An Improvement on Eigenfunction Restriction Estimates for Compact Boundaryless Riemannian Manifolds with Nonpositive Sectional Curvature

by

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A dissertation submitted to The Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy.

Baltimore, Maryland

May, 2013

Abstract

Let (M, g) be an n-dimensional compact boundaryless Riemannian manifold with nonpositive sectional curvature, then we are interested in the growth rate of the L^p norms of eigenfunctions of the Laplace-Beltrami operator restricted to smooth sunbmanifolds. Our conclusion is that we can give improved estimates for the L^p norms of the restrictions of eigenfunctions to smooth submanifolds of dimension k, for $p > \dfrac{2n}{n-1}$ when $k = n - 1$ and $p > 2$ when $k \leq n - 2$, compared to the general results of Burq, Gérard and Tzvetkov [2]. Earlier, Bérard [1] gave the same improvement for the case when $p = \infty$, for compact Riemannian manifolds without conjugate points for $n = 2$, or with nonpositive sectional curvature for $n \geq 3$ and $k = n - 1$. In this thesis, we give the improved estimates for $n = 2$, the L^p norms of the restrictions of eigenfunctions to geodesics. Our proof uses the fact that, the exponential map from any point in $x \in M$ is a universal covering map from $\mathbb{R}^2 \backsimeq T_xM$ to M, which allows us to lift the calculations up to the universal cover $(\mathbb{R}^2, \tilde{g})$, where $\tilde{g}$ is the pullback of g via the exponential map. Then we prove the main estimates by using the Hadamard parametrix for the wave equation on $(\mathbb{R}^2, \tilde{g})$, the stationary phase estimates, and the fact that the principal coefficient of the Hadamard parametrix is bounded, by observations of Sogge and Zelditch in [18]. The improved estimates also work for $n \geq 3$, with $p > \frac{4k}{n-1}$. We can then get the full result by interpolation. In the last chapter, we give some recent improved estimates that are not included in the main theorem.

Primary Reader: Christopher Sogge

Secondary Reader: Bernard Shiffman

Acknowledgments

I would like to express my profound appreciation to my advisor, Professor Christopher Sogge, for his efficient guidance, generous help and unlimited patience. This thesis could not have been completed without his help. His spirits and kindness have set up a good example for my life.

I would like to thank Professor Bernard Shiffman and his wife, Ms. Doris Shiffman, who provided me helpful advice and encouragement in my first year of graduate study. I would like to thank my colleges Peng Shao, Jui-En Chang, Xin Yu, Jin-Cheng Jiang, Sinan Ariturk, Xing Wang, Ling Xiao, Arash Karami, Timothy Tran, and Matthew McGonagle, for their support to my work with insightful discussions and warm company. I would like to thank the faculty and staff in the math department at Johns Hopkins: Dr. Jian Kong, Dr. Richard Brown, Morris Hunt, Sabrina Raymond, Linda Buckner, Charlene Poole, Christina Bannon and many others.

Dedication

This thesis is dedicated to my family, who make me feel so loved. They are my parents Shaoquan Chen and Likai Liang, and my husband Yuan Lu.

Contents

Chapter 1

Introduction

Let (M, g) be a compact, smooth n-dimensional boundaryless Riemannian manifold with nonpositive sectional curvature. Denote Δ_g the Laplace-Beltrami operator associated to the metric g, and $d_g(x, y)$ the geodesic distance between x and y associated with the metric g. We know that there exist $\lambda \geq 0$ and $\phi_\lambda \in L^2(M)$ such that $-\Delta_g \phi_\lambda = \lambda^2 \phi_\lambda$, and we call ϕ_λ an eigenfunction corresponding to the eigenvalue λ. Let $\{e_j(x)\}_{j\in\mathbb{N}}$ be an $L^2(M)$-orthonormal basis of eigenfunctions of $\sqrt{-\Delta_g}$, with eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$, and $\{E_j(x)\}_{j\in\mathbb{N}}$ be the projections onto the j-th eigenspace, restricted to Σ, i.e. $E_j f(x) = e_j(x) \int_M e_j(y) f(y) dy$, for any $f \in L^2(M)$, $x \in \Sigma$. Thus, we may write

$$\Delta_g|_\Sigma = \sum_{j=0}^{\infty} \lambda_j E_j. \tag{1.1}$$

We are interested in the spectral projection of Δ_g onto a certain window centered at the eigenspace with corresponding eigenvalue λ. We may consider only the positive λ's as we are interested in the asymptotic behavior of the eigenfunction projections. This is considered one of the major ways to measure the concentration of eigenfunctions of the Laplace-Beltrami operator on a manifold. The restriction theorem is stated as the following.

Theorem 1.1 (Burq, Gérard, Tzvetkov, [2]). *Let (M, g) be a compact smooth n-dimensional boundaryless Riemannian manifold associated with the metric g, and Σ be an k-dimensional smooth submanifold on M. There exists a constant $C > 0$ such that for any ϕ_λ, we have*

$$||\phi_\lambda||_{L^p(\Sigma)} \leq C(1+\lambda)^{\rho(\lambda,n)}||\phi_\lambda||_{L^2(M)}, \tag{1.2}$$

where

$$\begin{aligned} \rho(n-1,n) &= \begin{cases} \frac{n-1}{2} - \frac{n-1}{p} & \text{if} \quad \frac{2n}{n-1} < p \leq +\infty, \\ \frac{n-1}{4} - \frac{n-2}{p} & \text{if} \quad 2 \leq p < \frac{2n}{n-1}, \end{cases} \\ \rho(n-2,n) &= \frac{n-1}{2} - \frac{n-2}{p} \quad \text{if } 2 < p \leq +\infty, \\ \rho(k,n) &= \frac{n-1}{2} - \frac{k}{p} \quad \text{if } 1 \leq k \leq n-3. \end{aligned} \tag{1.3}$$

If $p = \frac{2n}{n-1}$ and $k = n-1$, we have

$$||\phi_\lambda||_{L^p(\Sigma)} \leq C(1+\lambda)^{\frac{n-1}{2n}}(\log\lambda)^{\frac{1}{2}}||\phi_\lambda||_{L^2(M)}, \tag{1.4}$$

and if $p = 2$ and $k = n-2$, we have

$$||\phi_\lambda||_{L^p(\Sigma)} \leq C(1+\lambda)^{\frac{1}{2}}(\log\lambda)^{\frac{1}{2}}||\phi_\lambda||_{L^2(M)}. \tag{1.5}$$

As an improvement, our main theorem is the following.

Theorem 1.2 (Chen, preprint [3]). *Let (M, g) be a compact smooth n-dimensional boundaryless Riemannian manifold with nonpositive curvature, and Σ be an k-dimensional smooth submanifold on M. Let $\{E_j(x)\}_{j\in\mathbb{N}}$ be the projections onto the j-th eigenspace, restricted to Σ. Given any $f \in L^2(M)$, we have the following estimate:*

When $k = n-1$,

$$||\sum_{|\lambda_j-\lambda|\le(\log\lambda)^{-1}} E_j f||_{L^p(\Sigma)} \lesssim \frac{\lambda^{\delta(p)}}{(\log\lambda)^{\frac{1}{2}}}||f||_{L^2(M)}, \quad \forall p > \frac{2n}{n-1}; \tag{1.6}$$

When $k \le n-2$,

$$||\sum_{|\lambda_j-\lambda|\le(\log\lambda)^{-1}} E_j f||_{L^p(\Sigma)} \lesssim \frac{\lambda^{\delta(p)}}{(\log\lambda)^{\frac{1}{2}}}||f||_{L^2(M)}, \quad \forall p > 2, \tag{1.7}$$

where $\delta(p) = \frac{n-1}{2} - \frac{k}{p}$.

Note that we may assume that (M, g) is also simply connected in the proof.

The following corollary is an immediate consequence of this theorem.

Corollary 1.3 (Chen, preprint [3]). *Let (M, g) be a compact smooth n-dimensional boundaryless Riemannian manifold with nonpositive curvature, and Σ be an k-dimensional smooth submanifold on M. For any eigenfunction ϕ_λ of Δ_g s.t. $-\Delta_g\phi_\lambda = \lambda^2\phi_\lambda$, we have the following estimate:*

When $k = n-1$,

$$||\phi_\lambda||_{L^p(\Sigma)} \lesssim \frac{\lambda^{\delta(p)}}{(\log\lambda)^{\frac{1}{2}}}||\phi_\lambda||_{L^2(M)}, \quad \forall p > \frac{2n}{n-1}; \tag{1.8}$$

When $k \leq n-2$,

$$||\phi_\lambda||_{L^p(\Sigma)} \lesssim \frac{\lambda^{\delta(p)}}{(\log\lambda)^{\frac{1}{2}}}||\phi_\lambda||_{L^2(M)}, \quad \forall p > 2, \tag{1.9}$$

where $\delta(p) = \frac{n-1}{2} - \frac{k}{p}$.

In [11], Reznikov achieved weaker estimates for hyperbolic surfaces, which inspired this current line of research. In Theorem 1.1, Burq, Gérard and Tzvetkov showed that given any k-dimensional submanifold Σ of an n-dimensional compact boundaryless manifold M, for any $p > \dfrac{2n}{n-1}$ when $k = n-1$ and for any $p > 2$ when $k \leq n-2$, one has

$$||\phi_\lambda||_{L^p(\Sigma)} \lesssim \lambda^{\delta(p)}||\phi_\lambda||_{L^2(M)}, \tag{1.10}$$

while for $p = \frac{2n}{n-1}$ when $k = n-1$ and for $p = 2$ when $k = n-2$ one has

$$||\phi_\lambda||_{L^p(\Sigma)} \lesssim \lambda^{\delta(p)}(\log\lambda)^{\frac{1}{2}}||\phi_\lambda||_{L^2(M)}. \tag{1.11}$$

Later on, Hu improved the result at one end point in [9], so that one has (1.10) for $p = \frac{2n}{n-1}$ when $k = n-1$. It is very possible that one can also improve the result at the other end point, where $p = 2$, $k = n-2$, so that we also have (1.10) there. Our Theorem 4.1 gives an improvement for (1.10) of $(\log\lambda)^{-\frac{1}{2}}$ for $p \geq 2$ for certain small k's (See Remark 4.2). The following proposition shows that it is impossible to improve (1.10) for manifolds with constant positive sectional curvature.

Proposition 1.4. *The estimate* (1.10) *is saturated by the Zonal spherical harmonics. Therefore, it cannot be improved on the standard sphere.*

Proof. For details, please refer to Section 3.4 in [14]. Generally speaking, the Zonal spherical harmonics, called $Z_m, m = 1, 2, ...$, are harmonic functions on the standard sphere such that for points near the pole, we have

$$|Z_m(x)| \approx m^{\frac{n-1}{2}} ||Z_m||_{L^2(\mathbb{S}^n)}, \tag{1.12}$$

with eigenvalues being $\lambda^2 = m(m + n - 1)$.

Consequently, for any submanifold $\Sigma \subset \mathbb{S}^n$, we may choose a pole on Σ and consider the family of corresponding Zonal eigenfunctions, then

$$||Z_m|_\Sigma||_{L^p(\Sigma)} \gtrsim m^{\frac{n-1}{2}-\frac{k}{p}} ||Z_m||_{L^2(\mathbb{S}^n)} \approx \lambda^{\frac{n-1}{2}-\frac{k}{p}} ||Z_m||_{L^2(\mathbb{S}^n)}, \tag{1.13}$$

which means that (1.10) is saturated. □

Note that their proof of Theorem 1.1 in [2] indicates that for any $f \in L^2(M)$,

$$|| \sum_{|\lambda_j - \lambda| < 1} E_j f||_{L^p(\Sigma)} \lesssim \lambda^{\delta(p)} ||f||_{L^2(M)}, \tag{1.14}$$

for any $p \geq \dfrac{2n}{n-1}$ when $k = n - 1$ and $p \geq 2$ when $k \leq n - 2$ except that there is an extra $(\log \lambda)^{\frac{1}{2}}$ on the right hand side when $p = 2$ and $k = n - 2$. In the proof, they constructed $\chi_\lambda = \chi(\sqrt{-\Delta_g} - \lambda)$ from $L^2(M)$ to $L^p(\Sigma)$, where $\chi \in \mathcal{S}(\mathbb{R})$ such that $\chi(0) = 1$, and showed that $\chi_\lambda(\chi_\lambda)^*$ is an operator from $L^p(\Sigma)$ to $L^{p'}(\Sigma)$ with norm $O(\lambda^{2\delta(p)})$. That means, there exists at least an $\varepsilon > 0$ such that

$$|| \sum_{|\lambda_j - \lambda| < \varepsilon} E_j f||_{L^p(\Sigma)} \lesssim \lambda^{\delta(p)} ||f||_{L^2(M)}. \tag{1.15}$$

The reason why (1.15) is true can be seen in this way. Consider the dual form of

$$||\chi(\lambda - \sqrt{-\Delta_g})f||_{L^p(\Sigma)} \lesssim \lambda^{\delta(p)}||f||_{L^2(M)}, \tag{1.16}$$

which says

$$||\sum_j \chi(\lambda - \lambda_j)E_j^* g||_{L^2(M)} \lesssim \lambda^{\delta(p)}||g||_{L^p(\Sigma)}, \tag{1.17}$$

where E_j^* is the conjugate operator of E_j such that $E_j^* g(x) = e_j(x)\int_\Sigma e_j(y)g(y)dy$, for any $g \in L^2(\Sigma)$ and $x \in M$. There exists an $\varepsilon > 0$ such that $\chi(t) > \frac{1}{2}$ when $|t| < \varepsilon$ because we assumed that $\chi(0) = 1$. Therefore, the square of the left hand side of (1.17) is

$$\sum_{|\lambda-\lambda_j|<\varepsilon} ||\chi(\lambda - \lambda_j)E_j^* g||_{L^2(M)}^2 + \sum_{|\lambda-\lambda_j|>\varepsilon} ||\chi(\lambda - \lambda_j)E_j^* g||_{L^2(M)}^2 \geq \frac{1}{4}\sum_{|\lambda-\lambda_j|<\varepsilon} ||E_j^* g||_{L^2(M)}^2. \tag{1.18}$$

That means

$$||\sum_{|\lambda-\lambda_j|<\varepsilon} E_j^* g||_{L^2(M)} \lesssim \lambda^{\delta(p)}||g||_{L^p(\Sigma)}, \tag{1.19}$$

which is the dual version of (1.15).

If we divide the interval $(\lambda-1, \lambda+1)$ into $\frac{1}{\varepsilon}$ sub-intervals whose lengths are 2ε, and apply the last estimate $\frac{1}{\varepsilon}$ times, we get (1.14). Thinking in this way, our estimates (1.6) and (1.7) are equivalent to the estimates for

$$||\sum_{|\lambda_j-\lambda|<\varepsilon \log^{-1}\lambda} E_j||_{L^2(M)\to L^p(\Sigma)}, \tag{1.20}$$

for some number $\varepsilon > 0$, which is equivalent to estimating

$$||\chi(T(\lambda - \sqrt{-\Delta_g}))||_{L^2(M)\to L^p(\Sigma)}, \tag{1.21}$$

for $T \approx \log^{-1}\lambda$.

By Proposition 1.4, the estimates (1.10) is sharp when M is the standard sphere $\mathbb{S}^n$ and Σ is any submanifold of dimension k. It is natural to try to improve it on Riemannian manifolds with nonpositive sectional curvature. Recently, Sogge and Zelditch in [18] showed that for any 2-dimensional compact boundaryless Riemannian manifold with nonpositive sectional curvature one has

$$\sup_{\gamma\in\Pi} ||\phi_\lambda||_{L^p(\gamma)}/||\phi_\lambda||_{L^2(M)} = o(\lambda^{\frac{1}{4}}), \quad \text{for } 2 \le p < 4, \tag{1.22}$$

where Π denotes the space of all unit-length geodesics in M. (1.14) is sharp for any compact manifolds, in the sense that we fix the scale of the spectral projection (See proof in [2]). If we are allowed to consider a smaller scale of spectral projection, then Theorem 1.2 is an improvement of $\sqrt{\log\lambda}$ for (1.14), with the extra assumption that M has nonpositive curvature. The corollary is an improvement of (1.10). Note that (1.8) and (1.22) improve (1.10) for the whole range of p in dimension 2 except for $p = 4$[1].

Theorem 1.2 is related to certain L^p-estimates for eigenfunctions. For example, for 2-dimensional Riemannian manifolds, Sogge showed in [15] that

$$||\phi_\lambda||_{L^p(M)}/||\phi_\lambda||_{L^2(M)} = o(\lambda^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})}) \tag{1.23}$$

for some $2 < p < 6$ if and only if

$$\sup_{\gamma\in\Pi} ||\phi_\lambda||_{L^2(\gamma)}/||\phi_\lambda||_{L^2(M)} = o(\lambda^{\frac{1}{4}}). \tag{1.24}$$

[1] For $2 \le p \le \frac{2n}{n-1}$ when $k = n-1$, it is showed that $\delta(p) = \frac{n-1}{4} - \frac{n-2}{2p}$ in [2]. This estimate is also saturated for submanifolds of the standard sphere by the highest weight spherical harmonics.

This indicates relations between the restriction theorem and the L^p-estimates for eigenfunctions in [12] by Sogge, which showed that for any compact Riemannian manifold of dimension n, one has

$$||\phi_\lambda||_{L^p(M)} \lesssim \lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{p})}||\phi_\lambda||_{L^2(M)}, \quad \text{for } 2 \leq p \leq \frac{2(n+1)}{n-1}, \tag{1.25}$$

and

$$||\phi_\lambda||_{L^p(M)} \lesssim \lambda^{n(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}||\phi_\lambda||_{L^2(M)}, \quad \text{for } \frac{2(n+1)}{n-1} \leq p \leq \infty. \tag{1.26}$$

There have been several results showing that (1.26) can be improved for $p > \frac{2(n+1)}{n-1}$ (see [16] and [17]) to bounds of the form $||\phi_\lambda||_{L^p(M)}/||\phi_\lambda||_{L^2(M)} = o(\lambda^{n(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}})$ for fixed $p > 6$. Recently, Hassell and Tacey [7], following Bérard's [1] estimate for $p = \infty$, showed that for fixed $p > 6$, this ratio is $O(\lambda^{n(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}/\sqrt{\log \lambda})$ on Riemannian manifolds with constant negative curvature, which inspired our work.

Chapter 2

Set up of the proof of the improved restriction theorem

Let us first analyze the situation for any dimension n, which we will use in Chapters 3 and 4.

Take a real-valued multiplier operator $\chi \in \mathcal{S}(\mathbb{R})$ such that $\chi(0) = 1$, and $\hat{\chi}(t) = 0$ if $|t| \geq \frac{1}{2}$. Let $\rho = \chi^2$, then $\hat{\rho}(t) = 0$ if $|t| \geq 1$. Here, $\hat{\chi}$ is the Fourier Transform of χ. Same notations in the following.

For some number T, which will be determined later, and is approximately $\log \lambda$, we have $\chi(T(\lambda - \sqrt{-\Delta_g}))\varphi_\lambda = \varphi_\lambda$. The theorem is proved if we can show that for any $f \in L^2(M)$,

$$||\chi_T^\lambda f||_{L^p(\Sigma)} \lesssim \frac{\lambda^{\delta(p)}}{(\log \lambda)^{\frac{1}{2}}} ||f||_{L^2(M)}, \tag{2.1}$$

where $\chi_T^\lambda = \chi(T(\lambda - \sqrt{-\Delta_g}))$ is an operator from $L^2(M)$ to $L^p(\Sigma)$.

This is equivalent to showing that, for any $g \in L^{p'}(\Sigma)$,

$$||\chi_T^\lambda(\chi_T^\lambda)^* g||_{L^p(\Sigma)} \lesssim \frac{\lambda^{2\delta(p)}}{\log \lambda}||g||_{L^{p'}(\Sigma)}, \tag{2.2}$$

where p' is the conjugate number of p such that $\frac{1}{p}+\frac{1}{p'}=1$. and $(\chi_T^\lambda)^*$ is the conjugate operator of χ_T^λ, which maps $L^{p'}(\Sigma)$ into $L^2(M)$.

If $\{e_j(x)\}_{j\in\mathbb{N}}$ is an $L^2(M)$ orthonormal basis of eigenfunctions of $\sqrt{-\Delta_g}$, with eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$, and $\{E_j(x)\}_{j\in\mathbb{N}}$ is the projections onto the j-th eigenspace restricted to Σ, then $I|_\Sigma = \sum_{j\in\mathbb{N}} E_j$, and $\sqrt{-\Delta_g}|_\Sigma = \sum_{j\in\mathbb{N}} \lambda_j E_j$. If we set $\rho_T^\lambda = \rho(T(\lambda - \sqrt{-\Delta_g})) : L^2(M) \to L^p(\Sigma)$, then the kernel of $\chi_T^\lambda(\chi_T^\lambda)^*$ is the kernel of ρ_T^λ restricted to $\Sigma\times\Sigma$. This can be seen in the following way.

Expand χ_T^λ and $(\chi_T^\lambda)^*$,

$$\chi_T^\lambda f(x) = \sum_{j\in\mathbb{N}} \chi(T(\lambda-\lambda_j))e_j(x)\int_M e_j(y)f(y)dy, \quad \forall f\in L^2(M), \tag{2.3}$$

and

$$(\chi_T^\lambda)^* g(x) = \sum_{j\in\mathbb{N}} \chi(T(\lambda-\lambda_j))e_j(x)\int_\Sigma e_j(y)g(y)dy, \quad \forall g\in L^{p'}(\Sigma). \tag{2.4}$$

Then

$$\begin{aligned}\chi_T^\lambda(\chi_T^\lambda)^* g(x) &= \sum_{i,j\in\mathbb{N}} \chi(T(\lambda-\lambda_i))\chi(T(\lambda-\lambda_j))e_j(x)\int_M e_j(y)e_i(y)\int_\Sigma e_i(z)g(z)dzdy \\ &= \sum_{j\in\mathbb{N}} \chi(T(\lambda-\lambda_j))^2 e_j(x)\int_\Sigma e_j(z)g(z)dz \\ &= \sum_{j\in\mathbb{N}} \rho(T(\lambda-\lambda_j))e_j(x)\int_\Sigma e_j(z)g(z)dz.\end{aligned} \tag{2.5}$$

On the other hand,

$$
\begin{aligned}
\rho_T^\lambda &= \sum_{j\in\mathbb{N}} \rho(T(\lambda-\lambda_j))E_j \\
&= \sum_{j\in\mathbb{N}} \frac{1}{2\pi}\int_{-1}^{1} \hat\rho(t)e^{it[T(\lambda-\lambda_j)]}E_j dt \\
&= \sum_{j\in\mathbb{N}} \frac{1}{2\pi T}\int_{-T}^{T} \hat\rho(\frac{t}{T})e^{it(\lambda-\lambda_j)}E_j dt \\
&= \frac{1}{2\pi T}\int_{-T}^{T} \hat\rho(\frac{t}{T})e^{it(\lambda-\sqrt{-\Delta_g})}dt \\
&= \frac{1}{\pi T}\int_{-T}^{T} \hat\rho(\frac{t}{T})\cos(t\sqrt{-\Delta_g})e^{it\lambda}dt - \rho(T(\lambda+\sqrt{-\Delta_g}))
\end{aligned} \tag{2.6}
$$

Here, $\rho(T(\lambda+\sqrt{-\Delta_g}))$ is an operator whose kernel is $O(\lambda^{-N})$, for any $N\in\mathbb{N}$, so that we only have to estimate the first term. We are not going to emphasize the restriction to Σ until we get to the point when we take the L^p norm on Σ.

Denote the kernel of $\cos(t\sqrt{-\Delta_g})$ as $\cos(t\sqrt{-\Delta_g})(x,y)$, for $x,y\in M$, then for any $g\in L^{p'}(\Sigma)$,

$$
\chi_T^\lambda(\chi_T^\lambda)^* g(x) = \frac{1}{\pi T}\int_\Sigma\int_{-T}^{T} \hat\rho(\frac{t}{T})\cos(t\sqrt{-\Delta_g})(x,y)e^{it\lambda}g(y)dtdy + O(1). \tag{2.7}
$$

Take the $L^p(\Sigma)$ norm on both sides,

$$
||\chi_T^\lambda(\chi_T^\lambda)^* g||_{L^p(\Sigma)} \le \frac{1}{\pi T}(\int_\Sigma |\int_\Sigma\int_{-T}^{T} \hat\rho(\frac{t}{T})\cos(t\sqrt{-\Delta_g})(x,y)e^{it\lambda}g(y)dtdy|^p dx)^{1/p}+O(1). \tag{2.8}
$$

We are going to use Young's inequality (see [13]), with $\frac{1}{r} = 1-[(1-\frac{1}{p})-\frac{1}{p}] = \frac{2}{p}$, and

$$
K(x,y) = \frac{1}{\pi T}\int_{-T}^{T} \hat\rho(\frac{t}{T})(\cos t\sqrt{-\Delta_g})(x,y))e^{it\lambda}dt. \tag{2.9}
$$

Denote K as the operator with kernel $K(x,y)$ from now on.[1]

Since $K(x,y)$ is symmetric in x and y, once we have

$$\sup_{x\in\Sigma} ||K(x,\cdot)||_{L^r(\Sigma)} \lesssim \frac{\lambda^{2\delta(p)}}{\log\lambda}, \tag{2.10}$$

where $r = \frac{p}{2}$, then by Young's inequality, the theorem is proved.

We can use the same argument as in [18] to lift the manifold to $\mathbb{R}^n$. As stated in Theorem IV.1.3 in [10], since (M,g) has non-positive curvature, considering x to be a fixed point on Σ, there exists a universal covering map $p = \exp_x : \mathbb{R}^n \to M$. In this way, (M,g) is lifted to $(\mathbb{R}^n, \tilde{g})$, with the metric $\tilde{g} = (\exp_x)^* g$ being the pullback of g via $\exp_x$. $\tilde{g}$ is a complete Riemannian metric on $\mathbb{R}^n$. Define an automorphism for $(\mathbb{R}^n, \tilde{g})$, $\alpha : \mathbb{R}^n \to \mathbb{R}^n$, to be a deck transformation if

$$p \circ \alpha = p,$$

when we shall write $\alpha \in \mathrm{Aut}(p)$. If $\tilde{x} \in \mathbb{R}^n$ and $\alpha \in \mathrm{Aut}(p)$, let us call $\alpha(\tilde{x})$ the translate of $\tilde{x}$ by α, then we call a simply connected set $D \subset \mathbb{R}^n$ a fundamental domain of our universal cover p if every point in $\mathbb{R}^n$ is the translate of exactly one point in D. We can then identify our submanifold Σ in (M,g) uniquely with a submanifold in $D \subset \mathbb{R}^n$ with one-to-one correspondence. Likewise, a function $f(x)$ in M is uniquely identified by one $f_D(\tilde{x})$ on D if we set $f_D(\tilde{x}) = f(x)$, where $\tilde{x}$ is the unique point in $D \cap p^{-1}(x)$. Using f_D we can define a "periodic extension", $\tilde{f}$,

[1] The definition of $K(x,y)$ may be changed in this paper, but we always call K the corresponding operator with the kernel $K(x,y)$.

of f to $\mathbb{R}^n$ by defining $\tilde{f}(\tilde{y})$ to be equal to $f_D(\tilde{x})$ if $\tilde{x} = \tilde{y}$ modulo $\mathrm{Aut}(p)$, i.e. if $(\tilde{x}, \alpha) \in D \times \mathrm{Aut}(p)$ are the unique pair so that $\tilde{y} = \alpha(\tilde{x})$.

In this setting, we shall exploit the relationship between solutions of the wave equation on (M, g) of the form

$$\begin{cases} (\partial_t^2 - \Delta_g)u(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times M \\ u(0,\,\cdot\,) = f,\ \partial_t u(0,\,\cdot\,) = 0, \end{cases} \tag{2.11}$$

and certain ones on $(\mathbb{R}^n, \tilde{g})$

$$\begin{cases} (\partial_t^2 - \Delta_{\tilde{g}})\tilde{u}(t,\tilde{x}) = 0, \quad (t,\tilde{x}) \in \mathbb{R}_+ \times \mathbb{R}^n \\ \tilde{u}(0,\,\cdot\,) = \tilde{f},\ \partial_t \tilde{u}(0,\,\cdot\,) = 0. \end{cases} \tag{2.12}$$

If $(f(x), 0)$ is the Cauchy data in (2.11) and $(\tilde{f}(\tilde{x}), 0)$ is the periodic extension to $(\mathbb{R}^n, \tilde{g})$, then the solution $\tilde{u}(t, \tilde{x})$ to (2.12) must be a periodic function of $\tilde{x}$ since $\tilde{g}$ is the pullback of g via p and $p \circ \alpha = p$. As a result, we have that the solution to (2.11) must satisfy $u(t, x) = \tilde{u}(t, \tilde{x})$ if $\tilde{x} \in D$ and $p(\tilde{x}) = x$. Thus, periodic solutions to (2.12) correspond uniquely to solutions of (2.11). Note that $u(t, x) = \big(\cos(t\sqrt{-\Delta_g})f\big)(x)$ is the solution of (2.11), so that

$$\cos(t\sqrt{-\Delta_g})(x, y) = \sum_{\alpha \in \mathrm{Aut}(p)} \cos(t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x}, \alpha(\tilde{y})), \tag{2.13}$$

if $\tilde{x}$ and $\tilde{y}$ are the unique points in D for which $p(\tilde{x}) = x$ and $p(\tilde{y}) = y$.

To conclude this chapter, let us introduce the oscillatory following phase estimate that we are going to use throughout the proof of the main theorem.

Theorem 2.1 (Sogge, [13]). *Suppose that $\Phi(y) \in C^\infty$ has a non-degenerate critical point at $y = y_0$, i.e.*

$$\nabla\Phi(y_0) = 0 \tag{2.14}$$

but

$$\det(\partial^2\Phi/\partial y_j \partial y_k) \neq 0 \quad \text{when } y = y_0, \tag{2.15}$$

then

$$\int_{\mathbb{R}^n} e^{i\lambda\Phi(x)}\eta(x)dx = (\lambda/2\pi)^{-n/2} e^{i\lambda\Phi(y_0)}\eta(y_0)|\det\Phi''(y_0)|^{-1/2} e^{\frac{\pi i}{4} sgn\Phi''(y_0)} + O(\lambda^{-n/2-1}) \tag{2.16}$$

if η has small support.

The proof can be found in [13], Chapter 1.

Chapter 3

Proof of the improved restriction theorem, for $n = 2$

While we can prove Theorem 1.2 for any dimension n, we will prove the case when $n = 2$ first separately, as it is the simplest case, and does not involve interpolation or various sub-dimensions. Here is what it says.

Theorem 3.1. *Let (M, g) be a compact smooth boundaryless Riemannian surface with nonpositive sectional curvature, and γ be a smooth curve with finite length, then for any $f \in L^2(M)$, we have the following estimate*

$$||\sum_{|\lambda_j-\lambda|<(\log\lambda)^{-1}} E_j f||_{L^p(\gamma)} \lesssim \frac{\lambda^{\frac{1}{2}-\frac{1}{p}}}{(\log\lambda)^{\frac{1}{2}}}||f||_{L^2(M)}, \quad \forall p > 4. \tag{3.1}$$

We will prove Theorem 3.1 by the end of this section. By a partition of unity, we can assume that we fix x to be the mid-point of γ, and parametrize γ by its arc length centered at x so that

$$\gamma = \gamma[-1,1] \quad \text{and} \quad \gamma(0) = x, \tag{3.2}$$

and we may assume that the geodesic distance between any x and $y \in \gamma$ is comparable to the arc length between them on γ.

We need to estimate the $L^r(\gamma)$ norm of

$$\int_{-T}^{T} \hat{\rho}(\frac{t}{T})(\cos t\sqrt{-\Delta_g})(x,y)e^{it\lambda}dt = \sum_{\alpha\in \mathrm{Aut}(p)} \int_{-T}^{T} \hat{\rho}(\frac{t}{T})(\cos t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x},\alpha(\tilde{y}))e^{it\lambda}dt. \tag{3.3}$$

We should have the following estimates:

Up to an error of $O(\lambda^{-1})\exp(O(d_{\tilde{g}}(\tilde{x},\tilde{y})))+O(e^{dT})$ and $O(\lambda^{-1})\exp(O(d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))))+O(e^{dT})$ respectively,

$$\int_{-T}^{T} \hat{\rho}(\frac{t}{T})(\cos t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x},\tilde{y})e^{it\lambda}dt = O(\lambda) \quad \textit{when} \quad d_{\tilde{g}}(\tilde{x},\tilde{y}) < \frac{1}{\lambda}, \tag{3.4}$$

$$\int_{-T}^{T} \hat{\rho}(\frac{t}{T})(\cos t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x},\tilde{y})e^{it\lambda}dt = O((\frac{\lambda}{d_{\tilde{g}}(\tilde{x},\tilde{y})})^{1/2}) \quad \textit{when} \quad d_{\tilde{g}}(\tilde{x},\tilde{y}) \geq \frac{1}{\lambda}, \tag{3.5}$$

$$\alpha \neq Id, \int_{-T}^{T} \hat{\rho}(\frac{t}{T})(\cos t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x},\alpha(\tilde{y}))e^{it\lambda}dt = O((\frac{\lambda}{d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))})^{\frac{1}{2}}) \tag{3.6}$$

To prove (3.5) and (3.6), we need the following lemma.

Lemma 3.2. *Assume that $w(\tilde{x},\tilde{x}')$ is a smooth function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$, and $\Theta \in \mathbb{S}^{n-1}$, then there exist constants $a_\pm$ such that*

$$\int_{\mathbb{S}^{n-1}} e^{iw(\tilde{x},\tilde{x}')\cdot\Theta}d\Theta = \sqrt{2\pi}^{n-1}\sum_{\pm} a_\pm \frac{e^{\pm i|w(\tilde{x},\tilde{x}')|}}{|w(\tilde{x},\tilde{x}')|^{\frac{n-1}{2}}} + O(|w(\tilde{x},\tilde{x}')|^{-\frac{n-1}{2}-1}), \tag{3.7}$$

when $|w(\tilde{x}, \tilde{x}')| \geq 1.$

Proof. Without loss of generality, we may assume that $\tilde{x} = 0$. Let $w(\tilde{x}, \tilde{x}') = d_{\tilde{g}}(\tilde{x}, \tilde{x}')\Xi(\tilde{x}, \tilde{x}')$, where $\Xi(\tilde{x}, \tilde{x}') \in \mathbb{S}^{n-1}$. Divide $\mathbb{S}^{n-1}$ into the upper half and the lower half,

$$\mathbb{S}_{+}^{n-1} = \{(h(\tilde{x}'), \tilde{x}') : \tilde{x}' = (\tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_n) \in \mathbb{R}^{n-1}, h(x') = \sqrt{1 - \tilde{x}_2^2 - ..., \tilde{x}_n^2}\}$$

and

$$\mathbb{S}_{-}^{n-1} = \{(h(\tilde{x}'), \tilde{x}') : \tilde{x}' = (\tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_n) \in \mathbb{R}^{n-1}, h(\tilde{x}') = -\sqrt{1 - \tilde{x}_2^2 - ..., \tilde{x}_n^2}\}.$$

We are going to estimate the integral on $\mathbb{S}_{+}^{n-1}$, and it is the same for the lower half.

$$\int_{\mathbb{S}_{+}^{n-1}} e^{iw(\tilde{x},\tilde{x}')\cdot\Theta} d\Theta = \int_{B^{n-1}(0,1)} e^{id_{\tilde{g}}(\tilde{x},\tilde{x}')<\Xi(\tilde{x},\tilde{x}'),(h(\tilde{x}'),\tilde{x}')>} \sqrt{1 + |\bigtriangledown h(\tilde{x}')|^2} d\tilde{x}'^{1} \tag{3.8}$$

The unit normals to $\mathbb{S}_{+}^{n-1}$ at $(h(\tilde{x}'), \tilde{x}')$ are $\pm v \in \mathbb{S}^{n-1}$ satisfying

$$\bigtriangledown_{\tilde{x}'}(< v, (h(\tilde{x}'), \tilde{x}') >) = 0 \tag{3.9}$$

Then if $\bigtriangledown h = 0$, in which case $(h(\tilde{x}'), \tilde{x}') = (1, 0, ..., 0)^2$, $v = (1, 0, ..., 0)$ would be one of the unit normals. Furthermore,

$$\frac{\partial^2 \phi}{\partial \tilde{x}'_j \partial \tilde{x}'_k} = \frac{\partial^2 h}{\partial \tilde{x}'_j \partial \tilde{x}'_k}, \tag{3.10}$$

where $\phi(v, \tilde{x}') =< v, (h(\tilde{x}'), \tilde{x}') >$.

[1] $B^{n-1}(0, 1)$ is the unit ball in $\mathbb{R}^{n-1}$.

[2] In the lower half of the sphere, this corresponds to $(h(\tilde{x}'), \tilde{x}') = (-1, 0, ..., 0)$

Recall that the curvature of $\mathbb{S}_+^{n-1}$,

$$K = (1 + |\bigtriangledown h|^2)^{-\frac{n-1}{2}} \det(\frac{\partial^2 h}{\partial \tilde{x}'_j \partial \tilde{x}'_k}) \neq 0 \tag{3.11}$$

By a rotation argument, we know that

$$\det(\frac{\partial^2 \phi}{\partial \tilde{x}'_j \partial \tilde{x}'_k}) \neq 0, \tag{3.12}$$

when v is normal to $\mathbb{S}_+^{n-1}$ at $(h(\tilde{x}'), \tilde{x}')$.

Therefore, there are only 2 non-degenerate critical points of the phase function $w(\tilde{x}, \tilde{x}') \cdot \Theta$, which are $\pm\Xi(\tilde{x}, \tilde{x}')$. Applying stationary phase estimates Theorem 2.1, we get (3.7). □

Let us return to estimating the kernel $K(x, y)$. Applying the Hadamard Parametrix,

$$\begin{aligned} \cos(t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x}, \alpha(\tilde{y})) = {} & \frac{w_0(\tilde{x}, \alpha(\tilde{y}))}{(2\pi)^n} \sum_{\pm} \int_{\mathbb{R}^n} e^{i\Phi(\tilde{x}, \alpha(\tilde{y}))\cdot\xi \pm it|\xi|} d\xi \\ & + \sum_{\nu=1}^{N} w_\nu(\tilde{x}, \alpha(\tilde{y})) \mathcal{E}_\nu(t, d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))) + R_N(t, \tilde{x}, \alpha(\tilde{y})), \end{aligned} \tag{3.13}$$

where $|\Phi(\tilde{x}, \alpha(\tilde{y}))| = d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))$, $\mathcal{E}_\nu, \nu = 1, 2, 3, ...$ are defined recursively by $2\mathcal{E}_\nu(t, r) = -t \int_0^t \mathcal{E}_{\nu-1}(s, r) ds$, where $\mathcal{E}_0(t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \cos(t|\xi|) d\xi$[3], and $w_\nu(\tilde{x}, \alpha(\tilde{y}))$ equals some constant times $u_\nu(\tilde{x}, \alpha(\tilde{y}))$ that satisfies:

$$\begin{cases} u_0(\tilde{x}, \alpha(\tilde{y})) = \Theta^{-\frac{1}{2}}(\alpha(\tilde{y})) \\ u_{\nu+1}(\tilde{x}, \alpha(\tilde{y})) = \Theta(\alpha(\tilde{y})) \int_0^1 s^\nu \Theta^{\frac{1}{2}}(\tilde{x}_s) \Delta_{\tilde{g}} u_\nu(\tilde{x}, \tilde{x}_s) ds, \quad \nu \geq 0. \end{cases} \tag{3.14}$$

[3]Since $\mathcal{E}\nu(t, x)$ is invariant under the same radius, we consider $\mathcal{E}\nu(t, x) = \mathcal{E}\nu(t, |x|)$.

where $\Theta(\alpha(\tilde{y})) = (\det g_{ij}(\alpha(\tilde{y})))^{\frac{1}{2}}$, and $(\tilde{x}_s)_{s\in[0,1]}$ is the minimizing geodesic from $\tilde{x}$ to $\alpha(\tilde{y})$ parametrized proportionally to arc length. (see [1] and [18])

First note that for $N \geq n + \frac{3}{2}$, by using the energy estimates (see [14] Theorem 3.1.5), one can show that $|R_N(t, \tilde{x}, \alpha(\tilde{y}))| = O(e^{dt})$ for some constant $d > 0$, which is at most $O(e^{dT}) = O(\lambda^{d\beta})$ after we choose T to be approximately $\beta \log \lambda$, so that it is small compared to the first N terms, since we may choose β as close to 0 as possible.

Theorem 3.3. *Given an n-dimensional compact Riemannian manifold (M, g) with nonpositive curvature, and let $(\mathbb{R}^n, \tilde{g})$ be the universal covering of (M, g). Then if $N \geq n + \frac{3}{2}$, in local coordinates,*

$$(\cos t\sqrt{-\Delta_{\tilde{g}}})f(\tilde{x}) = \int K_N(t, \tilde{x}; \tilde{y})f(\tilde{y})dV_{\tilde{g}}(\tilde{y}) + \int R_N(t, \tilde{x}; \tilde{y})f(\tilde{y})dV_{\tilde{g}}(\tilde{y}), \quad (3.15)$$

where

$$K_N(t, \tilde{x}; \tilde{y}) = \sum_{\nu=0}^{N} w_\nu(\tilde{x}, \tilde{y})\mathcal{E}_\nu(t, d_{\tilde{g}}(\tilde{x}, \tilde{y})), \quad (3.16)$$

with the remainder kernel R_N satisfying

$$|R_N(t, \tilde{x}; \tilde{y}))| = O(e^{dt}). \quad (3.17)$$

for some number $d > 0$.

This comes from Equation (42) in [1]. The proof can be found in [1].

Proof. Here we give a sketch of the proof given in [14].

Consider $w(t,x) = \int R_N(t,x;y)f(y)dV_g(y)$, and $F_N(t,x) = (\partial_t^2 - \Delta_g)w(t,x)$. By the construction of the Hadamard parametrix, one has that

$$(\partial_t^2-\Delta_g)\sum_{\nu=0}^{N}\alpha_\nu(x,y)\mathcal{E}_\nu(t,d_g(x,y)) = (\det g_{jk}(y))^{-\frac{1}{2}}\delta_{0y}(t,x)-(\Delta_g\alpha_N(x,y))\mathcal{E}_N(t,d_g(x,y)), \tag{3.18}$$

so that

$$F_N(t,x) = \int (\Delta_g\alpha_N(x,y))\mathcal{E}_N(t,d_g(x,y))f(y)dV_g(y). \tag{3.19}$$

Recall the definition of $\mathcal{E}_\nu$ that it is homogeneous of degree $2N-n$, then we have, for any $m \le N - \frac{n+1}{2}$,

$$|\partial_{t,x,y}^\alpha(\Delta_g\alpha_N(x,y))\mathcal{E}_N(t,d_g(x,y))| \le C_T t^{2N-n-|\alpha|} \le C_T t^{2N-n}, \quad |\alpha| \le m \tag{3.20}$$

for some constant $C_T = O(e^{d'T})$.

By Minkowski's integral inequality, this gives, for any $|\alpha| + j \le m \le N - \frac{n+1}{2}$,

$$||\partial_t^j\partial_x^\alpha F_N(t,\cdot)||_{L^2} \le C_T t^{2N-j-\frac{n}{2}}||f||_{L^1}, \tag{3.21}$$

and therefore,

$$\int_0^t ||\partial_s^m F_N(s,\cdot)||_{L^2}ds + \sum_{j=0}^{m-1}||\partial_t^j\partial_x^\alpha F_N(t,\cdot)||_{L^2} \le C_T t^{2N-\frac{n}{2}+1}||f||_{L^1}, \quad \forall|\alpha| \le m-j. \tag{3.22}$$

As a result, the energy estimates in Lemma 3.1.3 in [14],

$$\sum_{j=0}^{m+1}||\partial_t^j w(t,\cdot)||_{H^{m+1-j}} \le C_{m,T}(\int_0^t ||\partial_s^m F(x,\cdot)||_{L^2}ds + \sum_{j=0}^{m-1}||\partial_t^j F(t,\cdot)||_{H^{m-1-j}})^4 \tag{3.23}$$

[4] Here, since $C_{m,T} = o(e^{dT})$, we may replace it by C_T.

yields

$$||w(t,\cdot)||_{H^{m+1}} + ||\partial_t w(t,\cdot)||_{H^m} \leq C_T t^{2N-\frac{n}{2}+1}||f||_{L^1}. \quad (3.24)$$

Now since $N \geq n + \frac{3}{2}$, we can choose an $m \geq 0$ so that $\frac{n}{2} < m \leq N - \frac{n+1}{2}$, and $w(t,\cdot) \in H^m(\mathbb{R}^n)$. Use Lemma 3.1.4 in [14],

$$||f||_{L^\infty}(\mathbb{R}^n) \leq C_T(t^{m-\frac{n}{2}}||f||_{\dot{H}^m(\mathbb{R}^\times)} + t^{-\frac{n}{2}}||f||_{L^2(\mathbb{R}^n)}), \quad \forall f \in H^m(\mathbb{R}^n), \quad (3.25)$$

we conclude that

$$|w(t,\cdot)| \leq C_T(t^{m-\frac{n}{2}}||w(t,\cdot)||_{\dot{H}^m} + t^{-\frac{n}{2}}||w(t,\cdot)||_{L^2}) \leq C_T t^{2N-n+1+m}||f||_{L^1}. \quad (3.26)$$

That means,

$$|\int R_N(t,x;y)f(y)dV_g(y)| \leq Ct^{2N-n+1+m}||f||_{L^1}. \quad (3.27)$$

By taking the supremum over all $||f||_{L^1} \leq 1$, we see that this implies (3.17).

□

By this theorem,

$$\int_{-T}^{T} |R_N(t,\tilde{x},\alpha(\tilde{y}))|dt \leq C\int_0^T e^{dt}dt = O(e^{dT}). \quad (3.28)$$

Moreover, for $\nu = 1,2,3,...$, we have the following estimate for $\mathcal{E}_\nu(t,r)$.

Theorem 3.4. *For $\nu = 0,1,2,...$ and $\mathcal{E}_\nu(t,r)$ defined above, we have*

$$|\int \hat{\rho}(t)e^{it\lambda}\mathcal{E}_\nu(t,r)dt| = O(\lambda^{n-1-2\nu}), \quad \lambda \geq 1 \quad (3.29)$$

Proof. Recall that

$$\mathcal{E}_0(t,r) = \frac{H(t)}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\Phi(\tilde{x},\tilde{y})\cdot\xi} \cos t|\xi| d\xi, \tag{3.30}$$

so that

$$\begin{aligned} |\int \hat{\rho}(t) e^{it\lambda} \mathcal{E}_0(t,r) dt| &= |\frac{1}{2(2\pi)^n} \int \int_{\mathbb{R}^n} \hat{\rho}(t) e^{it(\lambda \pm |\xi|) + i\Phi(\tilde{x},\tilde{y})\cdot\xi} d\xi dt| \\ &\approx |\int_{\mathbb{R}^n} [\rho(\lambda + |\xi|) + \rho(\lambda - |\xi|)] e^{i\Phi(\tilde{x},\tilde{y})\cdot\xi} d\xi| \\ &\leq \int_{\mathbb{R}^n} |\rho(\lambda + |\xi|)| + |\rho(\lambda - |\xi|)| d\xi \\ &= O(\lambda^{n-1}). \end{aligned} \tag{3.31}$$

By the definition of $\mathcal{E}_\nu$ such that $\dfrac{\partial \mathcal{E}_\nu}{\partial t} = \frac{t}{2}\mathcal{E}_{\nu-1}$ and integrate by parts, we get that for any $\nu = 1, 2, 3, ...,$

$$\int \hat{\rho}(t) e^{it\lambda} \mathcal{E}_\nu(t,r) dt = O(\lambda^{n-1-2\nu}). \tag{3.32}$$

□

The following theorem has been shown by Bérard in [1] about the size of the coefficients $u_k(\tilde{x},\tilde{y})$.

Theorem 3.5. *Let (M,g) be a compact n-dimensional Riemannian manifold and let σ be its sectional curvature (hence, there is a number Γ such that $-\Gamma^2 \leq \sigma$). Assume that either*

1.$n = 2$, and M does not have conjugate points;

or

2.$-\Gamma^2 \leq \sigma \leq 0$; i.e. M has nonpositive sectional curvature.

Let $(\mathbb{R}^n, \tilde{g})$ *be the universal covering of* (M, g)*, and let* $\tilde{u}_\nu$*,* $\nu = 0, 1, 2, ...$ *be defined by the relations* (3.14)*, then for any integers* l *and* ν

$$\Delta_{\tilde{g}}^l \tilde{u}_\nu(\tilde{x}, \tilde{y}) = O(\exp(O(d_{\tilde{g}}(\tilde{x}, \tilde{y})))). \tag{3.33}$$

The proof can be found in [1] Appendix: Growth of the Functions $u_k(x, y)$.

Since $w_\nu(\tilde{x}, \alpha(\tilde{y}))$ is a constant times $\tilde{u}_\nu(\tilde{x}, \alpha(\tilde{y}))$, this theorem tells us that $|w_\nu(\tilde{x}, \alpha(\tilde{y}))| = O(\exp(c_\nu d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))))$, for some constant c_ν depending on ν.

Moreover, denote that $\psi(t) = \hat{\rho}(\frac{t}{T})$, and $\tilde{\psi}$ as the inverse Fourier Transform of ψ, we have $\tilde{\psi} \in \mathcal{S}(\mathbb{R})$ such that

$$|\tilde{\psi}(t)| \leq T(1 + T|t|)^{-N}, \quad \text{for all } N \in \mathbb{N}. \tag{3.34}$$

Therefore,

$$\begin{aligned}
&\sum_{\nu=1}^{N} |w_\nu(\tilde{x}, \alpha(\tilde{y})) \int_{-T}^{T} \hat{\rho}(\frac{t}{T}) e^{it\lambda} \mathcal{E}_\nu(t, d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})))dt| \\
=&\sum_{\nu=1}^{N} O(T(T\lambda)^{n-1-2\nu} \exp(c_\nu d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})))) \\
=&O(T^{n-2}\lambda^{n-3} \exp(C_N d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})))),
\end{aligned} \tag{3.35}$$

for some C_N depending on $c_1, c_2, ..., c_{N-1}$.

All in all, taking $n = 2$, and disregarding the integral of the remainder kernel,

$$\begin{aligned}
&|\int_{-T}^{T} \hat{\rho}(\frac{t}{T}) \cos(t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x}, \alpha(\tilde{y})) e^{it\lambda} dt| \\
=&|\int_{-T}^{T} \hat{\rho}(\frac{t}{T}) \frac{w_0(\tilde{x}, \alpha(\tilde{y}))}{4\pi^2} \sum_{\pm} \int_{\mathbb{R}^2} e^{i\Phi(\tilde{x}, \alpha(\tilde{y}))\cdot\xi \pm it|\xi|} e^{it\lambda} d\xi dt| + O(\lambda^{-1} \exp(C_N d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})))).
\end{aligned} \tag{3.36}$$

On the other hand, $w_0(\tilde{x}, \tilde{y})$ has a better estimate. By applying Günther's Comparison Theorem [5], with the assumption of nonpositive curvature, we can show that $|w_0(\tilde{x}, \tilde{y})| = O(1)$. The proof is given by Sogge and Zelditch in [18] for $n = 2$. Let's see the case for any dimension n. In the geodesic polar coordinates we are using, $t\Theta$, $t > 0$, $\Theta \in \mathbb{S}^{n-1}$, for $(\mathbb{R}^n, \tilde{g})$, the metric $\tilde{g}$ takes the form

$$ds^2 = dt^2 + \mathcal{A}^2(t, \Theta)\, d\Theta^2, \tag{3.37}$$

where we may assume that $\mathcal{A}(t, \Theta) > 0$ for $t > 0$. Consequently, the volume element in these coordinates is given by

$$dV_g(t, \theta) = \mathcal{A}(t, \Theta)\, dtd\Theta, \tag{3.38}$$

and by Günther's [5] comparison theorem if the curvature of (M, g), which is the same as that of $(\mathbb{R}^n, \tilde{g})$ is nonpositive, we have

$$\mathcal{A}(t, \theta) \geq t^{n-1}, \tag{3.39}$$

where t^{n-1} is the volume element of the Euclidean space. While in geodesic normal coordinates about x, we have

$$w_0(x, y) = \big(\det\, g_{ij}(y)\big)^{-\frac{1}{4}},$$

(see [1], [6] or §2.4 in [14]). If y has geodesic polar coordinates (t, Θ) about x, then $t = d_{\tilde{g}}(x, y)$, so that $w_0(x, y) = \sqrt{t^{n-1}/\mathcal{A}(t, \Theta)} \leq 1$.

Therefore,

$$|\sum_{\pm}\int_{\mathbb{R}^2}\int_{-T}^{T} e^{i\Phi(\tilde{x},\tilde{y})\cdot\xi\pm it|\xi|+it\lambda}\hat{\rho}(\frac{t}{T})dtd\xi| =|\int_{\mathbb{R}^2} e^{i\Phi(\tilde{x},\tilde{y})\cdot\xi}(\tilde{\psi}(\lambda+|\xi|)+\tilde{\psi}(\lambda-|\xi|))d\xi|$$
$$\leq \int_{\mathbb{R}^2}|\tilde{\psi}(\lambda+|\xi|)|d\xi + \int_{\mathbb{R}^2}|\tilde{\psi}(\lambda-|\xi|)|d\xi \tag{3.40}$$

Note that $\tilde{\psi}(\lambda+|\xi|) = O(T(1+\lambda+|\xi|)^{-N})$, for any $N \in \mathbb{N}$, so that $\int_{\mathbb{R}^2}|\tilde{\psi}(\lambda+|\xi|)|d\xi$ can be arbitrarily small, while $\tilde{\psi}(\lambda-|\xi|) = O(T(1+T|\lambda-|\xi||)^{-N})$, for any $N \in \mathbb{N}$, so that $\int_{\mathbb{R}^2}|\tilde{\psi}(\lambda-|\xi|)|d\xi \lesssim T\int_{\lambda-1\leq|\xi|\leq\lambda+1}(1+T|\lambda-|\xi||)^{-N}d\xi = O(\lambda)$, provided that $\lambda \geq 1$. So

$$\int_{-T}^{T}\hat{\rho}(\frac{t}{T})(\cos t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x},\tilde{y})e^{it\lambda}dt = O(\lambda) + O(\lambda^{-1}\exp(C_N d_{\tilde{g}}(\tilde{x},\tilde{y}))), \tag{3.41}$$

disregarding the integral of the remainder kernel.

However, this estimate can be improved when $d_{\tilde{g}}(\tilde{x},\tilde{y}) \geq \frac{1}{\lambda}$.

As we can see, the main contribution of

$$\cos(t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x},\tilde{y}) = \frac{w_0(\tilde{x},\tilde{y})}{4\pi^2}\sum_{\pm}\int_{\mathbb{R}^2} e^{i\Phi(\tilde{x},\tilde{y})\cdot\xi\pm it|\xi|}d\xi + \sum_{\nu=1}^{N} w_\nu(\tilde{x},\tilde{y})\mathcal{E}_\nu(t,d_{\tilde{g}}(\tilde{x},\tilde{y})) + R_N(t,\tilde{x},\tilde{y}) \tag{3.42}$$

comes from the first term, and the corresponding term in $\int_{-T}^{T}\hat{\rho}(\frac{t}{T})(\cos t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x},\tilde{y})e^{it\lambda}dt$ is bounded by

$$C|\sum_{\pm}\int_{-T}^{T}\int_{\mathbb{R}^2}\hat{\rho}(\frac{t}{T})e^{i\Phi(\tilde{x},\tilde{y})\cdot\xi\pm it|\xi|}e^{it\lambda}dtd\xi| = C|\sum_{\pm}\int_{-T}^{T}\int_{0}^{\infty}\int_{0}^{2\pi}\hat{\rho}(\frac{t}{T})e^{ir\Phi(\tilde{x},\tilde{y})\cdot\Theta\pm itr+it\lambda}rdtdrd\theta|. \tag{3.43}$$

Integrate with respect to t first, then the quantity above is bounded by a constant times

$$\sum_{\pm} \int_0^{\infty} \int_0^{2\pi} \tilde{\psi}(\lambda \pm r) e^{ir\Phi(\tilde{x},\tilde{y})\cdot\Theta} r d\theta dr. \tag{3.44}$$

Because $|\tilde{\psi}(\lambda \pm r)| \lesssim T(1 + T|\lambda \pm r|)^{-N}$ for any $N > 0$, the term with $\tilde{\psi}(\lambda + r)$ in the sum is $O(1)$, while the other term with $\tilde{\psi}(\lambda - r)$ is significant only when r is comparable to λ, say, $c_1\lambda < r < c_2\lambda$ for some constants c_1 and c_2. In this case, as we assumed that $d_{\tilde{g}}(\tilde{x}, \tilde{y}) \geq \frac{1}{\lambda}$, we can also assume that $d_{\tilde{g}}(\tilde{x}, \tilde{y}) \gtrsim \frac{1}{r}$.

By Lemma 3.2, $\int_0^{2\pi} e^{iw\cdot\Theta} d\theta = \sqrt{2\pi}|w|^{-1/2} \sum_{\pm} a_{\pm} e^{\pm i|w|} + O(|w|^{-3/2})$, $|w| \geq 1$, where $w = r\Phi(\tilde{x}, \tilde{y})$. Integrate up θ, the above quantity is then controlled by a constant times

$$\begin{aligned}
&|\sum_{\pm} \int_{c_1\lambda}^{c_2\lambda} \tilde{\psi}(\lambda - r)|r d_{\tilde{g}}(\tilde{x}, \tilde{y})|^{-1/2} e^{\pm i r d_{\tilde{g}}(\tilde{x},\tilde{y})} r dr + \int_{c_1\lambda}^{c_2\lambda} \tilde{\psi}(\lambda - r)|r d_{\tilde{g}}(\tilde{x}, \tilde{y})|^{-3/2} r dr| \\
\leq & d_{\tilde{g}}(\tilde{x}, \tilde{y})^{-1/2} \int_{c_1\lambda}^{c_2\lambda} |\tilde{\psi}(\lambda - r)| r^{1/2} dr + d_{\tilde{g}}(\tilde{x}, \tilde{y})^{-3/2} \int_{c_1\lambda}^{c_2\lambda} |\tilde{\psi}(\lambda - r)| r^{-1/2} dr \\
= & d_{\tilde{g}}(\tilde{x}, \tilde{y})^{-1/2} O(\lambda^{1/2}) + O(d_{\tilde{g}}(\tilde{x}, \tilde{y})^{-1}) \\
= & O\left(\left(\frac{\lambda}{d_{\tilde{g}}(\tilde{x}, \tilde{y})}\right)^{1/2}\right)
\end{aligned} \tag{3.45}$$

Note that these two equalities are still valid when c_1 and c_2 are changed to 0 and ∞.

The first equality is because of the following.

$$\begin{aligned}\int_{c_1\lambda}^{c_2\lambda} |\tilde{\psi}(\lambda-r)|r^{1/2}dr \leq & T\int_{c_1\lambda}^{c_2\lambda}(1+T|\lambda-r|)^{-N}r^{1/2}dr \quad \text{(for any } N, \text{ take } N=2)\\ =& T\lambda^{3/2}\int_{c_1}^{c_2}(1+T\lambda|1-r|)^{-2}r^{1/2}dr\\ =& T\lambda^{3/2}\int_{1}^{c_2}(T\lambda r-T\lambda+1)^{-2}r^{1/2}dr+T\lambda^{3/2}\int_{c_1}^{1}(T\lambda-T\lambda r+1)^{-2}r^{1/2}dr\\ \leq& [\lambda^{1/2}(T\lambda r-T\lambda+1)^{-1}r^{1/2}|_{c_2}^{1}+\lambda^{1/2}\int_1^{c_2}\frac{1}{2}r^{-1/2}(T\lambda r-T\lambda+1)^{-1}dr]\\ &+[\lambda^{1/2}(T\lambda-T\lambda r+1)^{-1}r^{1/2}|_{c_1}^{1}+\lambda^{1/2}\int_{c_1}^{1}\frac{1}{2}r^{-1/2}(T\lambda-T\lambda r+1)^{-1}dr]\\ =& O(\lambda^{1/2}).\end{aligned} \tag{3.46}$$

And the second one comes from

$$\begin{aligned}\int_{c_1\lambda}^{c_2\lambda} |\tilde{\psi}(\lambda-r)|r^{-1/2}dr \lesssim & T\int_{c_1\lambda}^{c_2\lambda}(1+T|\lambda-r|)^{-N}r^{-1/2}dr \quad \text{(for any } N, \text{ take } N=2)\\ \leq& Td_{\tilde{g}}(\tilde{x},\tilde{y})^{1/2}\lambda\int_{c_1}^{c_2}(1+T\lambda|1-r|)^{-N}dr \quad (d_{\tilde{g}}(\tilde{x},\tilde{y})\\ =& d_{\tilde{g}}(\tilde{x},\tilde{y})^{1/2}[T\lambda\int_1^{c_2}(T\lambda r-T\lambda+1)^{-2}dr+T\lambda\int_{c_1}^{1}(1+T\lambda-T\lambda r)^{-2}dr]\\ =& d_{\tilde{g}}(\tilde{x},\tilde{y})^{1/2}[(T\lambda r-T\lambda+1)^{-1}|_{c_2}^{1}+(1+T\lambda-T\lambda r)^{-1}|_{c_1}^{1}]\\ =& O(d_{\tilde{g}}(\tilde{x},\tilde{y})^{1/2}).\end{aligned} \tag{3.47}$$

Therefore, when $d_{\tilde{g}}(\tilde{x},\tilde{y}) \geq \frac{1}{\lambda}$,

$$|\frac{w_0(\tilde{x},\tilde{y})}{4\pi^2}\sum_{\pm}\int_{\mathbb{R}^2}\hat{\rho}(\frac{t}{T})e^{i\Phi(\tilde{x},\tilde{y})\cdot\xi\pm it|\xi|}e^{it\lambda}d\xi| = O((\frac{\lambda}{d_{\tilde{g}}(\tilde{x},\tilde{y})})^{\frac{1}{2}}). \tag{3.48}$$

Now we have finished the estimates for $\alpha = Id$. For $\alpha \neq Id$, note that we can find a constant C_p that is different from 0, depending on the universal cover, p, of the manifold M, such that

$$d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})) > C_p, \tag{3.49}$$

for all $\alpha \in \mathrm{Aut}(p)$ different from Id. The constant C_p comes from the fact that if we assume that the injectivity radius of M is greater than a number, say, 1, and that x is the center of some geodesic ball with radius one contained in M, then we can choose the fundamental domain D such that x is at least some distance, say, $C_p > 1$, away from any translation of D, which we denote as $\alpha(D)$, for any $\alpha \in \mathrm{Aut}(p)$ that is not identity. Therefore, we may use the estimates for $d_{\tilde{g}}(\tilde{x}, \tilde{y}) \geq \frac{1}{\lambda}$ before, assuming λ is larger than $\frac{1}{C_p}$. Use the Hadamard parametrix, (see [18]), similarly as before, estimating only the main term,

$$\begin{aligned}
&\left|\int_{-T}^{T} \hat{\rho}(\frac{t}{T})(\cos t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x}, \alpha(\tilde{y}))e^{it\lambda}dt\right| \\
\lesssim &\left|(2\pi)^{-2}\int_{\mathbb{R}^2}\int_{-T}^{T} \hat{\rho}(\frac{t}{T})e^{i\Phi(\tilde{x},\alpha(\tilde{y}))\cdot\xi}\cos(t|\xi|)e^{it\lambda}dt\right| \\
\lesssim &\sum_{\pm}\left|\int_0^{2\pi}\int_0^{\infty}\int_{-T}^{T} e^{ir\Phi(\tilde{x},\alpha(\tilde{y}))\cdot\Theta\pm itr+it\lambda}\hat{\rho}(\frac{t}{T})rdtdrd\theta\right| \\
= &\sum_{\pm}\left|\int_0^{\infty}\int_0^{2\pi}\tilde{\psi}(\lambda - r)e^{ir\Phi(\tilde{x},\alpha(\tilde{y}))\cdot\Theta}rd\theta dr\right| \\
\lesssim &\sum_{\pm}\left|\int_0^{\infty}\tilde{\psi}(\lambda - r)|rd_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))|^{-\frac{1}{2}}e^{ird_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))}rdr + \int_0^{\infty}\tilde{\psi}(\lambda - r)|rd_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))|^{-\frac{3}{2}}rdr\right| \\
= &O\Big(\big(\frac{\lambda}{d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))}\big)^{\frac{1}{2}}\Big).
\end{aligned} \tag{3.50}$$

Now we have shown all the estimates (3.4), (3.5), and (3.6). Totally, $K(x, y)$ is

$$O(\frac{1}{T}(\frac{\lambda}{\lambda^{-1} + d_{\tilde{g}}(\tilde{x}, \tilde{y})})^{\frac{1}{2}}) + \sum_{Id\neq\alpha\in\mathrm{Aut}(p)}[O(\frac{1}{T}(\frac{\lambda}{d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))})^{1/2}) + O(\frac{e^{ET}}{T})], \tag{3.51}$$

where $E = \max\{C_N, d\} + 1$.

Note that, by the finite propagation speed of the wave operator $\partial_t^2 - \Delta_{\tilde{g}}$, $d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})) \leq T$ in the support of $\cos(t\sqrt{-\Delta_g})(\tilde{x}, \alpha(\tilde{y}))$. While M is a compact manifold with non-positive sectional curvature, the number of terms of α's such that $d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})) \leq T$ is at most e^{cT}[5], for some constant c depending on the curvature, by the Bishop Comparison Theorem (see [10] [18]).

We take the $L^r(\gamma)$ norms of each individual term first, then by the Minkowski's inequality, $||K(x, \cdot)||_{L^r(\gamma[-1,1])}$ is bounded by the sum. Also note that we may consider the geodesic distance to be comparable to the arc length of the geodesic.

The first term is simple, and it is controlled by a constant times

$$\frac{1}{T}\left(\int_0^1 \left(\frac{\lambda}{\lambda^{-1} + \tau}\right)^{\frac{r}{2}} d\tau\right)^{1/r} = O\left(\frac{\lambda^{\frac{p-2}{p}}}{T}\right). \tag{3.52}$$

Accounting in the number of terms of those α's, the second term is bounded by a constant times

$$e^{cT} \cdot \frac{\lambda^{\frac{1}{2}}}{T}\left(\int_0^1 \left(\frac{1}{C_p}\right)^{\frac{r}{2}} d\tau\right)^{\frac{1}{r}} = O\left(e^{cT}\frac{\lambda^{\frac{1}{2}}}{T}\right). \tag{3.53}$$

Therefore,

$$\begin{aligned} ||K(x, \cdot)||_{L^r(\gamma[-1,1])} =& O\left(\frac{\lambda^{\frac{p-2}{p}}}{T}\right) + O\left(e^{cT}\frac{\lambda^{\frac{1}{2}}}{T}\right) + O\left(\frac{e^{(c+E)T}}{T}\right) \\ =& I + II + III. \end{aligned} \tag{3.54}$$

Now take $T = \beta \log \lambda$, where $\beta \leq \dfrac{p-4}{2(c+E)p}$. (Note that we can assume that

[5]The number of terms of α's such that $d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})) \leq T$ is also bounded below by $e^{c'T}$ for some constant c' depending on the curvature of the manifold, according to Günther and Bishop's Comparison Theorem in [10] (also see [18]).

$c \neq 0$, otherwise, there is only one α that we are considering, which is $\alpha = Id$.) Then

$$I = II = O(\frac{\lambda^{\frac{p-2}{p}}}{\log \lambda}), \tag{3.55}$$

and

$$III = o(\frac{\lambda^{\frac{p-2}{p}}}{\log \lambda}). \tag{3.56}$$

Summing up, we get that

$$||K(x, \cdot)||_{L^r(\gamma[-1,1])} = O(\frac{\lambda^{\frac{p-2}{p}}}{\log \lambda}). \tag{3.57}$$

Now apply Young's inequality, with $r = \frac{p}{2}$, we get that

$$\forall f \in L^{p'}(\gamma), ||\chi_T^\lambda (\chi_T^\lambda)^* f||_{L^p(\gamma)} \lesssim \frac{(1+\lambda)^{1-\frac{2}{p}}}{\log \lambda} ||f||_{L^{p'}(\gamma)}.$$

Therefore, Theorem 3.1 is proved.

Chapter 4

Higher dimensions, $n \geq 3$

Now we move on to the case for $n \geq 3$. While we want to show Theorem 1.2 for the full range of p directly, we can only show it under the condition that $p > \frac{4k}{n-1}$ using the same method as in the last chapter. Although we only need $p = \infty$ later to interpolate and get to the full version of Theorem 1.2, we will show the most as we can for the moment.

Theorem 4.1. *Let (M, g) be a compact smooth n-dimensional boundaryless Riemannian manifold with nonpositive curvature, and Σ be an k-dimensional compact smooth submanifold on M, then for any $f \in L^2(M)$, we have the following estimate*

$$|| \sum_{|\lambda_j - \lambda| \leq (\log \lambda)^{-1}} E_j f ||_{L^p(\Sigma)} \lesssim \frac{\lambda^{\delta(p)}}{(\log \lambda)^{\frac{1}{2}}} ||f||_{L^2(M)}, \quad \forall p > \frac{4k}{n-1}, \tag{4.1}$$

where

$$\delta(p) = \frac{n-1}{2} - \frac{k}{p}. \tag{4.2}$$

Remark 4.2. *Note that although this estimate is not complete (that works for all $p > 2$) for general numbers $k < n$, we get the complete range of $p \geq 2$ when k and n satisfy $\frac{4k}{n-1} < 2$. That means that we get the improvement for all $p \geq 2$ when $k = 1$, $n > 3$; $k = 2$, $n > 5$; etc..*

For $n \geq 3$, for the sake of using interpolation later, we need to insert a bump function[1]. Take $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi(t) = 1$ when $|t| \leq \frac{1}{2}$ and $\varphi(t) = 0$ when $|t| > 1$. Then we only have to consider the following kernel[2]

$$K(x,y) = \frac{1}{\pi T}\int_{-T}^{T}(1-\varphi(t))\hat{\rho}(\frac{t}{T})(\cos t\sqrt{-\Delta_g})(x,y))e^{it\lambda}dt, \tag{4.3}$$

which is non-zero only when $|t| > \frac{1}{2}$. In the following discussion, we may sometimes only show estimates for $K(x,y)$ when $t > \frac{1}{2}$, as the part for $t < -\frac{1}{2}$ can be done similarly.

The reason why we only consider the above kernel $K(x,y)$ is because of the following lemma.

Lemma 4.3. *For $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi(t) = 1$ when $|t| \leq \frac{1}{2}$ and $\varphi(t) = 0$ when $|t| > 1$. Let*

$$\tilde{K}(x,y) = \frac{1}{\pi T}\int_{-1}^{1}\varphi(t)\hat{\rho}(\frac{t}{T})(\cos t\sqrt{-\Delta_g})(x,y)e^{it\lambda}dt, \tag{4.4}$$

then

$$\sup_x ||\tilde{K}(x,\cdot)||_{L^r(\Sigma)} = O(\frac{\lambda^{2\delta(p)}}{\log\lambda}). \tag{4.5}$$

[1] We do not need the bump function if we simply want to prove Theorem 4.1.

[2] This kernel is different from the one in (2.9).

We will postpone the proof to the end of this section.

Now we are ready to prove Theorem 4.1, which is essentially the same as the lower dimension case, and what we need to show is (2.10). By a partition of unity, we may choose some point $x \in \Sigma$, and consider Σ to be within a ball with geodesic radius 1 centered at x, and under the geodesic normal coordinates centered at x, parametrize Σ as

$$\Sigma = \{(t, \Theta) | y = \exp_x(t\Theta) \in \Sigma, t \in [-1, 1], \Theta \in \mathbb{S}^{k-1}\}$$

Applying the Hadamard Parametrix, for any $\alpha \in \mathrm{Aut}(p)$,

$$\cos(t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x}, \alpha(\tilde{y})) = \frac{w_0(\tilde{x}, \alpha(\tilde{y}))}{(2\pi)^n} \sum_{\pm} \int_{\mathbb{R}^n} e^{i\Phi(\tilde{x},\alpha(\tilde{y}))\cdot\xi \pm it|\xi|} d\xi + \sum_{\nu=1}^{\infty} w_\nu(\tilde{x}, \tilde{y}) \mathcal{E}_\nu(t, d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))) + R_N(t, \tilde{x}, \alpha(\tilde{y})), \quad (4.6)$$

where $|\Phi(\tilde{x}, \alpha(\tilde{y}))| = d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))$, and $w_\nu, \mathcal{E}_\nu, \nu = 1, 2, 3, ...$ are those described in Chapter 3.

By Theorem 3.5,

$$\int_{-T}^{T} |R_N(t, \tilde{x}, \alpha(\tilde{y}))| dt \lesssim \int_0^T e^{dt} dt = O(e^{dT}). \quad (4.7)$$

Moreover, by (3.29), for $\nu = 1, 2, 3, ...$,

$$\left| \int_{-T}^{T} (1 - \varphi(t)) \hat{\rho}(\frac{t}{T}) e^{it\lambda} \mathcal{E}_\nu(t, d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))) dt \right| = O(T(T\lambda)^{n-1-2\nu}). \quad (4.8)$$

Since $|w_\nu(\tilde{x}, \alpha(\tilde{y}))| = O(\exp(c_\nu d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))))$ by [1], for some constant c_ν depending on ν,

$$
\begin{aligned}
&\sum_{\nu=1}^{N}|w_\nu(\tilde{x},\alpha(\tilde{y}))\int_{-T}^{T}(1-\varphi(t))\hat{\rho}(\frac{t}{T})e^{it\lambda}\mathcal{E}_\nu(t,d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y})))dt| \\
=&\sum_{\nu=1}^{N}O(T(T\lambda)^{n-1-2\nu}\exp(c_\nu d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y})))) \qquad (4.9)\\
=&O(T^{n-2}\lambda^{n-3}\exp(C_N d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y})))),
\end{aligned}
$$

for some C_N depending on $c_1, c_2, ..., c_{N-1}$.

All in all, disregarding the integral of the remainder kernel,

$$
\begin{aligned}
&|\int_{-T}^{T}(1-\varphi(t))\hat{\rho}(\frac{t}{T})\cos(t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x},\alpha(\tilde{y}))e^{it\lambda}dt| \\
&\quad =|\int_{-T}^{T}(1-\varphi(t))\hat{\rho}(\frac{t}{T})\frac{w_0(\tilde{x},\tilde{y})}{(2\pi)^n}\sum_{\pm}\int_{\mathbb{R}^2}e^{i\Phi(\tilde{x},\alpha(\tilde{y}))\cdot\xi\pm it|\xi|}e^{it\lambda}d\xi dt| \\
&\qquad\qquad + O(T^{n-2}\lambda^{n-3}\exp(C_N d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y})))). \qquad (4.10)
\end{aligned}
$$

On the other hand, $|w_0(\tilde{x},\tilde{y})| = O(1)$ (see [18]) by applying Günther's Comparison Theorem in [5], and for

$$
|\sum_{\pm}\int_{\mathbb{R}^n}\int_{-T}^{T}(1-\varphi(t))e^{i\Phi(\tilde{x},\alpha(\tilde{y}))\cdot\xi\pm it|\xi|+it\lambda}\hat{\rho}(\frac{t}{T})dtd\xi|, \qquad (4.11)
$$

as we may assume as before that $d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y})) > \frac{1}{2}$ by the stationary phase estimates in [13].

Denote that $\psi(t) = (1-\varphi(t))\hat{\rho}(\frac{t}{T})$, and $\tilde{\psi}$ is the inverse Fourier Transform of ψ.

Again we have, $\tilde{\psi}(\lambda+|\xi|) = O(T(1+\lambda+|\xi|)^{-N})$, for any $N \in \mathbb{N}$, so $\int_{\mathbb{R}^n}|\tilde{\psi}(\lambda+|\xi|)|d\xi$ can be arbitrarily small, while $\tilde{\psi}(\lambda-|\xi|) = O(T(1+T|\lambda-|\xi||)^{-N})$.

Integrate (4.11) with respect to t first, then it is bounded by a constant times

$$\sum_{\pm}\int_0^\infty\int_{\mathbb{S}^{n-1}}\tilde{\psi}(\lambda\pm r)e^{ir\Phi(\tilde{x},\alpha(\tilde{y}))\cdot\Theta}r^{n-1}d\Theta dr. \tag{4.12}$$

Because $|\tilde{\psi}(\lambda\pm r)|\le T(1+T|\lambda\pm r|)^{-N}$ for any $N>0$, the term with $\tilde{\psi}(\lambda+r)$ in the sum is $O(1)$, while the other term with $\tilde{\psi}(\lambda-r)$ is significant only when r is comparable to λ, say, $c_1\lambda<r<c_2\lambda$ for some constants c_1 and c_2. In this case, as we assumed that $d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))\ge D$, we can also assume that $d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))\gtrsim\frac{1}{r}$ for large λ.

By Lemma 3.2, $\int_{\mathbb{S}^{n-1}}e^{iw\cdot\Theta}d\Theta=\sqrt{2\pi}^{n-1}|w|^{-\frac{n-1}{2}}\sum_{\pm}a_{\pm}e^{\pm i|w|}+O(|w|^{-\frac{n+1}{2}})$, $|w|\ge 1$, where $w=r\Phi(\tilde{x},\alpha(\tilde{y}))$. Integrate up Θ, the above quantity is then controlled by a constant times

$$\begin{aligned}&|\sum_{\pm}\int_{c_1\lambda}^{c_2\lambda}\tilde{\psi}(\lambda-r)|rd_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))|^{-\frac{n-1}{2}}e^{\pm ird_{\tilde{g}}(\tilde{x},\tilde{y})}r^{n-1}dr+\int_{c_1\lambda}^{c_2\lambda}\tilde{\psi}(\lambda-r)|rd_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))|^{-\frac{n+1}{2}}r^{n-1}dr|\\ \le&d_{\tilde{g}}(x,y)^{-\frac{n-1}{2}}\int_{c_1\lambda}^{c_2\lambda}|\tilde{\psi}(\lambda-r)|r^{\frac{n-1}{2}}dr+d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))^{-\frac{n+1}{2}}\int_{c_1\lambda}^{c_2\lambda}|\tilde{\psi}(\lambda-r)|r^{\frac{n-3}{2}}dr\\ =&O((\frac{\lambda}{d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))})^{\frac{n-1}{2}})\end{aligned} \tag{4.13}$$

Therefore, disregarding the integral of the remainder kernel,

$$\begin{aligned}&\int_{-T}^{T}(1-\varphi(t))\hat{\rho}(\frac{t}{T})(\cos t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x},\alpha(\tilde{y}))e^{it\lambda}dt\\ &\qquad=O((\frac{\lambda}{d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))})^{\frac{n-1}{2}})+O(T^{n-2}\lambda^{n-3}\exp(C_Nd_{\tilde{g}}(\tilde{x},\alpha(\tilde{y})))).\end{aligned} \tag{4.14}$$

Now $K(x,y)$ is

$$\sum_{\alpha\in\mathrm{Aut}(p)}[O(\frac{1}{T}(\frac{\lambda}{d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))})^{\frac{n-1}{2}})+O(\frac{e^{ET}}{T})], \tag{4.15}$$

where $E = \max\{C_N, d\} + 1$.

Here we still have: the number of terms of α's such that $d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})) \leq T$ is at most e^{cT}, for some constant c depending on the curvature, and there exists a constant C_p such that $d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})) > C_p$ for any $\alpha \in \mathrm{Aut}(p)$ different from identity.

Now we take the $L^r(\Sigma)$ norms of each individual terms. By (3.49), and accounting in the number of terms of those α's, the first one is bounded by a constant times

$$\frac{e^{cT}\lambda^{\frac{n-1}{2}}}{T}(\int_0^1 C_p^{-\frac{n-1}{2}\cdot r}\tau^{k-1}d\tau)^{\frac{1}{r}} = O(\frac{e^{cT}\lambda^{\frac{n-1}{2}}}{T}). \tag{4.16}$$

Therefore,

$$\begin{aligned} ||K(x,\cdot)||_{L^r(\Sigma)} =& O(\frac{e^{cT}\lambda^{\frac{n-1}{2}}}{T}) + O(\frac{e^{(c+E)T}}{T}) \\ =& I + II. \end{aligned} \tag{4.17}$$

Now take $T = \beta \log \lambda$, where $\beta = \dfrac{\frac{n-1}{2} - \frac{2k}{p} - \delta}{c+E}$, where δ satisfies $0 < \delta < \frac{n-1}{2} - \frac{2k}{p}$. Note that $\frac{n-1}{2} - \frac{2k}{p} > 0$ when $p > \frac{4k}{n-1}$. Then

$$I = O(\frac{\lambda^{\beta c + \frac{n-1}{2}}}{\log \lambda}) = O(\frac{\lambda^{\frac{n-1}{2} - \frac{2k}{p} - \delta + \frac{n-1}{2}}}{\log \lambda}) = o(\frac{\lambda^{n-1-\frac{2k}{p}}}{\log \lambda}), \tag{4.18}$$

and

$$II = O(\frac{\lambda^{\beta(c+E)}}{\log \lambda}) = O(\frac{\lambda^{\frac{n-1}{2} - \frac{2k}{p} - \delta}}{\log \lambda}) = o(\frac{\lambda^{n-1-\frac{2k}{p}}}{\log \lambda}). \tag{4.19}$$

Summing up, we get that

$$||K(x,\cdot)||_{L^r(\Sigma)} = o(\frac{\lambda^{n-1-\frac{2k}{p}}}{\log \lambda}). \tag{4.20}$$

Now apply Young's inequality, with $r = \frac{p}{2}$, together with the estimate in Lemma 4.3, we have

$$\forall f \in L^{p'}(\Sigma), ||\chi_T^\lambda(\chi_T^\lambda)^* f||_{L^p(\Sigma)} \lesssim \frac{\lambda^{n-1-\frac{2k}{p}}}{\log\lambda}||f||_{L^{p'}(\Sigma)}. \tag{4.21}$$

Therefore, Theorem 4.1 is proved.

proof of Lemma 4.3. With similar approaches as the previous discussions, we can show that $\tilde{K}(x, y)$ is

$$O(\frac{1}{T}(\frac{\lambda}{\lambda^{-1} + d_{\tilde{g}}(\tilde{x}, \tilde{y})})^{\frac{n-1}{2}}) + \sum_{Id \neq \alpha \in \mathrm{Aut}(p)} [O(\frac{1}{T}(\frac{\lambda}{d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))})^{\frac{n-1}{2}}) + O(e^{ET})], \tag{4.22}$$

where $E = \max\{C_N, d\} + 1$.

Note that $|t| \leq 1$ for $\varphi(t) \neq 0$, and the number of terms such that $d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})) \leq 1$ is at most e^c, so that

$$||\tilde{K}(x, y)||_{L^r(\Sigma)} = O(\frac{\lambda^{2\delta(p)}}{\log\lambda}), \tag{4.23}$$

if we take $T = \log\lambda$ and calculate as before. □

Chapter 5

Proof of the main theorem in all dimensions

To show Theorem 1.2, we need to use interpolation. Recall that

$$
\begin{aligned}
K(x,y) =& \frac{1}{\pi T}\int_{-T}^{T}(1-\varphi(t))\hat{\rho}(\frac{t}{T})(\cos t\sqrt{-\Delta_g})(x,y))e^{it\lambda}dt \\
=& \frac{1}{2\pi T}\int_{-T}^{T}(1-\varphi(t))\hat{\rho}(\frac{t}{T})(e^{it\sqrt{-\Delta_g}}+e^{-it\sqrt{-\Delta_g}})(x,y)e^{it\lambda}dt
\end{aligned} \tag{5.1}
$$

is the kernel of the operator

$$
\begin{aligned}
& \frac{1}{2\pi T}[\sum_j \tilde{\psi}(\lambda-\lambda_j)E_j + \sum_j \tilde{\psi}(\lambda+\lambda_j)E_j] \\
=& \frac{1}{2\pi T}[\sum_j \tilde{\psi}(\lambda-\lambda_j)E_j] + O(1) \\
=& \frac{1}{2\pi T}\tilde{\psi}(\lambda-\sqrt{-\Delta_g}) + O(1),
\end{aligned} \tag{5.2}
$$

where $\tilde{\psi}(t)$ is the inverse Fourier transform of $(1-\varphi(t))\hat{\rho}(\frac{t}{T})$ so that $|\tilde{\psi}(t)| \le T(1+|t|)^{-N}$ for any $N \in \mathbb{N}$.

We have the following estimate for $\tilde{\psi}(\lambda - \sqrt{-\Delta_g})$.

Theorem 5.1. *For $k \neq n-2$,*

$$||\tilde{\psi}(\lambda - P)g||_{L^2(\Sigma)} \lesssim T\lambda^{2\delta(2)}||g||_{L^2(\Sigma)}, \quad \textit{for any } g \in L^2(\Sigma), \tag{5.3}$$

and for $k = n-2$,

$$||\tilde{\psi}(\lambda - P)g||_{L^2(\Sigma)} \lesssim T\lambda^{2\delta(2)} \log\lambda ||g||_{L^2(\Sigma)}, \quad \textit{for any } g \in L^2(\Sigma), \tag{5.4}$$

where $P = \sqrt{-\Delta_g}$.

Proof. Recall the proof of the corresponding restriction Theorem 1.1 in [2], they showed that for $\chi \in \mathcal{S}(\mathbb{R})$, and define

$$\chi_\lambda = \chi(\sqrt{-\Delta_g} - \lambda) = \sum_j \chi(\lambda_j - \lambda)E_j, \tag{5.5}$$

we have

$$||\chi_\lambda||_{L^2(M)\to L^2(\Sigma)} = O(\lambda^{\delta(2)}), \tag{5.6}$$

for $k \neq n-2$, and

$$||\chi_\lambda||_{L^2(M)\to L^2(\Sigma)} = O(\lambda^{\delta(2)}(\log\lambda)^{\frac{1}{2}}), \tag{5.7}$$

for $k = n-2$.

Now consider $\tilde{\psi}(\lambda - P)$ as $S\tilde{S}^*$, where

$$S = \sum_j (1 + |\lambda_j - \lambda|)^{-M} E_j \tag{5.8}$$

and

$$\tilde{S} = \sum_j (1 + |\lambda_j - \lambda|)^{M} \tilde{\psi}(\lambda_j - \lambda)E_j, \tag{5.9}$$

where M is some large number.

Recall that $|\tilde{\psi}(\tau)| \leq T(1+|\tau|)^{-N}$ for any $N \in \mathbb{N}$, we then have

$$|(1+|\lambda_j-\lambda|)^M \tilde{\psi}(\lambda_j-\lambda)| \leq T(1+|\lambda_j-\lambda|)^{-N} \tag{5.10}$$

for any N.

By (1.14), which we deduced from the proof of Theorem 3 in [2], for a given λ,

$$|| \sum_{\lambda_j \in (\lambda-1,\lambda+1)} E_j ||_{L^2(M)\to L^2(\Sigma)} = O(\lambda^{\delta(2)}), \quad \text{if } k \neq n-2 \tag{5.11}$$

and

$$|| \sum_{\lambda_j \in (\lambda-1,\lambda+1)} E_j ||_{L^2(M)\to L^2(\Sigma)} = O(\lambda^{\delta(2)}(\log\lambda)^{\frac{1}{2}}), \quad \text{if } k = n-2 \tag{5.12}$$

so that for any $f \in L^2(M)$,

$$\begin{aligned}
&|| \sum_j (1+|\lambda_j-\lambda|^{-M}) E_j f ||_{L^2(\Sigma)} \\
\leq & || \sum_{\lambda_j \in (\lambda-1,\lambda+1)} E_j f ||_{L^2(\Sigma)} + || \sum_{\lambda_j \notin (\lambda-1,\lambda+1)} (1+|\lambda_j-\lambda|^{-M}) E_j f ||_{L^2(\Sigma)} \\
\lesssim & \begin{cases} \lambda^{\delta(2)} ||f||_{L^2(M)} + \sum_{\lambda_j \notin (\lambda-1,\lambda+1)} (1+|\lambda_j-\lambda|)^{-M} ||E_j f||_{L^2(\Sigma)}, & \text{if } k \neq n-2, \\ \lambda^{\delta(2)} (\log\lambda)^{\frac{1}{2}} ||f||_{L^2(M)} + \sum_{\lambda_j \notin (\lambda-1,\lambda+1)} (1+|\lambda_j-\lambda|)^{-M} ||E_j f||_{L^2(\Sigma)}, & \text{if } k = n-2. \end{cases}
\end{aligned} \tag{5.13}$$

As

$$\begin{aligned}
&\sum_{\lambda_j \notin (\lambda-1,\lambda+1)} (1+|\lambda_j-\lambda|)^{-M} ||E_j f||_{L^2(\Sigma)} \\
\leq & \begin{cases} \sum_{\lambda_j \notin (\lambda-1,\lambda+1)} \lambda_j^{\delta(2)} (1+|\lambda_j-\lambda|)^{-M} ||E_j f||_{L^2(M)}, & \text{if } k \neq n-2, \\ \sum_{\lambda_j \notin (\lambda-1,\lambda+1)} \lambda_j^{\delta(2)} (\log\lambda_j)^{\frac{1}{2}} (1+|\lambda_j-\lambda|)^{-M} ||E_j f||_{L^2(M)}, & \text{if } k = n-2, \end{cases}
\end{aligned} \tag{5.14}$$

which can be made arbitrarily small when M is sufficiently large,

$$||\sum_j (1+|\lambda_j-\lambda|^{-M})E_j f||_{L^2(\Sigma)} \le \begin{cases} \lambda^{\delta(2)}||f||_{L^2(M)}, & \text{if } k \neq n-2, \\ \lambda^{\delta(2)}(\log\lambda)^{\frac{1}{2}}||f||_{L^2(M)}, & \text{if } k = n-2. \end{cases} \tag{5.15}$$

Similarly, we have

$$||\sum_j (1+|\lambda_j-\lambda|^{M})\tilde{\phi}(\lambda_j-\lambda)E_j f||_{L^2(\Sigma)} \le \begin{cases} T\lambda^{\delta(2)}||f||_{L^2(M)}, & \text{if } k \neq n-2, \\ T\lambda^{\delta(2)}(\log\lambda)^{\frac{1}{2}}||f||_{L^2(M)}, & \text{if } k = n-2. \end{cases} \tag{5.16}$$

Therefore,

$$\begin{aligned} ||\tilde{\psi}(\lambda-P)g||_{L^2(\Sigma)} =& ||S\tilde{S}^*g||_{L^2(\Sigma)} \\ \le& ||S||_{L^2(M)\to L^2(\Sigma)}||\tilde{S}^*||_{L^2(\Sigma)\to L^2(M)}||g||_{L^2(\Sigma)} \\ =& ||S||_{L^2(M)\to L^2(\Sigma)}||\tilde{S}||_{L^2(M)\to L^2(\Sigma)}||g||_{L^2(\Sigma)} \\ \lesssim& \begin{cases} T\lambda^{2\delta(2)}||g||_{L^2(\Sigma)}, & \text{if } k \neq n-2, \\ T\lambda^{2\delta(2)}\log\lambda||g||_{L^2(\Sigma)}, & \text{if } k = n-2. \end{cases} \end{aligned} \tag{5.17}$$

□

Now we may finish the proof of Theorem 1.2.

Recall that we denote K as the operator whose kernel is $K(x,y)$. The above theorem tells us that,

$$||K||_{L^2(\Sigma)\to L^2(\Sigma)} \le \begin{cases} O(\lambda^{2\delta(2)}), & \text{for } k \neq n-2; \\ O(\lambda^{2\delta(2)}\log\lambda), & \text{for } k = n-2. \end{cases} \tag{5.18}$$

Interpolating this with

$$||K||_{L^1(\Sigma)\to L^\infty(\Sigma)} = O(\frac{e^{cT}\lambda^{\frac{n-1}{2}}}{T}) \tag{5.19}$$

by Theorem 4.1 respectively, we get that for any p and $k \neq n-2$,

$$||K||_{L^{p'}(\Sigma)\to L^p(\Sigma)} = O(\frac{\lambda^{\frac{n-1}{2}(1-\frac{2}{p})}e^{cT(1-\frac{2}{p})}\lambda^{2\delta(2)\cdot\frac{2}{p}}}{T^{1-\frac{2}{p}}}) = O(\frac{\lambda^{\frac{n-1}{2}-\frac{n-1}{p}+\frac{4\delta(2)}{p}}e^{cT(1-\frac{2}{p})}}{T^{1-\frac{2}{p}}}), \tag{5.20}$$

and for $k = n-2$,

$$||K||_{L^{p'}(\Sigma)\to L^p(\Sigma)} = O(\frac{\lambda^{\frac{n-1}{2}-\frac{n-1}{p}+\frac{4\delta(2)}{p}}e^{cT(1-\frac{2}{p})}(\log\lambda)^{\frac{2}{p}}}{T^{1-\frac{2}{p}}}) = O(\frac{\lambda^{\frac{n-1}{2}-\frac{n-1}{p}+\frac{4\delta(2)}{p}}e^{cT(1-\frac{2}{p})}(\log\lambda)^{\frac{2}{p}}}{T^{1-\frac{2}{p}}}). \tag{5.21}$$

If $k = n-1$, then $\delta(2) = \frac{1}{4}$.

$$||K||_{L^{p'}(\Sigma)\to L^p(\Sigma)} = O(\frac{\lambda^{\frac{n-1}{2}-\frac{n-2}{p}}e^{cT(1-\frac{2}{p})}}{T^{1-\frac{2}{p}}}). \tag{5.22}$$

Since $\frac{n-1}{2} - \frac{n-2}{p} < 2\delta(p)$ if $p > \frac{2n}{n-1}$, say, $\frac{n-1}{2} - \frac{n-2}{p} + \delta < 2\delta(p)$ for some small number $\delta > 0$, then taking $\beta = \frac{\delta}{c(1-\frac{2}{p})}$, and $T = \beta\log\lambda$, we have

$$||K||_{L^{p'}(\Sigma)\to L^p(\Sigma)} = O(\frac{\lambda^{2\delta(p)-\delta}}{T^{1-\frac{2}{p}}}) = O(\frac{\lambda^{2\delta(p)-\delta}}{(\log\lambda)^{1-\frac{2}{p}}}) = o(\frac{\lambda^{2\delta(p)}}{\log\lambda}), \tag{5.23}$$

which indicates Theorem 1.2.

If $k = n-2$,

$$||K||_{L^{p'}(\Sigma)\to L^p(\Sigma)} = O(\frac{\lambda^{\frac{n-1}{2}-\frac{n-1}{p}+\frac{4\delta(2)}{p}}e^{cT(1-\frac{2}{p})}(\log\lambda)^{\frac{2}{p}}}{T^{1-\frac{2}{p}}}) = O(\frac{\lambda^{\frac{n-1}{2}-\frac{n-1}{p}+\frac{2}{p}}e^{cT(1-\frac{2}{p})}(\log\lambda)^{\frac{2}{p}}}{T^{1-\frac{2}{p}}}). \tag{5.24}$$

Now since $\frac{n-1}{2}-\frac{n-1}{p}+\frac{2}{p}<(n-1)-\frac{2(n-2)}{p}$ when $p>2$, we can take $\delta>0$ such that $\frac{n-1}{2}-\frac{n-1}{p}+\frac{2}{p}+\delta<(n-1)-\frac{2(n-2)}{p}$, and take $\beta=\frac{\delta}{c(1-\frac{2}{p})}$, $T=\beta\log\lambda$, then

$$||K||_{L^{p'}(\Sigma)\to L^p(\Sigma)}=O(\frac{\lambda^{2\delta(p)-\delta}(\log\lambda)^{\frac{2}{p}}}{(\log\lambda)^{1-\frac{2}{p}}})=o(\frac{\lambda^{2\delta(p)}}{\log\lambda}), \tag{5.25}$$

which is what we need.

If $k\le n-3$, $\delta(2)=\frac{n-1}{2}-\frac{k}{2}$, then

$$||K||_{L^{p'}(\Sigma)\to L^p(\Sigma)}=O(\frac{\lambda^{\frac{n-1}{2}-\frac{n-1}{p}+\frac{4\delta(2)}{p}}e^{cT(1-\frac{2}{p})}}{T^{1-\frac{2}{p}}})=O(\frac{\lambda^{\frac{n-1}{2}-\frac{n-1}{p}+\frac{2(n-1)-2k}{p}}e^{cT(1-\frac{2}{p})}}{T^{1-\frac{2}{p}}}). \tag{5.26}$$

Since $\frac{n-1}{2}-\frac{n-1}{p}+\frac{2(n-1)-2k}{p}<(n-1)-\frac{2k}{p}=2\delta(p)$ for $p>2$, we can take $\delta>0$ such that $\frac{n-1}{2}-\frac{n-1}{p}+\frac{2(n-1)-2k}{p}+\delta<(n-1)-\frac{2k}{p}$, and take $\beta=\frac{\delta}{c(1-\frac{2}{p})}$, $T=\beta\log\lambda$, then

$$||K||_{L^{p'}(\Sigma)\to L^p(\Sigma)}=O(\frac{\lambda^{2\delta(p)-\delta}}{(\log\lambda)^{1-\frac{2}{p}}})=o(\frac{\lambda^{2\delta(p)}}{\log\lambda}), \tag{5.27}$$

which finishes Theorem 1.2.

Chapter 6

Further results

Note that we cannot simply adopt the same method to improve the restriction estimates at any endpoints for p, especially for $k = n-2, p = 2$, when it seems hard to remove the extra $(\log\lambda)^{\frac{1}{2}}$ in (1.11). Fortunately, if we consider only the geodesic concentration of the eigenfunctions, we are able to get rid of it for the lowest dimension $k = 1, n = 3$.

Theorem 6.1 (Chen, Sogge, preprint [4]). *For a 3-dimensional compact Riemannian manifold (M, g), there exists a uniform constant $C = C(M, g)$ such that*

$$\sup_{\gamma\in\Pi}\left(\int_\gamma |\phi_\lambda|^2 ds\right)^{\frac{1}{2}} \leq C(1+\lambda)^{\frac{1}{2}}||\phi_\lambda||_{L^2(M)}, \tag{6.1}$$

where Π denotes the set of all unit-length geodesics.

Interestingly, although other restriction estimates for codimension 2 are temporarily difficult to refine (to remove the extra $\log^{\frac{1}{2}}(\lambda)$), this refined estimate comes from an application of purely the L^2 boundedness of Hilbert transform in addition to the Fourier inversion formulae, the standard TT^* argument and the oscillatory integral theorems. The reason that the $k = 1$ case is simpler may be that the submanifolds are just geodesics, which have many well-known properties, and it is just an integral of a single variable when we calculate the L^p norm of the restriction of the eigenfunctions. It turns out to be helpful to first consider the restriction to geodesics, and we obtained the following estimates for the two remaining cases for $k = 1$ recently.

Theorem 6.2 (Chen, Sogge, preprint [4]). *For a 2-dimensional compact manifold of nonpositive curvature, we have*

$$\limsup_{\lambda\to\infty}(\sup_{\gamma\in\Pi}\lambda^{-\frac{1}{4}}||\phi_\lambda||_{L^4(\gamma)}) = 0. \tag{6.2}$$

For a 3-dimensional compact manifold of constant nonpositive curvature, we have

$$\limsup_{\lambda\to\infty}(\sup_{\gamma\in\Pi}\lambda^{-\frac{1}{2}}||\phi_\lambda||_{L^2(\gamma)}) = 0. \tag{6.3}$$

Remark that (6.2) is outstanding in the sense that it is the first result establishing an improvement under general assumptions of an estimate that is saturated both by the Zonal functions and the highest weight spherical harmonics. The proof of these two estimates are essentially the same as Theorem 1.2, except one breakthrough step of showing that the set of points where the mixed Hessian of the distance function between two points on two different geodesics is zero has zero measure. By applying this fact which leads to a better application of the oscillatory integral theorem, we have obtained these results which we did not get in [3]. The extra assumption of constant curvature for the 3-dimensional case comes from the difficulty of showing the above fact.

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Vita

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