_0))} \int_{B_R(x_0)} f^2,$$

where C_4 *is a constant depending on* k *and* h*.*

Proof. Let $N = \Sigma \times [-R, R]$, we define a function $g(x, t) = f(x)e^{t/R}$. Then

$$\triangle_N \, g(x, t) = e^{t/R} \triangle_\Sigma f + e^{t/R} R^{-2} f \geq 0.$$

Also by the Gauss Equation,

$$K_\Sigma \geq K_M - 2h^2 \geq -(k^2 + 2h^2).$$

In the product space N, we have $Ric_N \geq K_\Sigma \geq -K$, where $K = k^2 + 2h^2$. Then for a nonnegative subharmonic function g on such a manifold, we use Theorem 2.1 to obtain

$$g^2(x,t) \leq c_1 e^{c_2(1+\sqrt{K}R)} \frac{\int_{B_R(x,t)\subset N} g^2}{Vol(B_R(x,t))}, \tag{26}$$

where c_1 and c_2 are positive constants.

Now we set $x = x_0$ and $t = 0$, then use $R < \pi/|h|$ to obtain

$$f^2(x_0) = g^2(x_0, 0) \leq c \frac{\int_{B_R(x_0,0)\subset N} f^2}{Vol(B_R(x_0,0))}.$$

By using

$$\{B_R(x_0,0) \subset N\} \subseteq \{B_R(x_0) \subset \Sigma\} \times [-R, R],$$

and

$$\{B_R(x_0,0) \subset N\} \supseteq \{B_{\frac{R}{2}}(x_0) \subset \Sigma\} \times [-R/2, R/2],$$

we obtain

$$f^2(x_0) \leq ce^2 \frac{R \int_{B_R(x_0)\subset\Sigma} f^2}{R \cdot Vol(B_{\frac{R}{2}}(x_0))}.$$

Moreover, by the Bishop volume comparison theorem, we have

$$\frac{Vol(\Sigma, B_{R/2})}{V(K, R/2)} \geq \frac{Vol(\Sigma, B_R)}{V(K,R)}.$$

Here $V(K,R)$ is the volume of the geodesic ball B_R in the space form with constant sectional curvature K.

At last, we have

$$f^2(x_0) \leq \frac{C_4}{Vol(B_R(x_0))} \int_{B_R(x_0)} f^2.$$

Q.E.D.

Remark 2.2 *In the minimal surface theory, the mean value inequality is a key tool. Here we can only have a local version because the mean curvature h restricts the radius of the disk as is stated in Theorem 1.2.*

2.3 Estimate of elliptic integrands

T. Colding and W. Minicozzi II showed that the nonnegativity of certain Schrödinger operators on a surface implied certain upper bounds on intrinsic balls [CM1]. This result played a key role in their study of embedded minimal surfaces in $\mathcal{R}^3$. It was new and useful because it applied to general surfaces.

In particular, the stability operator is such a Schrödinger operator. We can obtain estimates for area and total curvature on general stable surfaces without a priori bounds. We are going to use their result to obtain our estimates for CMC surfaces in §3.1.

Now we prove a Theorem in [CM1].

Theorem 2.2 *([CM1], Thm2.1) For any intrinsic ball $B_R = B_R(x)$ in Σ^2, where $B_R(x) \cap Cutlocus(x) = \emptyset$ and $B_R \cap \partial\Sigma = \emptyset$. Let $\kappa \geq 0$ be a constant. And let $\nu, \omega \geq 0$ be functions. For some differential operator L_1, L_2 on Σ,*

if $-L_1 = -\triangle_\Sigma - \nu + 3\kappa + c_1 K_\Sigma \geq 0$, then

$$R^{-2} Area(B_R) + \frac{c_2}{2\pi c_1} \int_{B_R} \nu(1 - \frac{s}{R})^2 ds \leq c_2, \tag{27}$$

and if $-L_2 = -\triangle_\Sigma - \omega + 2\kappa \geq 0$, then for all $0 < \mu < 1$

$$\int_{B_{\mu^2 R}} \omega \leq c_2(\log \mu)^{-2} - 2c_2/\log \mu + 2\kappa c_2 R^2 \mu^2, \tag{28}$$

where $c_1 > (1 + 3\kappa R^2)/2$, $c_2 = 2\pi c_1/(2c_1 - 1 - 3\kappa R^2)$.

Proof. Let $l(s)$ be the length of $\partial B_s(x_0)$ and $K(s) = \int_{B_s} K_\Sigma$.

By the Gauss-Bonnet theorem,

$$l'(s) = \int_{\partial B_s} k_g(s)ds = 2\pi\chi(B_s) - K(s) = 2\pi - K(s). \tag{29}$$

We can choose a cut-off function $f = \eta(s)$ such that $\eta : [0, R] \to \mathcal{R}^+$ is smooth and satisfies $\eta(0) = 1, \eta(R) = 0, \eta' \leq 0$.

Using the nonnegativity of L_1 and coarea formula, we have

$$\begin{aligned} 0 &\leq -\int_{B_R} fL(f) = \int_{B_R} \left(|\nabla f|^2 - f^2\nu + 3\kappa f^2 + c_1 f^2 K_\Sigma\right) \\ \int_{B_R} \nu f^2 &\leq \int_{B_R} |\nabla f|^2 + 3\kappa \int_{B_R} f^2 + c_1 \int_{B_R} f^2 K_\Sigma \\ &= \int_{s=0}^{s=R} (\eta')^2 l(s) + 3\kappa \int_{s=0}^{s=R} \eta^2 l(s) + c_1 \int_{s=0}^{s=R} \eta^2 \int_{\partial B_s} K_\Sigma. \end{aligned} \tag{30}$$

Using (29) and integrating by parts, we obtain

$$\begin{aligned} \int_{s=0}^{s=R} \eta^2 \int_{\partial B_R} K_\Sigma &= \int_{s=0}^{s=R} \eta^2 K'(s) = -\int_{s=0}^{s=R} (\eta^2)' K(s) \\ &= \int_{s=0}^{s=R} (\eta^2)'(l'(s) - 2\pi). \end{aligned}$$

Now choose $\eta(s) = 1 - s/R$, so $(\eta')^2 = \frac{1}{R^2}$ and $(\eta^2)' = \frac{2(s-R)}{R^2}$.

We plug them into (30) to obtain

$$\begin{aligned} \int_{B_R} \nu f^2 &\leq R^{-2} \int_{s=0}^{s=R} l(s) + 3\kappa \int_{s=0}^{s=R} l(s) + c_1 \int_{s=0}^{s=R} (\eta^2)'(l'(s) - 2\pi) \\ \int_{B_R} \nu f^2 &+ 2c_1 R^{-1} \int_{s=0}^{s=R} (1 - \frac{s}{R}) l'(s) \leq (R^{-2} + 3\kappa) \int_{s=0}^{R} l(s) + 2\pi c_1 \end{aligned}$$

Integrating by parts again,

$$\int_{B_R} \nu(1 - s/R)^2 + 2c_1 R^{-2} \int_{s=0}^{s=R} l(s) \leq (R^{-2} + 3\kappa) \int_{s=0}^{s=R} l(s) + 2\pi c_1. \tag{31}$$

By the coarea formula we have $Area(B_R) = \int_{s=0}^{s=R} l(s)$. Substituting it in (31) and after simplifying, we obtain (27).

To show (28) we define the cutoff function $f = \eta(s)$ on $[0, R]$ by

$$\eta(s) = \begin{cases} 1 & : \quad 0 \le s \le \mu^2 R \\ \frac{\log(sR^{-1})}{\log \mu} - 1 & : \quad \mu^2 R < s \le \mu R \quad (0 < \mu < 1) \\ 0 & : \quad \mu R < s \le R \end{cases} \tag{32}$$

Substituting (32) into an analogous inequality of (30),

$$\begin{aligned} \int_{B_{\mu^2 R}} \omega &\le \int_{B_{\mu^2 R}} \omega f^2 \le \int_{s=0}^{s=R} (\eta')^2 l(s) + 2\kappa \int_{s=0}^{s=R} \eta^2 l(s) \\ &\le (\log \mu)^2 \int_{s=\mu^2 R}^{s=\mu R} l(s) s^{-2} + 2\kappa \int_{s=0}^{s=\mu R} l(s). \end{aligned}$$

Using (27) and integrating by parts twice, we obtain (28)

$$\begin{aligned} \int_{B_{\mu^2 R}} \omega &\le (\log \mu)^{-2} \left[Area(B_s) s^{-2} \right]_{\mu^2 R}^{\mu R} \\ &\quad + 2(\log \mu)^{-2} \int_{s=\mu^2 R}^{s=\mu R} s^{-3} Area(B_s) + 2\kappa c_2 R^2 \mu^2 \\ &\le c_2 (\log \mu)^{-2} - \frac{2c_2}{\log \mu} + 2\kappa c_2 R^2 \mu^2. \end{aligned}$$

Q.E.D.

Remark 2.3 *Because of the choices of cutoff functions in the proof, we have to use strong stability in our estimates in §3.1.*

Remark 2.4 *Recently we learned from H. Rosenberg that the condition* $B_R \cap Cutlocus(x) = \emptyset$ *can be relaxed in the above proof, which means we can choose a large ball for our curvature estimates later.*

3 Main result

3.1 Area and total curvature estimates

We will see in §3.2 that *small total curvature* implies curvature estimates, which is similar to the minimal surface case. Here we show area and total curvature estimates by using the estimates for elliptic integrands in §2.3.

Theorem 3.1 *Let Σ be a strongly stable immersed CMC h-surface with trivial normal bundle in a three manifold M, where $|K_M| \le k^2$. If $B_R(x) \subset \Sigma$, $B_R(x) \cap \partial\Sigma = \emptyset$ and $B_R(x) \cap Cutlocus(x) = \emptyset$, then for any $R < \frac{1}{\sqrt{6(h^2+k^2)}}$,*

$$Area(B_R(x)) \le 4\pi R^2 \tag{33}$$

and for $0 < \mu < 1/e$

$$\int_{B_{\mu^2 R(x)}} |A^0|^2 \le -12\pi(log\mu)^{-1} + \frac{4}{3}\pi\mu^2 \tag{34}$$

Proof. For a stable constant mean curvature h-surface Σ in M, the Gauss equation is

$$K_\Sigma = K_M - \frac{1}{2}|A|^2 + 2h^2 = K_M - \frac{1}{2}|A^0|^2 + h^2 \tag{35}$$

As in (12), the stability operator is $L = \triangle_\Sigma + |A|^2 + Ric_M(n,n)$.

Rewrite it as

$$-L_2 = -\triangle_\Sigma - |A^0|^2 - (4h^2 + Ric_M(n,n) + 2k^2) + 2k^2 + 2h^2$$

and

$$-L_1 = -\triangle_\Sigma - \frac{1}{2}|A^0|^2 - (3h^2 + Ric_M(n,n) + K_M + 3k^2 + 3h^2) + 3(k^2+h^2) + K_\Sigma.$$

Now we can use(27) and (28) in Theorem 2.2, where $c_1 = 1$, $\omega \geq |A^0|^2$,$\nu > 0$ and $\kappa = h^2 + k^2$. Moreover, by choosing $R < \frac{1}{\sqrt{6(h^2+k^2)}}$ such that $c_2 < 4\pi$, we obtain (33)

$$Area(B_R) \leq 4\pi R^2.$$

Also take $\mu < 1/e$ and use $c_2 < 4\pi$, we obtain (34)

$$\int_{B_{\mu^2 R}} |A^0|^2 \leq -12\pi(log\mu)^{-1} + \frac{4}{3}\pi\mu^2.$$

Q.E.D.

3.2 Curvature estimates

Choi and Schoen[ChSc] proved that small total curvature yields a curvature estimate for minimal surfaces in a three manifold M with $|K_M| \leq k^2$. We will use a similar argument to give curvature estimates for CMC surfaces.

Throughout this subsection, we restrict our investigation to a geodesic ball $B_\lambda(x)$ of the CMC h-surface. Here the constant λ can be chosen as $\frac{\pi}{\sqrt{6(h^2+k^2)}}$ as we need in Theorem 0.1 and Theorem 3.1.

First we obtain a local version of Simons' inequality (Lemma 2.1).

Lemma 3.1 *For any geodesic ball $B_r(x) \subset B_\lambda(x)$, if $\sup|A^0| < \frac{1}{r}$, then*

$$\triangle_\Sigma |A^0|^2 \geq -C_6 r^{-2}(|A^0|^2 + C_5),$$

where C_5 depends on λ, h, the curvature tensor of M and its covariant derivative, C_6 depends on λ and h.

Proof.

We just plug $|A^0| < \frac{1}{r}$ and $r < \lambda$ into Lemma 2.1 to obtain

$$\begin{aligned}
\triangle_\Sigma |A^0|^2 &\geq -2C_2|A^0|^2 - 2|A^0|^4 - 8|h||A^0|^3 - 2C_3|A^0| \\
&\geq -2|A^0|^2\left(|A^0|^2 + C_2\right) - 8|h||A^0|\left(|A^0|^2 + \frac{C_3}{2|h|}\right) \\
&\geq -2r^{-2}\left(|A^0|^2 + C_5\right) - 8|h|r^{-1}\left(|A^0|^2 + C_5\right) \\
&\geq -2r^{-2}\left(|A^0|^2 + C_5\right)(1 + 4|h|r) \\
&\geq -C_6 r^{-2}\left(|A^0|^2 + C_5\right),
\end{aligned}$$

where $C_5 = max(C_2, \frac{C_3}{2|h|})$ and $C_6 = 2(1 + 4|h|\lambda)$. Q.E.D.

Then we give a lower bound for the volume of $B_r(x)$, which is a simple corollary of the Rauch Comparison theorem.

Proposition 3.1 *On the CMC h-surface Σ in M with $K_M \leq k^2$, if a geodesic ball $B_r(x) \cap Cutlocus(x) = \emptyset$, then $Vol(B_r(x)) \geq \frac{8}{\pi} r^2$.*

Proof. By the Gauss equation (35),

$$K_\Sigma = K_M - \frac{1}{2}|A|^2 + 2h^2 \leq k^2 + 2h^2 = K.$$

Using the Rauch Comparison theorem, we have

$$Vol(B_r) \geq Vol_K(B_r).$$

Here $Vol_K(B_r)$ is the volume of a geodesic ball B_r in the space form with constant sectional curvature K. We will use the sphere with radius $\frac{1}{\sqrt{K}}$ in $\mathcal{R}^3$ as the model.

Now from [ScY] Proposition 4.3 on pape 49, we know $Vol_K(B_r)/r^2$ is a non-increasing function about r. Since $B_r(x) \cap Cutlocus(x) = \emptyset$, on the sphere we have $r < \frac{\pi}{2\sqrt{K}}$. Thus we have

$$Vol_K(B_r)/r^2 \geq \frac{Vol_K\left(B_{\frac{\pi}{2\sqrt{K}}}\right)}{\pi^2/4K} = \frac{2\pi/K}{\pi^2/4K} = \frac{8}{\pi}.$$

The result follows.

Q.E.D.

Finally we give the curvature estimate in $B_\lambda(x)$.

Theorem 3.2 *In a geodesic ball $B_\lambda(x) \subset \Sigma \backslash \partial\Sigma$, where $B_\lambda(x) \cap Cutlocus(x) = \emptyset$, if for all ε, there exists $R < min(\lambda, \varepsilon)$, such that $\int_{B_{R(x)}} |A^0(x)|^2 \leq \varepsilon$, then there exists $0 < r_0 < \lambda$, for all $0 < \sigma \leq r_0$, we have*

$$sup_{B_{r_0-\sigma}} |A^0(x)|^2 \leq \sigma^{-2}. \tag{36}$$

Here r_0 depends on h, the curvature tensor of M and its covariant derivative.

Proof.

Let $F(x) = (R - r(x))^2 |A^0(x)|^2$, clearly $F(x) \geq 0$ and it achieves its maximum at some point x_0 on B_R. If $F(x_0) < 1$, it is easy to see $|A^0|^2 \leq \sigma^{-2}$ for $|x| \leq R - \sigma$, so that (36) holds.

If $F(x_0) \geq 1$, choose $2\sigma < R - r(x_0)$ such that $4\sigma^2 |A^0|^2(x_0) = 1$. We have

$$\begin{aligned}
\sup_{B_\sigma(x_0)} \sigma^2 |A^0|^2 &= \sup_{B_\sigma(x_0)} \sigma^2 \frac{F(x)}{(R - r(x))^2} \\
&\leq \frac{4\sigma^2}{(R - r(x_0))^2} \sup_{B_\sigma} F(x) \\
&\leq \frac{4\sigma^2}{(R - r(x_0))^2} F(x_0) = 1,
\end{aligned}$$

that is

$$\sup_{B_\sigma(x_0)} |A^0|^2 \leq \sigma^{-2}. \tag{37}$$

Now using Lemma 3.1 in the ball B_σ, we have

$$\triangle_\Sigma |A^0|^2 \geq -C_6\sigma^{-2}(|A^0|^2 + C_5).$$

Let $\mu = |A^0|^2 + C_5$, we have

$$\triangle_\Sigma \mu \geq -C_6\sigma^{-2}\mu.$$

Then by using a little different form of the mean value inequality of Proposition 2.4, we obtain

$$\mu^2(x_0) \;\; \leq \;\; \frac{C_4}{Vol(B_\sigma(x_0))}\int_{B_\sigma(x_0)} \mu^2. \tag{38}$$

With $B_\lambda(x_0) \cap Cutlocus(x_0) = \emptyset$ and $\sigma \leq \lambda$, we can get a lower bound for $Vol(B_\sigma)$ by Proposition (3.1):

$$Vol(B_\sigma(x_0)) \geq \frac{8}{\pi}\sigma^2. \tag{39}$$

We combine (37)-(39), and area estimate (33) in *Theorem* 2.2 to give

$$\begin{aligned}
|A^0|^4(x_0) &\leq \frac{C_4}{Vol(B_\sigma(x_0))}\int_{B_\sigma(x_0)}(|A^0|^2 + C_5)^2 d\nu \\
&\leq C_4\frac{\pi}{8}\sigma^{-2}\int_{B_\sigma(x_0)}(|A^0|^2 + C_5)(\sigma^{-2} + C_5)d\nu \\
&\leq c_*\sigma^{-4}\int_{B_\sigma(x_0)}(|A^0|^2 + C_5)d\nu \\
&\leq c_*\sigma^{-4}\left(\int_{B_\sigma(x_0)}|A^0|^2 d\nu + C_5 Vol(B_\sigma)\right) \\
&\leq c_*\sigma^{-4}\left(\int_{B_\sigma(x_0)}|A^0|^2 d\nu + C_5 4\pi\sigma^2\right) \text{ (by (33)).}
\end{aligned}$$

Now plug in $|A^0|^2(x_0) = \frac{\sigma^{-2}}{4}$ and $\sigma \le R \le \varepsilon$, we have

$$\frac{\sigma^{-4}}{16} \le c_* \sigma^{-4}(C\varepsilon + C_5 4\pi\varepsilon^2) \le c_* \sigma^{-4}\varepsilon.$$

Choose ε small enough, we get a contradiction. Hence we prove (36).

Q.E.D.

Corollary 3.1 *Under the same condition as in Theorem 3.2, we have*

$$sup_{B_{r_0-\sigma}} |K_\Sigma(x)| \le C_1 \sigma^{-2}, \tag{40}$$

where r_0 depends on h, the curvature tensor of M and its covariant derivative, C_1 is a constant depending on h and k.

Proof.

To show (40), notice that the Guass equation (35) yields

$$|K_\Sigma| = |K_M - \frac{1}{2}|A^0|^2 + h^2| \le \frac{1}{2}|A^0|^2 + k^2 + h^2.$$

Using Theorem3.2 and $\sigma < \lambda$, we can get (40)

$$|K_\Sigma| \le (1/2 + |h|^2\sigma^2 + k^2\sigma^2)\sigma^{-2} \le C_1\sigma^{-2}.$$

Q.E.D.

3.3 Proof of the main theorem

Now we combine *Theorem* 3.1 and *Theorem* 3.2 to prove the main theorem.

Proof. We prove the theorem in two steps for the sake of simplicity.

- Suppose that $B_R(x)$ is a topological disk. First we can show $B_R(x) \cap Cutlocus(x) = \emptyset$ if $R < \frac{\pi}{\sqrt{(2h^2+k^2)}}$. In fact, by the Gauss equation (35), we have $K_\Sigma \leq k^2 + 2h^2$. From the Rauch comparison theorem, the conjugate locus is empty when $R < \frac{\pi}{\sqrt{(2h^2+k^2)}}$. Thus for any point $p \in Cutlocus(x)$, there exist two minimal geodesics ι_1, ι_2, such that $\iota_1 \cap \iota_2 = x \cup p$. From the Gauss-Bonnet theorem, the loop is an essential loop. Now we have an essential loop in the stable disk $B_R(x)$. This is impossible. Then using $B_R(x) \cap Cutlocus(x) = \emptyset$, we can apply *Theorem* 3.1 to get area and total curvature estimates for all topological disks. Here we need to choose $R < \lambda = \frac{\pi}{\sqrt{6(h^2+k^2)}}$. Then for any ε, we can choose small μ, such that $\int_{B_R(x)} |A^0(x)|^2 < \varepsilon$, where R depends on μ. Lastly applying *Theorem* 3.2, we obtain the curvature estimate.

- If $B_R(x)$ is not a topological disk, we will look at the universal covering of B_R, which is a topological disk. Since B_R is stable, the stability operator $-L$ is nonnegative on B_R. From Theorem 1.3 [FiSc](see also [CM1] Lemma 1.26), there exists some positive function f with $Lf = 0$. That means the pullback of $-L$ is also nonnegative, so we obtain a stable topological disk on the covering. Now we repeat the above proof to give the curvature estimate for the covering disk. The curvature estimate for B_R follows.

Q.E.D.

Corollary 3.2 *(J. Spruck[Sp],1974) If a CMC surface Σ is a graph in $\mathcal{R}^3$, curvature estimates (4) and (5) hold.*

Proof. We only need to show Σ is strongly stable.

Suppose Σ is a graph over the plane $z = 0$. We can choose the unit vector $N = (0, 0, 1)$ and define a function $f = g(N, \overrightarrow{n})$, where g is the inner product of $\mathcal{R}^3$ and $\overrightarrow{n}$ is the unit normal vector field on Σ. Since Σ is a graph, f is positive.

Now we define a local orthonormal frame $\{E_1, E_2, \overrightarrow{n}\}$ in the neighborhood of a point p. We can obtain

$$\begin{aligned} \triangle_\Sigma f(p) &= \sum_i E_i E_i g(N, \overrightarrow{n})(p) = \sum_i g(N, \nabla_{E_i}\nabla_{E_i}\overrightarrow{n})(p) \\ &= g(N, \overrightarrow{n}) g(\overrightarrow{n}, \sum_i \nabla_{E_i}\nabla_{E_i}\overrightarrow{n}) = -f(p)\cdot |A|^2(p) \end{aligned}$$

From (12), the stability operator here is $L = \triangle + |A|^2$, thus we have $Lf = 0$. Since f is positive, by Theorem 1.3, Σ is strongly stable.

The rest follows from the main theorem 0.1.

Q.E.D.

Remark 3.1 *The curvature estimate for CMC surfaces in a space form [BeHa] can be obtained from our main theorem with different constants.*

Remark 3.2 *In fact, without much change of our proof, we can show similar curvature estimates (Theorem 0.1) when the stability operator L is bounded from above by some non-negative constant $2l$. A similar result has been proved for CMC surfaces in a space form by P. Bérard and L. Hauswirth [BeHa].*

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Vita

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