STRICHARTZ ESTIMATES AND STRAUSS CONJECTURE ON VARIOUS SETTINGS

by

Xin Yu

A dissertation submitted to The Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy

Baltimore, Maryland

April, 2011

Abstract

In this thesis, I include two works both focusing on existence and regularity of solutions of semilinear wave equations. In the first work I provide general Strichartz estimates for certain perturbed wave equation, equipped with these estimates I am able to verify the Strauss Conjecture for semilinear wave equations with 2 obstacles when $n = 3$. I also obtain the sharp life span for the subcritical case $n = 3, 2 < p < p_c$, by using a real interpolation method. In the second work (joint with Chengbo Wang), we verify the Strauss conjecture for semilinear wave equations on asymptotically Euclidean manifolds when n=3,4, we also give an almost sharp life span for the subcritical case when n=3. The main ingredients include a KSS type estimate with $0 < \mu < 1/2$ and weighted Strichartz estimates of order two.

READERS: Dr. Christopher Sogge (Advisor) and Dr. Chengbo Wang.

Acknowledgments

First and for most, I would like to express my deep gratitude to my advisor, Dr. Christopher Sogge. He has always been patient, warm-hearted and encouraging during this study.

Secondly, I would like to thank my collaborator Chengbo Wang. His knowledge and great personality impressed me, and I have learned a lot from him. I would also like to thank Sinan Ariturk, I owe him eternal gratitude for proofreading two of my papers.

Thirdly, I would like to thank the many staff and faculty at Johns Hopkins who have assisted in large ways and small throughout this process: Jian Kong, Sabrina Raymond, Charlene Poole, Linda Buckner, Morris Hunt, Williams Minicozzi, Chikako Mese, Richard Brown, Hans Lindblad and many others.
Additionally, I would like to extend heartfelt thanks to all my fellow graduate students including post-graduates Jin-Cheng Jiang ,Yifei Chen and Qi Zhong who have made the graduate life easier and more fun.

I dedicate this dissertation to my parents as well as Jiming and Aria.

Contents

Chapter 1

Introduction

1.1 Background

The wave equation is a hyperbolic partial differential equation which occurs in many fields such as acoustics, electromagnetics, and fluid dynamics. For the Cauchy problem of linear wave equations in Minkowski space $\mathbb{R}^{1+n}$,

$$\begin{cases} \Box u(t,x) = (\partial_t^2 - \Delta)u(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = f, \quad \partial_t u(0,x) = g, \quad x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

it is known that we have a direct formula for the solution. Explicitly, suppose $(f,g) \in C^{(n+3)/2} \times C^{(n+1)/2}$, when the dimension n is odd, by using the method of spherical means we can get a unique C^2 solution as follows

$$u(t,x) = \frac{1}{1 \cdot 3 \cdots (n-2)} [\frac{\partial}{\partial_t} (\frac{1}{t} \frac{\partial}{\partial_t})^{\frac{n-3}{2}} t^{n-2} A_t f(x) + (\frac{1}{t} \frac{\partial}{\partial_t})^{\frac{n-3}{2}} t^{n-2} A_t g(x)].$$

When the dimension n is even, we can also get the solution by Hadamard's method of descent,

$$(1.2) \quad u(t,x) = \frac{1}{1\cdot 3\cdots(n-1)\omega_n}[\frac{\partial}{\partial_t}(\frac{1}{t}\frac{\partial}{\partial_t})^{\frac{n-2}{2}}t^{n-1}\int_{|y|<1}\frac{f(x+ty)}{\sqrt{1-|y|^2}}dy+ (\frac{1}{t}\frac{\partial}{\partial_t})^{\frac{n-2}{2}}t^{n-1}\int_{|y|<1}\frac{g(x+ty)}{\sqrt{1-|y|^2}}dy].$$

One can obtain much information such as Huygen's principle and Comparison Theorem on the wave from the fundamental solutions above. However, in practise, the initial data (f, g) usually do not have the required smoothness, in which case we need the following solution expression in distribution form

$$u(t,x) = \cos(tD)f + \frac{\sin(tD)}{D}g + \int_0^t \frac{\sin(t-s)D}{D}F(s,\cdot)ds,$$

where $D = \sqrt{-\Delta}$, and we have added the forcing term F, i.e. we have $\Box u = F$. This expression is obtained simply by Fourier transform on the equation. Note that when the initial data is smooth enough the two expressions of solutions coincide with each other. When the initial data has weaker regularity, we can still set up fine existence theory. Specifically, for the Cauchy problem of the wave equation in Minkowski space

$$(1.3) \quad \begin{cases} \Box u(t,x) = F(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ u(0,x) = f, \quad \partial_t u(0,x) = g, & x \in \mathbb{R}^n, \end{cases}$$

by utilizing the idea of weak solution and an energy estimate, we have a unique solution with Sobolev regularity H^s as long as (f, g, F) has corresponding norm bound.

A more complicated case is when the forcing term F relies on the wave function itself.

In my graduate study I mainly study the system of semilinear wave equations,

$$(1.4) \qquad \begin{cases} (\partial_t^2 - \Delta_g)u(t,x) = F(u), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ u(0,x) = f, \quad \partial_t u(0,x) = g, \quad x \in \mathbb{R}^n, \end{cases}$$

where $F(u)$ acts like $|u|^p$. There are two main kinds of existence problems to consider, one is when the initial data is large, with techniques involving conservation laws and monotonicity formula methods. The other is when the initial data is small, with techniques involving perturbation methods and continuity arguments. I mainly work on the latter problem with small initial data, which has the advantage that we can control the nonlinear term by employing the finite speed of propagation property of wave, then combined with the result for linear systems, we can expect at least a local existence result of the nonlinear system.

In Minkowski space $\mathbb{R}^{1+n}$, the work is initiated by John [11], who showed that for small C_0^∞ data global solutions of (1.4) always exist when $p > 1 + \sqrt{2}$ but not when $p < 1 + \sqrt{2}$ when the dimension is 3. Later on Strauss [22] conjectured that for a general n there should be a global solution for small data when $p > p_c$, but blow up below p_c. Here the critical power p_c is defined as the positive root of the equation

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

This conjecture is surprising but has been verified by subsequent works on Euclidean space, and ended by Geogiev, Lindblad and Sogge [4] and Tataru [23]. It is known that $p > p_c$ is necessary for global existence, even with small data, see [18], [25], [29] and reference therein. Moreover, when $n \leqslant 3$ and $p \leqslant p_c$, the sharp life span is known in Zhou [28] (see also [12] for lower bound of the life span $p \leqslant p_c$ and $n \geqslant 3$, and [30] for upper bound of the life span when $p < p_c$ and $n \geqslant 3$). On more complicated manifolds, however, there has not been much work until very recently. My first work is to study the semilinear equation perturbed by obstacles, so we consider the initial-boundary problem

$$
\begin{cases}
(\partial_t^2 - \Delta_g)u(t,x) = F(u), \quad (t,x) \in \mathbb{R}_+ \times \Omega \\
Bu|_{\partial\Omega} = 0 \\
u(0,x) = f, \quad \partial_t u(0,x) = g,
\end{cases}
\tag{1.5}
$$

where Ω is the complement of several obstacles, and B is identity or inward normal derivative operator. This work is inspired by the work of Hidano, Metcalfe, Smith, Sogge and Zhou (HMSSZ)[6], who obtained global solutions for subcritical powers when there is one nontrapping obstacle and $n = 3, 4$. With an obstacle near the origin, we do not automatically get the same estimates as in Euclidean space due to reflections and refractions, but with a star-shaped or nontrapping assumption, many works have shown that there will be a local energy decay. By utilizing the local decay one can get around the difficulty near the origin and verify the Strauss conjecture by combining the Strichartz estimates in classical theory. This is basically what HMSSZ [6] presents. My work generalizes the results by allowing some trapped rays, the key point is to use a weaker energy decay proved by Ikawa [9]. Moreover, I obtained the sharp life span for the local solution when $p < p_c$ and $n = 3$ on $\mathbb{R} \times \Omega$, which corresponds the result in Euclidean space by Lindblad, while uses a different method of real interpolation.

My second work is a collaboration with Chengbo Wang. We focus on another kind of perturbation of wave equations and consider variable coefficients wave equation. Specifically, we consider the problem in asymptotically Euclidean space with metric g, in which $\mathfrak{g}_{ij} - \delta_{ij}$ acts like $\langle x \rangle^{-\rho}$ with $\rho > 0$, hence the perturbation is bounded near the origin and tends to be flat away from the origin. There is not much activities on this topic yet, following previous works of [2] and [21], we verified the Strauss conjecture for solutions of (1.4) in low dimensions. The precise results will be stated in the next section.

1.2 Preliminaries and Main Results

We use ∂ to denote the spatial gradient $\partial u = (\partial_{x_1} u, \cdots, \partial_{x_n} u)$. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is multi-indexed and f is a function in $\mathbb{R}^n$, then

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}.$$

The homogeneous Sobolev space $\dot{H}^\gamma$ is defined as

$$\|h\|_{\dot{H}^\gamma(\mathbb{R}^n)} = \||D|^\gamma h\|_{L^2_x(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left| |\xi|^\gamma \hat{h}(\xi) \right|^2 d\xi \right)^{1/2}.$$

We also define the mixed-norm space

$$\|h\|_{L^q_r L^p_\omega} = \Big(\int_0^\infty \Big(\int_{S^{n-1}} |h(r\omega)|^p \, d\sigma(\omega) \Big)^{q/p} r^{n-1} dr \Big)^{1/q}$$

for finite exponents and

$$\|h\|_{L^\infty_r L^p_\omega} = \sup_{r>0} \Big(\int_{S^{n-1}} |h(r\omega)|^p \, d\sigma(\omega) \Big)^{1/p}.$$

Let Z be generators of Lorentz group excluding the boost operators, i.e.

$$Z = \{\partial_i, \Omega_{jk}, 1 \leqslant i \leqslant n, 1 \leqslant j < k \leqslant n\},$$

where $\Omega_{ij} = x_i \partial_j - x_j \partial_i$ is the rotational vector field. Set

$$\Gamma = \{\partial_t, Z\}.$$

We write $f\lesssim g$ if $f \leqslant Cg$, where C is a constant. We also assume $p > 1$, and set

$$s_c = \frac{n}{2} - \frac{2}{p-1}, \quad s_d = \frac{1}{2} - \frac{1}{p}$$

In my first work we will use the following notations for the system (1.5)

- $\Omega = \mathbb{R}^n$

  or

  $\mathbb{R}^n \backslash \Omega$ is a subset of $|x| < R$ with smooth boundary such that a special local energy decay is satisfied.

- $\Delta_{\mathfrak{g}} = \sum_{ij} \frac{1}{\sqrt{\det \mathfrak{g}}} \partial_i \sqrt{\det \mathfrak{g}} \mathfrak{g}^{ij} \partial_j$ is the Laplace-Beltrami operator.

- The Riemannian metric $g_{jk}(x) = \delta_{jk}(x)$ for $|x| > R$.

- $B = I$ or ∂_v, v is the inward normal vector.

We also pose the assumption on the domain below.

Hypothesis B If (f, g, F) vanish for $|x| > R$, then for $\forall \epsilon > 0$,

$$\|u\|_{L_t^2 H^1([0,S]\times\{|x|<R\})} + \|\partial_t u\|_{L_{t,x}^2([0,S]\times\{|x|<R\})} \lesssim \|f\|_{\dot{H}^{1+\epsilon}} + \|g\|_{\dot{H}^\epsilon} + \|F\|_{L_t^2 \dot{H}^\epsilon}.$$

Here $S = T$ or $+\infty$. Later on we will see that this Hypothesis is satisfied in the case I shall consider. Next we provide a technical definition of a Sobolev-type norm,

$$\|h\|_{\tilde{H}_\epsilon^\gamma(\mathbb{R}^n)} = \||D|^\gamma (1-\Delta)^{\epsilon/2} h\|_{L_x^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left| |\xi|^\gamma (1+|\xi|^2)^{\epsilon/2} \hat{h}(\xi) \right|^2 d\xi.$$

Now the main existence result in my first work is as follows.

Main Theorem 1. Let $n = 3, 4$, assume Hypothesis B. Also assume

$$\sum_{i=1}^{2} |u|^i |\partial_u^i F(u)| \lesssim |u|^p.$$

Then if $p > p_c, \gamma = s_c$, there exists a global solution with $(Z^\alpha u, \partial_t Z^\alpha u) \in \dot{H}^\gamma \times \dot{H}^{\gamma-1}$, whenever

$$\sum_{|\alpha|\leqslant 2} \left(\|Z^\alpha f\|_{\tilde{H}^\gamma_{2\epsilon}} + \|Z^\alpha g\|_{\tilde{H}^{\gamma-1}_{2\epsilon}} \right) < \epsilon'$$

ϵ' is sufficiently small. Moreover, when $n = 3$, $2 < p < p_c$ and $\gamma = s_d$, we still have an almost global solution with life span $T = C\epsilon^{p(p-1)/(p^2-2p-1)}$.

As a remark, we state that Hypothesis B is satisfied when there are two obstacles far apart. Explicitly, Ikawa [9] managed to show that solutions of (1.5) with $n = 3,\ \Delta_g = \Delta$, $B = I$, and $F(u) = 0$ have exponential decay estimates with a loss of 2 derivatives of data. Interpolating between that estimate and the energy estimate we get an estimate of the form:

$$\|u'(t,x)_{L^2_x(|x|<1)}\| \lesssim e^{-ct} \|u'(0,x)\|_{\dot{H}^\epsilon(|x|<1)}, \quad \text{for any positive number } \epsilon,$$

which implies the local energy decay (Hypothesis B), so we will have the global and local existence results in this case. We also note that the life span given here for $p < p_c$ is sharp based on previous works of John, Lindblad and Zhou on Euclidean space.

The proof of the existence result relies on some generalized Strichartz estimates, which will be discussed in Chapter 2. For the moment we turn to the second work and present the main result.

We study the semilinear wave equation system (1.4) on asymptotically Euclidean manifolds $(\mathbb{R}^n, g)$ with small data. Specifically, for the metric

$$g = \sum_{i,j=1}^{n} \mathfrak{g}_{ij}(x)\, dx^i\, dx^j,$$

we suppose $\mathfrak{g}_{ij}(x) \in C^\infty(\mathbb{R}^n)$ and, for some $\rho > 0$,

$$\text{(H1)} \qquad \forall \alpha \in \mathbb{N}^n \qquad \partial_x^\alpha(\mathfrak{g}_{ij} - \delta_{ij}) = O(\langle x \rangle^{-|\alpha|-\rho}),$$

with $\delta_{ij} = \delta^{ij}$ being the Kronecker delta function. We also assume that

$$\text{(H2)} \qquad g \text{ is non-trapping.}$$

Let $g(x) = (\det(\mathfrak{g}))^{1/4}$. The Laplace–Beltrami operator associated with g is given by

$$\Delta_g = \sum_{ij} \frac{1}{g^2} \partial_i \mathfrak{g}^{ij} g^2 \partial_j,$$

where $\mathfrak{g}^{ij}(x)$ denotes the inverse matrix of $\mathfrak{g}_{ij}(x)$. We also define the corresponding vector fields $\tilde{\partial}_i = \partial_i g^{-1}, \tilde{\Omega}_{ij} = \Omega_{ij} g^{-1}$.

Now we can state our main results.

Main Theorem 2. Suppose (H1) and (H2) hold with $\rho > 2$, $n = 3, 4$, and $p_c < p < 1 + 4/(n-1)$. Then for any $\epsilon > 0$ such that

$$\text{(1.6)} \qquad s = s_c - \epsilon \in (s_d, 1/2)$$

there is a $\delta > 0$ depending on p so that (1.4) has a global solution satisfying $(Z^\alpha u(t, \cdot), \partial_t Z^\alpha u(t, \cdot)) \in \dot{H}^s \times \dot{H}^{s-1}$, $|\alpha| \leqslant 2$, $t \in \mathbb{R}_+$, whenever the initial data satisfies

$$\text{(1.7)} \qquad \sum_{|\alpha| \leqslant 2} \left(\|Z^\alpha f\|_{\dot{H}^s} + \|Z^\alpha g\|_{\dot{H}^{s-1}} \right) < \delta.$$

On the other hand, if $n = 3$ and $2 \leqslant p < p_c = 1 + \sqrt{2}$. Then there still exists a solution in $[0, T_\delta] \times \mathbb{R}^3$ under the same data assumption, with

$$s = s_d, \; T_\delta = c\, \delta^{\frac{p(p-1)}{p^2-2p-1}+\epsilon'}, \tag{1.8}$$

for any small $\epsilon' > 0$. As a remark, we state that if we relax the condition on ρ to $\rho > 1$, we still have the corresponding existence result when $n = 3$ with smallness of Cauchy data with first order derivatives. We will state this work in details in Chapter 3.

Chapter 2

Part I Strauss Conjecture on Perturbed Wave Equations

As in the classical existence results, the key ingredient of the proof of Main theorem 1 is different variations of Strichartz estimates. Recall that the classical mix-norm Strichartz estimates for the solution of (1.3) in Euclidean space is

$$\|u\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

where $n \geqslant 2$, (q, r, γ) and $(\tilde{q}, \tilde{r}, 1-\gamma)$ are admissible pairs, i.e. $2/q + (n-1)/r = (n-1)/2$ and $1/q + n/r = n/2 - \gamma$.

The next two sections are devoted to local and global Strichartz estimates to be used.

2.1 Local in time Strichartz Estimates in $\mathbb{R}^n$

The finite time Strichartz Estimates will be used to prove the local existence result of solutions, and is stated below.

Theorem 2.1.1. *Let u be a solution of* (1.3). *Let $\gamma = s_d$, if $0 < a \leqslant 1/2$, $2 < p < \infty$, and* *let* $S_T = [0,T] \times \mathbb{R}^n$, *then*

$$\||x|^{(-n+2a)/p+(n-1)/2} u\|_{L^p_t L^p_r L^2_\omega(S_T)} \lesssim T^{2a/p} \left(\|f\|_{\dot H^\gamma} + \|g\|_{\dot H^{\gamma-1}} + \||x|^{-n/2+1-\gamma} F\|_{L^1_t L^1_r L^2_\omega(S_T)}\right).$$

The proof of this Theorem involves an interpolation between the following two estimates.

- KSS estimate: $\||x|^{-1/2+a} e^{itD} f\|_{L^2_{t,x}} \lesssim T^a \|f\|_{L^2_x}$, $\quad 0 < a \leqslant 1/2$.
- Endpoint Trace lemma: $\||x|^{(n-1)/2} e^{itD} f\|_{L^\infty_{t,r} L^2_\omega} \lesssim \|f\|_{\dot B^{1/2}_{2,1}}$.

The KSS estimate originated by Keel, Smith and Sogge is a direct result of local energy decay and a scaling argument for a partition of $\{x : 0 < |x| < 1\}$. See [5] for details. The endpoint Trace lemma is due to Fang and Wang [3]. Also note that the the homogeneous Besov space $\dot B^s_{p,q}$ is defined as

$$\|f\|_{\dot B^s_{p,q}} = \|2^{js} P_j f\|_{l^q_j (j \in \mathbb{Z}) L^p_x},$$

where $f = \sum_j P_j f$ is the Littlewood-Paley decomposition. Next we will cite some results and notations on real interpolation theory, which can be found in [1] and [24]. The advantage of the real interpolation over the complex interpolation is that we can get a Sobolev norm out of two Besov norms. Let A_0, A_1 be Banach spaces, define the real interpolation space $(A_0, A_1)_{\theta,q}$ for $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$ via the norm,

$$\|a\|_{(A_0,A_1)_{\theta,q}} = \|a\|_{(A_0,A_1)_{\theta,q;K}} = \left(\int_0^\infty (t^{-\theta} K(t,a))^q dt/t \right)^{1/q},$$

where

$$K(t,a) = \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + \|a_1\|_{A_1}).$$

Then we have the fact

$$\Big(\big(L^{p_0}_{t,r}L^2_w, w_0(r)dtdrd\omega\big), \big(L^{p_1}_{t,r}L^2_\omega, w_1(r)dtdrd\omega\big)\Big)_{\theta,p}$$

$$= \big(L^p_{t,r}L^2_\omega, w(r)dtdrd\omega\big),$$

if $1/p = (1-\theta)/p_0 + \theta/p_1, w(r) = w_0^{p(1-\theta)/p_0} w_1^{p\theta/p_1}$. And

$$(B^{s_0}_{pq_0}, B^{s_1}_{pq_1})_{\theta,r} = B^{s^*}_{pr}, \text{ if } s_0 \neq s_1, r, q_0, q_1 \geqslant 1, s^* = (1-\theta)s_0 + \theta s_1.$$

$$(B^{s_0}_{pq_0}, B^{s_1}_{pq_1})_{\theta,2} \subset (B^{s_0}_{pq_0}, B^{s_1}_{pq_1})_{\theta,r}, \text{ if } r \geqslant 2.$$

Now we turn to the proof of Theorem 2.1.1. Let $\theta = 1-2/p$, note that $\dot{H}^s = \dot{B}^s_{2,2}$, then by the real interpolation between the KSS estimate and Endpoint Trace lemma given above with exponents (θ, p), we get for the homogeneous part,

$$\begin{aligned} \text{Homogeneous part of LHS} &= \||x|^{(-n+2a)/p+(n-1)/2}u\|_{L^p_t L^p_r L^2_\omega(S_T)} \\ &\lesssim T^{2a/p}\left(\|f\|_{(\dot{B}^0_{2,2},\dot{B}^{1/2}_{2,1})_{\theta,p}} + \|g\|_{(\dot{B}^{-1}_{2,2},\dot{B}^{-1/2}_{2,1})_{\theta,p}}\right) \\ &\lesssim T^{2a/p}\left(\|f\|_{(\dot{B}^0_{2,2},\dot{B}^{1/2}_{2,1})_{\theta,2}} + \|g\|_{(\dot{B}^{-1}_{2,2},\dot{B}^{-1/2}_{2,1})_{\theta,2}}\right) \\ &= T^{2a/p}\left(\|f\|_{\dot{H}^{1/2-1/p}} + \|g\|_{\dot{H}^{-1/2-1/p}}\right) = RHS. \end{aligned}$$

As for the inhomogeneous part F, we first apply Duhamel's principle to get

$$\text{Inhomogeneous part of LHS} \lesssim \|F\|_{L^1_t \dot{H}^{-1/2-1/p}}.$$

Then recall the Trace lemma

$$\|r^{n/2-s}v\|_{L^\infty_r L^2_\omega} \lesssim \|v\|_{\dot{H}^s}, 1/2 < s < n/2,$$

by duality we get

$$\|F\|_{L_t^1 \dot H^{-1/2-1/p}} \lesssim \||x|^{-1+p/2} F\|_{L_t^1 L_r^1 L_\omega^2(S_T)},$$

which finishes the proof of the local in time Strichartz estimates.

2.2 Strichartz Estimates on Ω

In Minkowski space, it is seen from [3] and [6] that Strauss conjecture when $n = 3, 4$ can be verified by utilizing the following weighted Strichartz estimates

$$\||x|^{\frac{n}{2} - \frac{n+1}{p} - \gamma} u\|_{L_t^p L_r^p L_\omega^2} \lesssim \|(f, g)\|_{(\dot H^\gamma, \dot H^{\gamma-1})} + \||x|^{-\frac{n}{2}+1-\gamma} F\|_{L_t^1 L_r^1 L_\omega^2}$$

if $1/2 - 1/p < \gamma < n/2 - 1/p$. When there is an obstacle, however, the classical Strichartz estimate does not naturally hold in the long run due to reflection rays. Borrowing the idea of [6], we observe that if we localize the time, by finite propogation property of wave, the solution away from the origin will behave as in the Minkowski space. As for the solution near the origin, there is an energy estimate available. Combining the two kinds of estimates, we will have control of the solution at least local in time. Next based on the argument in Smith and Sogge [19], by using the local energy decay, we expect to get a Strichartz-type estimate for perturbed wave equation from the corresponding one local in time and global in time in $\mathbb{R}^{1+n}$. Now I will present the idea discussed above in details, starting with the definition of the generalized admissible pair.

Definition 2.2.1. We say that (X, γ, η, p) is almost admissible if it satisfies

i), Minkowski almost Strichartz estimates

$$\|u\|_{L_t^p X([0,S] \times \mathbb{R}^n)} \lesssim A(S) \left(\|f\|_{\dot H^\gamma(\mathbb{R}^n)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^n)} \right), \tag{2.1}$$

where $A(S)$ is a function of S and equals a constant when $S = \infty$.

ii), Local almost Strichartz estimates for Ω

$$\|u\|_{L_t^p X([0,1]\times\Omega)} \lesssim \|f\|_{\tilde{H}_\eta^\gamma(\Omega)} + \|g\|_{\tilde{H}_\eta^{\gamma-1}(\Omega)}. \tag{2.2}$$

The generalized Strichartz estimates to be obtained is as follows.

Theorem 2.2.2. Assume Hypothesis B, $n > 2$, $p > 2$, $\gamma \in [-\frac{n-3}{2}, \frac{n-1}{2})$, and (X, γ, η, p) admissible, then

$$\|u\|_{L_t^p X([0,S]\times\Omega)} \lesssim A(S) \left(\|f\|_{\tilde{H}_{\epsilon+\eta}^\gamma} + \|g\|_{\tilde{H}_{\epsilon+\eta}^{\gamma-1}} \right). \tag{2.3}$$

When the forcing term F is added, by a $TT*$ argument and the fact

$$(\tilde{H}_\epsilon^\gamma)' = \tilde{H}_{-\epsilon}^{-\gamma},$$

we can easily get the inhomogeneous version of the Strichartz estimates.

Corollary 2.2.3. Under the same condition, and $(Y, 1-\gamma, \eta, r)$ admissible, then

$$\|u\|_{L_t^p X([0,S]\times\Omega)} \lesssim A(S) \left(\|f\|_{\tilde{H}_{\epsilon+\eta}^\gamma} + \|g\|_{\tilde{H}_{\epsilon+\eta}^{\gamma-1}} \right) + A^2(S) \|(1-\Delta_{\mathfrak{g}})^{\epsilon+\eta} F\|_{L_t^{r'} Y'([0,S]\times\Omega)}.$$

We will use the idea stated at the begining of the section to prove Theorem 2.2.2, so the difficulty lies in the control of so-called commutator terms, which comes out when we try to commute $\Box$ and function multiplier β. For example, if $\Box u = 0$, then $(1-\beta)u$ solves the wave equation

$$\Box(1-\beta)u = \Delta\beta \cdot u + 2\nabla\beta \cdot \nabla u,$$

which produces extra commutator term $G = \Delta\beta \cdot u + 2\nabla\beta \cdot \nabla u$ to be controlled. We mainly use the following two Propositions to fulfil this job.

Proposition 2.2.4. Let w be a solution of (1.3) with $f = g = 0$, and assume that (2.1) is valid whenever v is a solution of the homogeneous wave equation. Assume further that $p > 2, \gamma \geqslant (n-3)/2$. Then, if

$$F(t,x) = 0 \quad \text{if } |x| > 2R,$$

we have

$$\|w\|_{L_t^p X([0,S]\times\mathbb{R}^n)} \lesssim A(S) \|F\|_{L_t^2 \dot{H}^{\gamma-1}([0,S]\times\mathbb{R}^n)}. \tag{2.4}$$

To prove the proposition, we use the distributional form of solution

$$u(t,x) = \cos(tD)f + \sin\frac{tD}{D}g + \int_0^t \frac{\sin(t-s)D}{D} F(s,\cdot)ds.$$

By Christ-Kiselev Lemma we can apply the Minkowski Strichartz estimates, then it suffices to show

$$\| \int_0^S e^{-is|D|}|D|^{-1+\gamma}\beta(\cdot)(1-\Delta)^{(1-\gamma)/2}H(s,\cdot)ds\|_{L^2(\mathbb{R}^n)} \lesssim \|H\|_{L^2(\mathbb{R}_+\times\mathbb{R}^n)}.$$

But this is just the duality of the local energy decay

$$\|\beta e^{itD} f\|_{L_t^2 H^\gamma} \lesssim \|f\|_{\dot{H}^\gamma}, \text{ if } \gamma \leqslant \frac{n-1}{2},$$

which is provided in [19].

The second proposition to bound the commutator terms reply on Hypothesis B.

Proposition 2.2.5. (**Energy Decay**) If F is supported in $|x| < R$, $\gamma < \frac{n-1}{2}$. Assume

Hypothesis B. Let $\beta \in C_0^\infty(\mathbb{R}^n)$ equals 1 on a neighborhood of $\mathbb{R}^n \backslash \Omega$.

$$
\begin{aligned}
&\|u\|_{L_t^\infty \dot{H}_B^\gamma([0,S]\times\Omega)} + \|\partial_t u\|_{L_t^\infty \dot{H}_B^{\gamma-1}([0,S]\times\Omega)} \\
&\qquad + \|\beta u\|_{L_t^2 H_B^\gamma([0,S]\times\Omega)} + \|\beta\partial_t u\|_{L_t^2 H_B^{\gamma-1}([0,S]\times\Omega)} \\
&\lesssim \|f\|_{\tilde{H}_\epsilon^\gamma(\Omega)} + \|g\|_{\tilde{H}_\epsilon^{\gamma-1}(\Omega)} + \|F\|_{L_t^2 \dot{H}_B^{\gamma+\epsilon-1}([0,S]\times\Omega)}.
\end{aligned}
\tag{2.5}
$$

First note that the space $H_B^\gamma(\Omega)$ is the usual Dirichlet space with compatibility conditions on boundary of Ω satisfied, and we are assuming that the required compatibility conditions on data are meet throughout the work(see for example in [19]), therefore write $H^\gamma(\Omega)$ for short elsewhere.

To prove this estimate, note that f, g, and F are supported in a ball, the L_t^2 estimate in the case $\gamma = 1$ is just Hypothesis B, and then by elliptic regularity of the operator Δ_g,

$$
\begin{aligned}
\|\beta u\|_{L_t^2 H_x^3} + \|\beta\partial_t u\|_{L_t^2 H_x^2} &\lesssim \|\Delta_g(\beta u)\|_{L_t^2 H_x^1} + \|\beta u\|_{L_t^2 H_x^1} \\
&\quad + \|\Delta_g(\beta\partial_t u)\|_{L_t^2 L_x^2} + \|\beta\partial_t u\|_{L_t^2 L_x^2} \\
&\lesssim \|\beta\Delta_g u\|_{L_t^2 H_x^1} + \|[\Delta_g, \beta]u\|_{L_t^2 H_x^1} + \|\beta u\|_{L_t^2 H_x^1} \\
&\quad + \|\beta\partial_t\Delta_g u\|_{L_t^2 L_x^2} + \|[\Delta_g, \beta]\partial_t u\|_{L_t^2 L_x^2} + \|\beta\partial_t u\|_{L_t^2 L_x^2}
\end{aligned}
\tag{2.6}
$$

Since $\Delta_g u$ solves the equation with data $(\Delta_g f, \Delta_g g)$ and forcing term $\Delta_g F$, we get

$$
\begin{aligned}
\|\beta\Delta_g u\|_{L_t^2 H_x^1} + \|\beta\partial_t\Delta_g u\|_{L_t^2 L_x^2} &\lesssim \|\Delta_g f\|_{\dot{H}_x^{1+\epsilon}} + \|\Delta_g g\|_{\dot{H}_x^\epsilon} + \|\Delta_g F\|_{L_t^2 \dot{H}_x^\epsilon} \\
&\lesssim \|f\|_{\dot{H}_x^{3+\epsilon}} + \|g_{\dot{H}_x^{2+\epsilon}} + \|F\|_{L_t^2 \dot{H}_x^{2+\epsilon}}.
\end{aligned}
\tag{2.7}
$$

Also, notice that $[\Delta_g, \beta]u = \beta_1\partial_x u + \beta_2 u$, where $\beta_i \in C_0^\infty,\ i = 1, 2$ have support belonging to supp(β). Thus

$$\begin{aligned} \|[\Delta_g,\beta]u\|_{L^2_t H^1_x} + \|[\Delta_g,\beta]\partial_t u\|_{L^2_t L^2_x} &\lesssim \|\beta_3 u\|_{L^2_t H^2_x} + \|\beta_3\partial_t u_{L^2_t H^1_x} \\ &\lesssim \|\beta_3 u\|^{\theta}_{L^2_t H^1}\|\beta_3 u\|^{1-\theta}_{L^2_t H^3} \qquad (2.8) \\ &\quad + \|\beta_3\partial_t u\|^{\theta}_{L^2_{t,x}}\|\beta_3\partial_t u\|^{1-\theta}_{L^2_t H^2_x}, \qquad (2.9) \end{aligned}$$

where $\beta_3 \in C_0^\infty$ has support in supp(β_1) $\cap$ supp(β_2), and θ is any real number in $(0,1)$. Based on (3.51), (3.52) and (2.8), we get that the L^2_t estimate is true for $\gamma = 3$, and similarly holds for $\gamma = 5, 7, 9...$ and moreover for $\gamma \in \mathbb{R}$ by duality and interpolation, i.e.

$$\begin{aligned} \|\beta u\|_{L^2_t H^\gamma_B([0,S]\times\Omega)} + \|\beta\partial_t u\|_{L^2_t H^{\gamma-1}_B([0,S]\times\Omega)} \\ \lesssim \|f\|_{\dot{H}^{\gamma+\epsilon}(\Omega)} + \|g\|_{\dot{H}^{\gamma+\epsilon-1}(\Omega)} + \|F\|_{L^2_t \dot{H}^{\gamma+\epsilon-1}_B([0,S]\times\Omega)}. \end{aligned} \tag{2.10}$$

By Duhamel's principle, the inhomogeneous solution v satisfies

$$\|\beta v\|_{L^2_t H^\gamma_B([0,S]\times\Omega)} + \|\beta\partial_t v\|_{L^2_t H^{\gamma-1}_B([0,S]\times\Omega)} \lesssim \|F\|_{L^1_t \dot{H}^{\gamma+\epsilon-1}_B([0,S]\times\Omega)},$$

by duality of the above estimate, energy estimates and elliptic regularity, we get

$$\begin{aligned} \|u\|_{L^\infty_t \dot{H}^\gamma_B([0,S]\times\Omega)} + \|\partial_t u\|_{L^\infty_t \dot{H}^{\gamma-1}_B([0,S]\times\Omega)} \\ \lesssim \|f\|_{\dot{H}^{\gamma}(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}(\Omega)} + \|F\|_{L^2_t \dot{H}^{\gamma+\epsilon-1}_B([0,S]\times\Omega)}. \end{aligned} \tag{2.11}$$

Now (2.5) is a result of (2.10) and (2.11). We have finished the proof of Proposition 2.2.5.

Now we turn to the proof of Theorem 2.2.2. Fix $\beta \in C_0^\infty(\mathbb{R}^n)$ satisfying $\beta(x) = 1$, $|x| \leqslant 3R$ and write

$$u = v + w, \quad \text{where } v = \beta u,\ w = (1-\beta)u.$$

Then w solves the free wave equation

$$\begin{cases} (\partial_t^2 - \Delta)w = [\beta, \Delta]u \\ w|_{t=0} = (1-\beta)f, \partial_t w|_{t=0} = (1-\beta)g. \end{cases}$$

Notice that $[\beta, \Delta]u$ is compactly supported, so an application of Proposition 2.2.4 and the Minkowski Strichartz estimates shows that $\|w\|_{L_t^p X}$ is dominated by $A(S)\|\rho u\|_{L_t^2 \dot{H}_B^\gamma}$ plus good terms on the Cauchy data, if $\rho \in C_0^\infty$ equals one on the support of β. Therefore, by (2.5), $\|w\|_{L_t^p X}$ is dominated by the right hand side of (2.3) with $\eta = 0$.

For $v = \beta u$, we decompose it in time t and write $v = \sum_{j=-\infty}^{\infty} \varphi(t-j)v$, where $\varphi \in C_0^\infty((-1,1))$. Let $v_j = \varphi(t-j)v$ for $j \geqslant 1$ and $v_0 = v - \sum_{j=1}^{\infty} v_j$. Then v_j solves

$$\begin{cases} (\partial_t^2 - \Delta_g)v_j = G_j \\ Bv_j(t,x) = 0, \quad x \in \partial\Omega \\ v_j(0,\cdot) = \partial_t v_j(0,\cdot) = 0, \end{cases}$$

where $G_j = -\varphi(t-j)[\Delta_g, \beta]u + [\partial_t^2, \varphi(t-j)]\beta u + \varphi(t-j)F$. Also v_0 solves the equation with $G_0 = -\tilde{\varphi}[\Delta_g, \beta]u + [\partial_t^2, \tilde{\varphi}]\beta u + \tilde{\varphi}F$ and initial date $(\beta f, \beta g)$.

Since G_j with $j \geqslant 0$ vanishes if t is not in $[j-1, j+1]$ or if $|x| > 3R$, by the local Strichartz estimates (2.2) and Duhamel's Principle, we get for $j = 1, 2, \ldots,$

$$\|v_j\|_{L_t^p X([0,S]\times\Omega)} \lesssim \int_0^S \|G_j(s,\cdot)\|_{\tilde{H}_\eta^{\gamma-1}}\, ds \lesssim \|G_j\|_{L_t^2 H_B^{\gamma+\eta-1}}.$$

Similarly,

$$\|v_0\|_{L_t^p X(\mathbb{R}_+\times\Omega)} \lesssim \|f\|_{\tilde{H}_\eta^\gamma} + \|g\|_{\tilde{H}_\eta^{\gamma-1}} + \|G_0\|_{L_t^2 H_B^{\gamma+\eta-1}}.$$

Since $p > 2$, by (2.5) and the disjoint support of G_j, we have

$$\begin{aligned}\|v\|^2_{L^p_t X([0,S]\times\Omega)} &\lesssim \sum_{j=0}^{\infty} \|v_j\|^2_{L^p_t X([0,S]\times\Omega)} \\ &\lesssim \sum_{j=1}^{\infty} \|G_j\|^2_{L^2_t H^{\gamma+\eta-1}_B([0,S]\times\Omega)} + \|v_0\|_{L^p_t X(\mathbb{R}_+\times\Omega)} \\ &\lesssim \|f^2\|_{\dot H^{\gamma+\epsilon+\eta}} + \|g^2\|_{\dot H^{\gamma+\epsilon+\eta-1}} + \|F^2\|_{L^2_t \dot H^{\gamma+\epsilon+\eta-1}}, \\ &\lesssim A^2(S)(\|f^2\|_{\dot H^{\gamma+\epsilon+\eta}} + \|g^2\|_{\dot H^{\gamma+\epsilon+\eta-1}} + \|F^2\|_{L^2_t \dot H^{\gamma+\epsilon+\eta-1}}),\end{aligned}$$

which finishes the proof of Theorem 2.2.2.

2.3 Strauss Conjecture when $n = 3, 4$

In this section we provide the proof of Main Theorem 1. We use the classical iteration methods with suitable space X chosen, then apply the Strichartz estimates proved in the previous two sections to get the results we want. Since the argument is somewhat the same for local and global cases, we only present the subcritical case $p > p_c$ and $n = 3, 4$.

Define $X = X_{\gamma,p}(\mathbb{R}^n)$ to be the space with the norm defined by

$$\|h\|_{X_{\gamma,p}} = \|h\|_{L^{s_\gamma}(|x|<2R)} + (A(S))^{-1}\||x|^{-n/2+1-\gamma/p}h\|_{L^p_r L^2_\omega(\{|x|>2R\})}, \tag{2.12}$$

where $s_\gamma = 2n/(n-2\gamma)$. When $n = 3, p < p_c, \gamma = \frac{1}{2} - \frac{1}{p}$, we have $S = T$ and $A(T)$ is as defined in the last section; When $n = 3, 4,\ p > p_c$ and $\gamma = n/2 - 2/p - 1$ we have $S = \infty$ and $A(S)$ is a constant.

Using the space X defined just now, we can prove the following estimate:

$$\|u\|_{L^p_t X(\mathbb{R}_+\times\mathbb{R}^3)} \lesssim \|f\|_{\dot H^\gamma} + \|g\|_{\dot H^{\gamma-1}}, \tag{2.13}$$

when u solves $\Box u = 0$ with initial data (f, g).

Indeed, the contribution of the second part of the norm in (2.12) is proved by interpolation between Morawetz estimates and Trace Lemma, and the contribution of the first term is due to Sobolev estimates and an interpolation between L_t^2 and L_t^∞ bound in the following energy estimate

$$\begin{aligned}\|u\|_{L_t^\infty \dot{H}_B^\gamma(\mathbb{R}_+\times\Omega)} + \|\partial_t u\|_{L_t^\infty \dot{H}_B^{\gamma-1}(\mathbb{R}_+\times\Omega)} + \|\beta u\|_{L_t^2 H_B^\gamma(\mathbb{R}_+\times\Omega)} + \|\beta\partial_t u\|_{L_t^2 H_B^{\gamma-1}(\mathbb{R}_+\times\Omega)} \\ \lesssim \|f\|_{\tilde{H}_\epsilon^\gamma(\Omega)} + \|g\|_{\tilde{H}_\epsilon^{\gamma-1}(\Omega)} + \|F\|_{L_t^2 \dot{H}_B^{\gamma+\epsilon-1}(\mathbb{R}_+\times\Omega)},\end{aligned} \tag{2.14}$$

where F is supported in $|x| < R$, $\gamma < \frac{n-1}{2}$. The estimate is a variation of the energy estimate (2.5) and has similar proof, which we will neglect here.

Furthermore, by finite propagation speed of the wave equation, Sobolev estimates and interpolation between (2.14), we have the local estimate for solutions of (1.5) with $F = 0$:

$$\|u\|_{L_t^p X([0,1]\times\Omega)} \lesssim (\|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}}), \tag{2.15}$$

where $p \geqslant 2$.

From (2.13) and (2.15), we see that $(X, \gamma, 0, p)$ is admissible. By Theorem 2.2.2, we therefore obtain the following Proposition:

Proposition 2.3.1. Under the conditions of Main Theorem 1,

$$\|u\|_{L_t^p X(\mathbb{R}_+\times\Omega)} \lesssim \|f\|_{\tilde{H}_\epsilon^\gamma} + \|g\|_{\tilde{H}_\epsilon^{\gamma-1}}. \tag{2.16}$$

Based on the above proposition, it is easy to get the following corollary with forcing term added.

Corollary 2.3.2. For $n = 3, 4$, let u be a solution of (1.5), and assume Hypothesis B,

$\gamma = s_c$, $p > p_c$. Then

$$\|u\|_{L^p_t L^{s_\gamma}_x(\mathbb{R}_+\times\{|x|<2R\})} + \||x|^{-n/2+1-\gamma/p} u\|_{L^p_t L^p_r L^2_\omega(\mathbb{R}_+\times\{|x|>2R\})} \lesssim \|f\|_{\tilde{H}^\gamma_\epsilon} + \|g\|_{\tilde{H}^{\gamma-1}_\epsilon} + \|F\|_{L^1_t L_x^{s'_{1-\gamma-\epsilon}}(\mathbb{R}_+\times\{|x|<2R\})} + \||x|^{-n/2+1-\gamma} F\|_{L^1_t L^1_r L^2_\omega(\mathbb{R}_+\times\{|x|>2R\})}. \tag{2.17}$$

Proof. By (2.3.1), we can assume $f = g = 0$. By Duhamel's principle, we have

$$\begin{aligned} LHS &\lesssim \|F\|_{L^1_t \tilde{H}^{\gamma-1}_\epsilon(\mathbb{R}_+\times\Omega)} \\ &\lesssim \|F\|_{L^1_t \dot{H}^{\gamma-1}(\mathbb{R}_+\times\Omega)} + \|F\|_{L^1_t \dot{H}^{\gamma+\epsilon-1}(\mathbb{R}_+\times\Omega)}. \end{aligned}$$

Recall that the dual version of the trace lemma and Sobolev embedding gives (see (3.16) of [6]):

$$\|g\|_{\dot{H}^{\gamma-1}} \lesssim \left\||x|^{-n/2+1-\gamma} g\right\|_{L^1_r L^2_\omega(\{|x|>2R\})} + \|g\|_{L^{s'_{1-\gamma}}(\{|x|<2R\})}, \quad \text{if } 1/2 < 1-\gamma < n/2. \tag{2.18}$$

Here the condition $1/2 < 1-\gamma < n/2$ is satisfied owing to $\gamma = s_c$ and $p > p_c$.

If we use (2.18), then we get

$$\begin{aligned} &\|F\|_{L^1_t \dot{H}^{\gamma-1}(\mathbb{R}_+\times\Omega)} + \|F\|_{L^1_t \dot{H}^{\gamma+\epsilon-1}(\mathbb{R}\times\Omega)} \lesssim \\ &\qquad \left\||x|^{-n/2+1-\gamma} F\right\|_{L^1_t L^1_r L^2_\omega(\mathbb{R}_+\times\{|x|>2R\})} + \|F\|_{L^1_t L_x^{s'_{1-\gamma}}(\mathbb{R}_+\times\{|x|<2R\})} \\ &\qquad + \left\||x|^{-n/2+1-\gamma-\epsilon} F\right\|_{L^1_t L^1_r L^2_\omega(\mathbb{R}_+\times\{|x|>2R\})} + \|F\|_{L^1_t L_x^{s'_{1-\gamma-\epsilon}}(\mathbb{R}_+\times\{|x|<2R\})} \\ &\qquad \lesssim \left\||x|^{-n/2+1-\gamma} F\right\|_{L^1_t L^1_r L^2_\omega(\mathbb{R}_+\times\{|x|>2R\})} + \|F\|_{L^1_t L_x^{s'_{1-\gamma-\epsilon}}(\mathbb{R}_+\times\{|x|<2R\})}, \end{aligned}$$

when $\epsilon > 0$ is small enough, which completes the proof.

□

Now we are only left with adding the 2nd order derivatives on the solution for the

Strichartz estimates and energy estimates above. Specifically, we want to prove that

$$\sum_{|\alpha|\leqslant 2}\Big(\|\,\Gamma^\alpha u\|_{L_t^\infty \dot H_B^\gamma}+\|\partial_t\Gamma^\alpha u\|_{L_t^\infty \dot H_B^{\gamma-1}}\Big)\lesssim \sum_{|\alpha|\leqslant 2}\Big(\|Z^\alpha f\|_{\tilde H^\gamma_{2\epsilon}}+\|Z^\alpha g\|_{\tilde H^{\gamma-1}_{2\epsilon}}\Big)$$
$$+\sum_{|\alpha|\leqslant 2}\Big(\|\,|x|^{-\frac n2+1-\gamma}\Gamma^\alpha F\|_{L^1_tL^1_rL^2_\omega(\mathbb{R}_+\times\{|x|>2R\})}+\|\Gamma^\alpha F\|_{L^1_tL_x^{s'_{1-\gamma-2\epsilon}}(\mathbb{R}_+\times\{x\in\Omega:|x|<2R\})}\Big), \tag{2.19}$$

and

$$\sum_{|\alpha|\leqslant 2}\Big(\|\,|x|^{\frac n2-\frac{n+1}{p}-\gamma}\Gamma^\alpha u\|_{L^p_tL^p_rL^2_\omega(\mathbb{R}_+\times\{|x|>2R\})}+\|\Gamma^\alpha u\|_{L^p_tL_x^{s_\gamma}(\mathbb{R}_+\times\{x\in\Omega:|x|<2R\})}\Big)$$
$$\lesssim \sum_{|\alpha|\leqslant 2}\Big(\|Z^\alpha f\|_{\tilde H^\gamma_{2\epsilon}}+\|Z^\alpha g\|_{\tilde H^{\gamma-1}_{2\epsilon}}\Big)$$
$$+\sum_{|\alpha|\leqslant 2}\Big(\|\,|x|^{-\frac n2+1-\gamma}\Gamma^\alpha F\|_{L^1_tL^1_rL^2_\omega(\mathbb{R}_+\times\{|x|>2R\})}+\|\Gamma^\alpha F\|_{L^1_tL_x^{s'_{1-\gamma-2\epsilon}}(\mathbb{R}_+\times\{x\in\Omega:|x|<2R\})}\Big). \tag{2.20}$$

The idea is based on the fact that Γ commute with $\Box_g$ away from the origin, elliptic property of $\Delta_{\mathfrak{g}}$ and local energy decay near the origin. We will first deal with the Cauchy data for $\Gamma^\alpha u$. This is clear if Γ^α is replaced by Z^α. On the other hand, the Cauchy data is $(g,\Delta_g f+F(0,\cdot))$ for $\partial_t u$ and $(\Delta_g f+F(0,\cdot),\Delta_g g+\partial_t F(0,\cdot))$ for $\partial_t^2 u$, so we have

$$\|g\|_{\tilde H^\gamma_\epsilon}+\|\Delta_g f\|_{\tilde H^{\gamma-1}_\epsilon\cap\tilde H^\gamma_\epsilon}+\|F\|_{L^\infty_t\tilde H^{\gamma-1}_\epsilon\cap L^\infty_t\tilde H^\gamma_\epsilon}+\|\partial_t F\|_{L^\infty_t\tilde H^{\gamma-1}_\epsilon}+\|\Delta_g g\|_{\tilde H^{\gamma-1}_\epsilon}\lesssim$$
$$\sum_{|\alpha|\leqslant 2}\Big(\|Z^\alpha f\|_{\tilde H^\gamma_\epsilon}+\|Z^\alpha g\|_{\tilde H^{\gamma-1}_\epsilon}\Big)+\sum_{|\alpha|\leqslant 2}\|\Gamma^\alpha F\|_{L^1_t\tilde H^{\gamma-1}_\epsilon},$$

where we use Sobolev embedding in the time variable t for $(F,\partial_t F)$. If we use (2.18) to control the last term $\sum_{|\alpha|\leqslant 2}\|\Gamma^\alpha F\|_{L^1_t\tilde H^{\gamma-1}_\epsilon}$, then we get (2.19) and (2.20) for the Cauchy data part of Γu.

Let us now give the argument for (2.20). Fix $\beta_0\in C_0^\infty$ satisfying $\beta_0=1$ for $|x|\leqslant R$

and vanishing for $|x| > 2R$. Let

$$\Gamma^\alpha u = (1-\beta_0)\Gamma^\alpha u + \beta_0 \Gamma^\alpha u = v + w.$$

Since Γ commutes with $\Box_g$ when $|x| \geqslant R$, we have

$$\begin{cases} \Box_g v = (1-\beta_0)\Gamma^\alpha F - [\beta_0, \Delta_g]\Gamma^\alpha u\,, \\ v(0,\cdot) = ((1-\beta_0)\Gamma^\alpha u(0,\cdot),\ \partial_t v(0,\cdot) = \partial_t (1-\beta_0)\Gamma^\alpha u(0,\cdot). \end{cases}$$

The initial data has been taken care of from the discussion above, and the first nonlinear term is dominated by the right hand side of (2.20) by (2.17). For the second nonlinear term, we use Proposition 2.2.4 and control it by

$$\sum_{|\alpha|\leqslant 2} \| [\beta_0, \Delta_g]\Gamma^\alpha u\|_{L_t^2 H_B^{\gamma-1}} \lesssim \sum_{j\leqslant 2} \|\beta_1 \partial_t^j u\|_{L_t^2 H_B^{\gamma+2-j}}\,, \tag{2.21}$$

assuming that β_1 equals one on the support of β_0 and is supported in $R < |x| < 2R$. Note that $[\Box_g, \partial_t^2] = 0$, if we use (2.14) for $\partial_t^2 u$ and Duhamel's principle for the forcing term $\partial_t^2 F$, we can control $\|\beta_1 \partial_t^2 u\|_{L_t^2 H_B^\gamma}$ by the right hand side of (2.20). On the other hand, by Cauchy-Schwarz and Parseval's Formula,

$$\|\beta_1 \partial_t u\|^2_{L_t^2 H_B^{\gamma+1}} \lesssim \|\beta_1 \partial_t^2 u\|_{L_t^2 H_B^\gamma}\, \|\beta_1 u\|_{L_t^2 H_B^{\gamma+2}}.$$

So it suffices to dominate $\|\beta_1 u\|_{L_t^2 H_B^{\gamma+2}}$. By elliptic regularity of the operator Δ_g, we have

$$\begin{aligned} \|\beta_1 u\|_{L_t^2 H_B^{\gamma+2}} &\lesssim \|\beta_2 \Delta_g u\|_{L_t^2 H_B^\gamma} + \|\beta_2 u\|_{L_t^2 H_B^\gamma} \\ &\lesssim \|\beta_2 \partial_t^2 u\|_{L_t^2 H_B^\gamma} + \|\beta_2 u\|_{L_t^2 H_B^\gamma} + \|\beta_2 F\|_{L_t^2 H_B^\gamma}, \end{aligned}$$

where $\beta_2 \in C_0^\infty$ equals one on support of β_1 and is supported in the set where $|x| < 2R$.

The first two terms are dominated as above using (2.14) and Duhamel's principle. For the last term, Sobolev embedding and duality yields

$$\begin{aligned}(2.22)\qquad \|\beta_2 F\|_{L^2_t H^\gamma_B} &\lesssim \sum_{|\alpha|\leqslant 1} \|\partial^\alpha_x F\|_{L^2_t L^{s'_{1-\gamma}}(\mathbb{R}_+\times\{x\in\Omega:|x|\leqslant 2R\})} \\ &\lesssim \sum_{|\alpha|\leqslant 2} \|\partial^\alpha_{t,x} F\|_{L^1_t L^{s'_{1-\gamma-\epsilon}}(\mathbb{R}_+\times\{x\in\Omega:|x|\leqslant 2R\})}.\end{aligned}$$

Thus we are done with the proof of (2.20) when $\Gamma^\alpha u$ is replaced by v.

For $w = \beta_0\Gamma^\alpha u$, the coefficients of Γ are bounded on support of β_0, so by Sobolev embedding

$$\begin{aligned}\sum_{|\alpha|\leqslant 2} \|\beta_0\Gamma^\alpha u\|_{L^p_t L^{s_\gamma}_x(\mathbb{R}_+\times\Omega)} &\lesssim \sum_{|\alpha|\leqslant 2} \|\beta_1\Gamma^\alpha u\|_{L^p_t \dot H^\gamma_B} \\ &\lesssim \sum_{|j|\leqslant 2} \Big(\|\beta_1\partial^j_t u\|_{L^2_t H^{\gamma+2-j}_B} + \|\beta_1\Gamma^j u\|_{L^\infty_t \dot H^\gamma_B}\Big).\end{aligned}$$

The first term is dominated as above, and the bound for the second term comes from (2.19), so we are done with proof of (2.20).

Now we turn to the proof of (2.19).

As before we first consider the inequality where $\Gamma^\alpha u$ is replaced by $v = (1-\beta_0)\Gamma^\alpha u$ in (2.19). The inequality involving initial data has been taken care of in the first paragraph of the proof, and the first nonlinear term is from energy estimates in $\mathbb{R}^n$, Duhamel's principle and (2.18). For the remaining term by (2.14) we see that it is controlled by

$$(2.23)\qquad \sum_{|\alpha|\leqslant 2} \|\,[\beta_0,\Delta_g]\Gamma^\alpha u\|_{L^2_t H^{\gamma+\epsilon-1}_B} \lesssim \sum_{j\leqslant 2} \|\beta_1\partial^j_t u\|_{L^2_t H^{\gamma+\epsilon+2-j}_B}.$$

By almost the same argument as above we get the desired bound in (2.19).

Now we are only left with $w = \beta_0\Gamma^\alpha u$. First notice that the left hand side of (2.19) with w is dominated by $\sum_{j\leqslant 3}\|\beta_1\partial_t^j u\|_{L_t^\infty H_B^{2+\gamma-j}}$. For the case $j=0,1$, since

$$\begin{cases}\Box_g(\beta_1 u)=\beta_1 F+[\Delta_g,\beta_2]u\\ (\beta_1 u,\partial_t\beta_1 u)|_{t=0}=(\beta_1 f,\beta_1 g),\end{cases}$$

we use (2.5) with the Duhamel formula to bound

$$\begin{aligned}&\|\beta_1 u\|_{L_t^\infty H_B^{\gamma+2}}+\|\beta_1\partial_t u\|_{L_t^\infty H_B^{\gamma+1}}\\ &\qquad\lesssim\|\beta_1 f\|_{H_B^{\gamma+2}}+\|\beta_1 g\|_{H_B^{\gamma+1}}+\|\beta_2 u\|_{L_t^2 H_B^{\gamma+\epsilon+2}}+\|\beta_1 F\|_{L_t^1 H_B^{\gamma+\epsilon+1}}.\end{aligned}$$

The term on the right involving u was controlled previously; on the other hand, by Sobolev embedding,

$$\|\beta_1 F\|_{L_t^1 H_B^{\gamma+\epsilon+1}}\lesssim\sum_{|\alpha|\leqslant 2}\|\partial_x^\alpha F\|_{L_t^1 L_x^{s'_{1-\gamma-\epsilon}}}.$$

To handle the terms for $j=2,3$ we use the equation to bound

$$\sum_{j=2,3}\|\beta_1\partial_t^j u\|_{L_t^\infty H_B^{2+\gamma-j}}\leqslant\sum_{j=0,1}\Big(\|\beta_1\partial_t^j\Delta_g u\|_{L_t^\infty H_B^{\gamma-j}}+\|\beta_1\partial_t^j F\|_{L_t^\infty H_B^{\gamma-j}}\Big).$$

The terms involving $\Delta_g u$ are dominated by $\|\beta_2\partial_t^j u\|_{L_t^\infty H_B^{\gamma+2-j}}$ with $j=0,1$. The terms involving F are controlled for $j=1$ by Sobolev Embedding Theorem, and for $j=0$ by observing that (2.22) holds with L_t^2 replaced by L_t^∞. This completes the proof of (2.19).

Now we finally get to the proof of Strauss conjecture stated in Main Theorem 1. As stated we only present the case when $n=3$ and $p>p_c$. The argument is adapted from [6].

First, let f solve the Cauchy problem (1.5) with $F=0$. We iteratively define u_k, for

$k \geqslant 1$, by solving

$$\begin{cases} (\partial_t^2 - \Delta_g)u_k(t,x) = F_p(u_{k-1}(t,x))\,, \quad (t,x) \in \mathbb{R}_+ \times \Omega \\ u_k(0,\cdot) = f,\ \partial_t u_k(0,\cdot) = g \\ (Bu_k)(t,x) = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases}$$

Our aim is to show that if the constant $\varepsilon' > 0$ in Cahchy data bound is small enough, then so is

$$M_k = \sum_{|\alpha|\leqslant 2} \Big(\|\Gamma^\alpha u_k\|_{L_t^\infty \dot{H}_B^\gamma(\mathbb{R}_+\times\Omega)} + \|\partial_t \Gamma^\alpha u_k\|_{L_t^\infty \dot{H}_B^{\gamma-1}(\mathbb{R}_+\times\Omega)} \\ + \big\| |x|^{\frac{-\frac{n}{2}+1-\gamma}{p}} \Gamma^\alpha u_k \big\|_{L_t^p L_r^p L_\omega^2(\mathbb{R}_+\times\{|x|>2R\})} + \|\Gamma^\alpha u_k\|_{L_t^p L_x^{s_\gamma}(\mathbb{R}_+\times\{x\in\Omega:\,|x|<2R\})} \Big)$$

for every $k = 0, 1, 2, \ldots$.

For $k = 0$, it follows by (2.19) and (2.20) that $M_0 \leqslant C_0\varepsilon'$, with C_0 a fixed constant. More generally, (2.19) and (2.20) yield that

$$M_k \leqslant C_0\varepsilon' + C_0 \sum_{|\alpha|\leqslant 2} \Big(\big\| |x|^{-\frac{n}{2}+1-\gamma} \Gamma^\alpha F_p(u_{k-1}) \big\|_{L_t^1 L_r^1 L_\omega^2(\mathbb{R}_+\times\{|x|>2R\})} \\ + \|\Gamma^\alpha F_p(u_{k-1})\|_{L_t^1 L_x^{s'_{1-\gamma-2\epsilon}}(\mathbb{R}_+\times\{x\in\Omega:\,|x|<2R\})} \Big). \tag{2.24}$$

Note that our assumption on the nonlinear term F_p implies that for small v

$$\sum_{|\alpha|\leqslant 2} |\Gamma^\alpha F_p(v)| \lesssim |v|^{p-1} \sum_{|\alpha|\leqslant 2} |\Gamma^\alpha v| + |v|^{p-2} \sum_{|\alpha|\leqslant 1} |\Gamma^\alpha v|^2\,.$$

Furthermore, since u_k will be locally of regularity $H_B^{\gamma+2} \subset L^\infty$ and F_p vanishes at 0, it follows that $F_p(u_k)$ satisfies the B boundary conditions if u_k does.

Since the collection Γ contains vectors spanning the tangent space to S^{n-1}, by Sobolev embedding for $n=3,4$ we have

$$\|v(r\cdot)\|_{L^\infty_\omega}+\sum_{|\alpha|\leqslant 1}\|\Gamma^\alpha v(r\cdot)\|_{L^4_\omega}\lesssim\sum_{|\alpha|\leqslant 2}\|\Gamma^\alpha v(r\cdot)\|_{L^2_\omega}\,.$$

Consequently, for fixed $t,r>0$,

$$\sum_{|\alpha|\leqslant 2}\|\Gamma^\alpha F_p(u_{k-1}(t,r\cdot))\|_{L^2_\omega}\lesssim\sum_{|\alpha|\leqslant 2}\|\Gamma^\alpha u_{k-1}(t,r\cdot)\|^p_{L^2_\omega}\,.$$

Thus the first summand in the right hand side of (2.24) is dominated by $C_1M^p_{k-1}$.

We next observe that, since $s_\gamma>2$ and $n\leqslant 4$, it follows by Sobolev embedding on $\{\Omega\cap|x|<2R\}$ that

$$\|v\|_{L^\infty(x\in\Omega:|x|<2R)}+\sum_{|\alpha|\leqslant 1}\|\Gamma^\alpha v\|_{L^4(x\in\Omega:|x|<2R)}\lesssim\sum_{|\alpha|\leqslant 2}\|\Gamma^\alpha v\|_{L^{s_\gamma}(x\in\Omega:|x|<2R)}\,.$$

Since $s'_{1-\gamma-2\epsilon}<2$, it holds for each fixed t that

$$\begin{aligned}\sum_{|\alpha|\leqslant 2}\|\Gamma^\alpha F_p(u_{k-1}(t,\cdot))\|_{L^{s'_{1-\gamma-2\epsilon}}(x\in\Omega:|x|<2R)}&\lesssim\sum_{|\alpha|\leqslant 2}\|\Gamma^\alpha F_p(u_{k-1}(t,\cdot))\|_{L^2(x\in\Omega:|x|<2R)}\\&\lesssim\sum_{|\alpha|\leqslant 2}\|\Gamma^\alpha u_{k-1}(t,\cdot)\|^p_{L^{s_\gamma}(x\in\Omega:|x|<2R)}\,.\end{aligned}\tag{2.25}$$

The second summand in the right side of (2.24) is thus dominated by $C_1M^p_{k-1}$, and we conclude that $M_k\leqslant C_0\epsilon'+2C_0\,C_1M^p_{k-1}$. For ϵ' sufficiently small, by the definition of $A(S)$, we obtain

$$M_k\leqslant 2\,C_0\varepsilon',\quad k=1,2,3,\dots\tag{2.26}$$

To finish the proof of Strauss Conjecture we need to show that u_k converges to a solution of the equation (1.5). For this it suffices to show that

$$A_k = (A(S))^{-1} \big\| \, |x|^{\frac{-\frac{n}{2}+1-\gamma}{p}} (u_k - u_{k-1}) \, \big\|_{L^p_t L^p_r L^2_\omega(\mathbb{R}_+ \times \{|x|>2R\})} \\ + \|u_k - u_{k-1}\|_{L^p_t L^{s_\gamma}_x(\mathbb{R}_+ \times \{x\in\Omega:\, |x|<2R\})}$$

tends geometrically to zero as $k \to \infty$. Since $|F_p(v) - F_p(w)| \lesssim |v - w|(\,|v|^{p-1} + |w|^{p-1}\,)$ when v and w are small, the proof of (2.26) can be adapted to show that, for small $\varepsilon' > 0$, there is a uniform constant C so that

$$A_k \leqslant C A_{k-1}(M_{k-1} + M_{k-2})^{p-1},$$

which, by (2.26), implies that $A_k \leqslant \frac{1}{2} A_{k-1}$ for small ε'. Since A_1 is finite, the claim follows, which finishes the proof of Main Theorem 1.

Chapter 3

Part II Strauss Conjecture on Asymptotically Euclidean Manifolds

This chapter is devoted to the joint work with Chengbo Wang, which is subsequent to [21]. In the work of Sogge and Wang [21], a global existence result is obtained for the system of (1.4) with symmetrical metric $\mathfrak{g}_{ij}$. In our work we remove the radial assumption and also show the supercritical case $p < p_c$ when $n = 3$ by proving a local in time Strichartz estimate. We will first go over the argument presented in [21] in Section 3.1, then get the required estimates without radial assumption in Section 3.2 and 3.3. Finally we briefly give the argument to prove the existence results presented in Main Theorem 2.

3.1 A special Case: 3-D, $p > 1 + \sqrt{2}$, the metric g is spherically symmetric

Theorem 3.1.1. *Suppose* (H1) *and* (H2) *hold with* $\rho > 1$, $n = 3$, *and* $p > 1 + \sqrt{2}$. *Assume* $\mathfrak{g}_{ij}(x) = \mathfrak{g}_{ij}(|x|), \forall x$. *Then for any* $\epsilon > 0$ *such that*

$$s = s_c - \epsilon \in (s_d, 1/2) \tag{3.1}$$

there is a $\delta > 0$ *depending on* p *so that* (1.4) *has a global solution satisfying* $(Z^\alpha u(t,\cdot), \partial_t Z^\alpha u(t,\cdot)) \in \dot H^s \times \dot H^{s-1}$, $|\alpha| \leqslant 1$, $t \in \mathbb{R}_+$, *whenever the initial data satisfies*

$$\sum_{|\alpha|\leqslant 1} \left(\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}} \right) < \delta. \tag{3.2}$$

We expect to apply the idea in [6] to get the existence results in Theorem 3.1.1. Thus, it suffices to obtain the generalized Strichartz estimates as follows.

$$\sum_{|\alpha|\leqslant 1} \|Z^\alpha u\|_{L^2_t Y_{s,\epsilon}} + \||x|^{n/2-(n+1)/p-s-\epsilon} Z^\alpha u\|_{L^p_t L^p_{|x|} L^{2+\eta}_\omega(\{|x|>1\})} \lesssim \sum_{|\alpha|\leqslant 1} \left(\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}\right) \ , \tag{3.3}$$

and for $s \in [0,1]$,

$$\sum_{|\alpha|\leqslant 1} \Big(\|Z^\alpha u\|_{L^\infty_t \dot H^s} + \|\partial Z^\alpha u\|_{L^\infty_t \dot H^{s-1}} + \|Z^\alpha u\|_{L^p_t L^{q_s}_x(|x|\leqslant 1)}\Big) \lesssim \sum_{|\alpha|\leqslant 2} \left(\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}\right) \ , \tag{3.4}$$

where $q_s = 2n/(n-2s)$. Note that for convenience we have defined the norm $Y_{s,\epsilon}$ as

$$\|f(x)\|_{Y_{s,\epsilon}} = \|\langle x\rangle^{-(1/2)-s-\epsilon} f(x)\|_{L^2_x} \ .$$

On Asymptotically Euclidean manifolds we do not automatically have the above estimates which are proved to be true in the flat metric case, due to the lack of Fourier transform techniques. However, in the work of [2] and [21], an important Keel-Smith-Sogge (KSS) type estimate is proven for (M, g) by spectral methods. Equipped with the KSS estimates and energy estimates, we are able to proceed and eventually obtain the estimates (3.3) and (3.4).

We first present the KSS estimates on this setting. KSS estimates was originated by Keel, Smith and Sogge [15] and state that

$$(\log(2+T))^{-1/2}\|\langle x\rangle^{-1/2}u'\|_{L^2([0,T]\times\mathbb{R}^3)}\lesssim\|u'(0,\cdot)\|_{L^2(\mathbb{R}^3)}+\int_0^T\|F(s,\cdot)\|_{L^2(\mathbb{R}^3)}\,ds, \tag{3.5}$$

where u solves the equation $\Box u = F$ and $u' = (\partial_t u, \partial_x u)$. This estimate has been generalized for general weight of form $\langle x\rangle^{-a}$ with $a \geqslant 0$ (see [10] and references therein). Recently, Bony and Häfner [2] obtained a weaker version of the KSS estimates for asymptotically Euclidean space when the metric is non-trapping. With this estimate, they were able to show the global and long time existence for quadratic semilinear wave equations with dimension $n \geqslant 4$ and $n = 3$. Then Sogge and Wang [21] proved the almost global existence for 3-D quadratic semilinear equations by obtaining the sharp KSS estimates for $a = 1/2$. Now we present the KSS estimates as a lemma here.

Lemma 3.1.2 (KSS estimates). *Assume that* (H1) *and* (H2) *hold with* $\rho > 1$. *Let* $N \geqslant 0$, $\mu \geqslant 1/2$ *and*

$$A_\mu(T) = \begin{cases} (\log(2+T))^{-1/2} & \mu = 1/2, \\ 1 & \mu > 1/2. \end{cases}$$

Then the solution of (1.1) *satisfies*

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\sum_{1\leqslant k+j\leqslant N+1}\left\|\partial_t^k P^{j/2} g u(t,\cdot)\right\|_{L^2_x} + \sum_{|\alpha|\leqslant N} A_\mu(T)\left\|\langle x\rangle^{-\mu}\left(|(\Gamma^\alpha u)'| + \frac{|\Gamma^\alpha u|}{\langle x\rangle}\right)\right\|_{L^2_T L^2_x} \\ \lesssim \sum_{|\alpha|\leqslant N}\left\|(Z^\alpha u)'(0,\cdot)\right\|_{L^2_x} + \sum_{|\alpha|\leqslant N}\left\|\Gamma^\alpha F(s,\cdot)\right\|_{L^1_T L^2_x},\end{aligned} \tag{3.6}$$

where $L^q_T L^r_x = L^q([0,T]; L^r(\mathbb{R}^n))$.

We also make the following variation of solutions.

Remark 1. Set $P = -g\Delta_{\mathfrak{g}} g^{-1} = -g^{-1}\partial\mathfrak{g}^{ij}g^2\partial g^{-1}$, where $g = (\det\mathfrak{g})^{1/4}$. We will prove the estimates if u is the solution of $(\partial^2 + P)u = F$, which has the benefit that the solution can be represented by the following formula

$$u(t) = \cos(tP^{1/2})f + P^{-1/2}\sin(tP^{1/2})g + \int_0^t P^{-1/2}\sin((t-s)P^{1/2})F(s)ds \ .$$

All of the operators occurring in this formula commutates with the wave operator $\partial^2 + P$. In general, an estimate for $-\Delta_g$ will corresponds another estimate for P. For example, if we have the estimate (3.3) for P, consider the equation

$$\begin{cases} (\partial_t^2 - \Delta_g)v(t,x) = G(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ u(0,x) = v_0(x), \quad \partial_t v(0,x) = v_1(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{3.7}$$

Notice that if we let $u = gv$ and $F = gG$, then

$$(\partial^2 - \Delta_g)v = G \Leftrightarrow (\partial^2 + P)u = F. \tag{3.8}$$

Thus we have also the estimate (3.3) for $-\Delta_g$.

Now we turn to the proof of (3.3) and (3.4). When $|\alpha| = 0$, i.e. the order is zero, the $L_t^2 Y_{s,\epsilon}$ bound in (3.3) is from interpolation between

$$\|\langle x\rangle^{-1/2-\epsilon} e^{itP^{1/2}} f\|_{L_t^2 L_x^2} \lesssim \|f\|_{L_x^2},$$

and

$$\|\langle x\rangle^{-3/2-\epsilon} e^{itP^{1/2}} f\|_{L_t^2 L_x^2} \lesssim \|f\|_{\dot{H}_x^1},$$

which hold due to the KSS estimates. The $L_t^p L_{|x|}^p L_\omega^{2+\eta}$ bound is from interpolation between the $L_t^2 Y_{s,\epsilon}$ and the following Sobolev inequalities with angular regularity (see Corollary

1.2 in [3]),

$$\||x|^{\frac{d}{2}-\alpha}e^{itP^{1/2}}f(x)\|_{L^\infty_{t,|x|}L^{2+\eta}_\omega}\lesssim\|e^{itP^{1/2}}f(x)\|_{L^\infty_t\dot{H}^\alpha_x}\lesssim\|f\|_{\dot{H}^\alpha_x} \tag{3.9}$$

for $\alpha\in(1/2,1]$ and some $\eta>0$. As for (3.4), the first two terms in the left hand side are just the energy estimates. For the last term we first use a Sobolev embedding, then apply an interpolation between $p=2$ and $p=+\infty$. The case when $p=\infty$ is the energy estimates. The case when $p=2$ is from the local energy decay below.

Lemma 3.1.3 (Local Energy Decay). For the linear equation (1.4), if $F(t,x)=0$ for $|x|>R$ with R fixed, then for fixed $\beta\in C_0^\infty(\mathbb{R}^d)$, we have

$$\left\|\beta u\right\|_{L^2_tH^1}\lesssim\|f\|_{\dot{H}^1}+\|g\|_{L^2_x}+\|F\|_{L^2_tL^2_x}. \tag{3.10}$$

Moreover, if $F\equiv 0$ and $s\in[0,1]$, then

$$\left\|\beta u\right\|_{L^2_tH^s}\lesssim\|f\|_{\dot{H}^s}+\|g\|_{\dot{H}^{s-1}_x}. \tag{3.11}$$

The local energy decay also plays an important role in the proof of the KSS estimates when $\mu=1/2$ and can be proven by the KSS estimates when $\mu>1/2$. So we are finished with the case $|\alpha|=0$.

When $|\alpha|=1$, since we assume a radial metric, it is trivial for $Z=\Omega_{ij}$ due to the fact that $[P,\Omega_{ij}]=0$, so we are left with $Z=\partial$. The proof relies largely on the equivalence of $P^{1/2}$ and ∂ in specifical norms, which we state here as a lemma.

Lemma 3.1.4. [Relation between $P^{1/2}$ and ∂]

i, If $s\in[-1,1]$, then $\|u\|_{\dot{H}^s}\simeq\|P^{s/2}u\|_{L^2_x}$.

ii, If $s\in[0,1]$, then $\|\tilde{\partial}_ju\|_{\dot{H}^{-s}}\lesssim\|P^{1/2}u\|_{\dot{H}^{-s}}$, and $\|P^{1/2}u\|_{\dot{H}^s}\lesssim\sum_j\|\tilde{\partial}_ju\|_{\dot{H}^s}$.

iii, If $s\in(0,2]$ and $1<q<d/s$, then $\|P^{s/2}u\|_{L^q_x}\lesssim\|u\|_{\dot{H}^{s,q}}$.

iv, If $-3/2 \leqslant \widetilde{\mu} < \mu \leqslant 3/2$, then

$$\left\|\langle x\rangle^{-\mu}\widetilde{\partial}_\ell u\right\|_{L^2(\mathbb{R}^d)} \lesssim \left\|\langle x\rangle^{-\widetilde{\mu}} P^{1/2} u\right\|_{L^2(\mathbb{R}^d)}, \tag{3.12}$$

$$\left\|\langle x\rangle^{-\mu} P^{1/2} u\right\|_{L^2(\mathbb{R}^d)} \lesssim \sum_{\ell=1}^{d} \left\|\langle x\rangle^{-\widetilde{\mu}}\widetilde{\partial}_\ell u\right\|_{L^2(\mathbb{R}^d)}. \tag{3.13}$$

iv, If $u \in H^1(\mathbb{R}^d)$,

$$\|P^{1/2}u\|_{L^2(\mathbb{R}^d)} \lesssim \|\nabla g^{-1} u\|_{L^2(\mathbb{R}^d)} \lesssim \|P^{1/2}u\|_{L^2(\mathbb{R}^d)}. \tag{3.14}$$

With the above lemma and the estimates with order 0, we can estimate $L_t^2 Y_{s,\epsilon}$ part as follows.

$$\begin{aligned}
\sum_{|\alpha|\leqslant 1} \|\partial_x^\alpha u\|_{L_t^2 Y_{s,\epsilon}} &\lesssim \sum_{|\alpha|\leqslant 1} \|\tilde{\partial}_x^\alpha u\|_{L_t^2 Y_{s,\epsilon}} \\
&\lesssim \sum_{j\leqslant 1} \|P^{j/2} u\|_{L_t^2 Y_{s,\epsilon/2}} \\
&\lesssim \sum_{j\leqslant 1} \left(\|P^{j/2} f\|_{\dot{H}^s} + \|P^{j/2} g\|_{\dot{H}_x^{s-1}}\right) \\
&\lesssim \sum_{j\leqslant 1} \left(\|P^{(j+s)/2} f\|_{L_x^2} + \|P^{(j+s-1)/2} g\|_{L_x^2}\right) \\
&\lesssim \sum_{|\alpha|\leqslant 1} \left(\|\partial_x^\alpha f\|_{\dot{H}^s} + \|\partial_x^\alpha g\|_{\dot{H}_x^{s-1}}\right).
\end{aligned}$$

The proof for the first order energy norm $L_t^\infty \dot{H}^s$ is similar so we neglect it here. The $L_t^\infty L_r^\infty L_\omega^{2+\eta}$ estimate is direct consequence of the energy estimates and (3.9). So we are left with the local decay for $\|\phi\partial_x u\|_{L_t^2 \dot{H}^s}$, where $\phi \in C_0^\infty$, but this is again from the KSS estimates for $\mu > 1/2$. Now we are finished with the proof of (3.3) and (3.4), with these two estimates at hand, we are able to use similar argument as in [6] to obtain the existence result with $n = 3$ under symmetrical assumption on the metric.

3.2 Weighted Strichartz and Energy Estimates

In what follows, "remainder terms", r_j, $j \in \mathbb{N}$, will denote any smooth functions such that

$$\partial_x^\alpha r_j(x) = O\big(\langle x\rangle^{-\rho-j-|\alpha|}\big),\ \forall\alpha\ , \tag{3.15}$$

thus $P = -g\Delta_g g^{-1} = -\Delta + r_0\partial^2 + r_1\partial + r_2$.

In order to extend the existence result in Section 3.1 to $n = 4$ and nonsymmetric metric, there are several difficulties to overcome. Firstly, we no longer have $[\Omega, P] = 0$, thus have to figure out a way to handle with the extra commutator terms to get (3.3) and (3.4). Secondly, in order to extend the result to dimension $n = 4$, we hope to obtain the estimates with second order derivatives, hence need explore the relation between P and Z. Specifically, we are aimed at showing the following estimates.

Theorem 3.2.1. *Let u be the solution of* (1.4) *with $F = 0$. Assume that* (H1) *and* (H2) *hold with $\rho > 2$, $n \geqslant 3$, $2 < p \leqslant \infty$ and $s \in (s_d, 1)$. For all $\epsilon > 0$ and $\eta > 0$ small enough, we have*

$$\begin{aligned}\sum_{|\alpha|\leqslant 2}\|Z^\alpha u\|_{L_t^2 Y_{s,\epsilon}} &+ \||x|^{n/2-(n+1)/p-s-\epsilon} Z^\alpha u\|_{L_t^p L_{|x|}^p L_\omega^{2+\eta}(\{|x|>1\})} \\ &\lesssim \sum_{|\alpha|\leqslant 2}\left(\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}\right)\ ,\end{aligned} \tag{3.16}$$

and for $s \in [0,1]$,

$$\begin{aligned}\sum_{|\alpha|\leqslant 2}\Big(\|Z^\alpha u\|_{L_t^\infty \dot H^s} + \|\partial Z^\alpha u\|_{L_t^\infty \dot H^{s-1}}\Big) &+ \sum_{|\alpha|\leqslant 2}\|Z^\alpha u\|_{L_t^p L_x^{q_s}(|x|\leqslant 1)} \\ &\lesssim \sum_{|\alpha|\leqslant 2}\left(\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}\right)\ ,\end{aligned} \tag{3.17}$$

where $q_s = 2n/(n-2s)$.

Note that Theorem 3.2.1, with order 0 ($|\alpha| = 0$) and $\rho > 0$, has been proved in Section3.1. In order to deal with the commutators coming from the commutator of P and the Z, we will need the following three lemmas to gain control on forcing terms.

Lemma 3.2.2. Let u solve the wave equation (1.4). Then for any $s \in [0,1]$ and $\epsilon > 0$, we have:

$$\|u\|_{L^2_t Y_{s,\epsilon}} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}} + \|\langle x\rangle^{(1/2)+\epsilon} F\|_{L^2_t \dot{H}^{s-1}} \tag{3.18}$$

The homogeneous part is proved in Section 3.1. For the inhomogeneous part, when $s = 1$, it is just Remark 2.1 in [21] which states that

$$\left\|\langle x\rangle^{-3/2-\epsilon} u\right\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|\langle x\rangle^{(1/2)+\epsilon} F\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}. \tag{3.19}$$

When $s = 0$, it is equivalent to

$$\left\|\langle x\rangle^{-1/2-\epsilon} u\right\|_{L^2_t \dot{H}^1(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|\langle x\rangle^{(1/2)+\epsilon} F\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}$$

which is true due to (3.19) and the KSS estimates. Now (3.18) is from interpolation between $s = 0$ and $s = 1$.

Lemma 3.2.3. Let w solve the wave equation (1.4) with $f = g = 0$. Then for $s \in [0,1]$ and $\epsilon > 0$,

$$\|w\|_{L^\infty_t \dot{H}^s_x} \lesssim \|\langle x\rangle^{1/2+\epsilon} F\|_{L^2_t \dot{H}^{s-1}_x}. \tag{3.20}$$

Again we use interpolation. When $s = 1$, note that $\|w\|_{\dot{H}^1} \sim \|P^{1/2} w\|_{L^2}$, we can get the corresponding estimate by the KSS estimate with $\mu > 1/2$ and TT^* argument. When $s = 0$, we use duality as in the proof of (3.18), then it is just the KSS estimates.

On the basis of the above two lemmas, we can control the commutator terms by a kind of weighted $L^2_t \dot{H}^{s-1}_x$ norm. Then with the following lemma we will be able to bound this norm by the good terms, thus we can use the argument as in [21] to get over the difficulty on error terms.

Lemma 3.2.4. Let $n \geqslant 3$, $N \geqslant 1$ and u be the solution to (1.4) with $F = 0$. Then for any $s \in [0, 1]$, $\epsilon > 0$ and $|\alpha| = N$, we have

$$\sum_{|\alpha|=N} \|\langle x\rangle^{-(1/2)-\epsilon}\partial_x^\alpha u\|_{L^2_t \dot{H}^{s-1}} \lesssim \|f\|_{\dot{H}^{N+s-1}\cap \dot{H}^s} + \|g\|_{\dot{H}^{N+s-2}\cap \dot{H}^{s-1}} . \tag{3.21}$$

The estimate for $s = 1$ follows directly from the KSS estimates (3.6). Moreover, we have the following estimate

$$\begin{aligned} \|\langle x\rangle^{-(1/2)-\epsilon} u\|_{L^2_t L^2_x} &= \|\langle x\rangle^{-(1/2)-\epsilon} P^{1/2}(P^{-1/2}u)\|_{L^2_t L^2_x} \\ &\lesssim \|P^{-1/2}f\|_{\dot{H}^1} + \|P^{-1/2}g\|_{L^2_x} \lesssim \|f\|_{L^2_x} + \|g\|_{\dot{H}^{-1}} . \end{aligned} \tag{3.22}$$

For $s = 0$, first notice that since $n \geqslant 3$, we have Hardy's inequality

$$\| \langle x\rangle^{-2} xh\|_{L^2_x} \lesssim \|h\|_{\dot{H}^1} ,$$

and the duality gives

$$\| \langle x\rangle^{-2} xf\|_{\dot{H}^{-1}} \lesssim \|f\|_{L^2_x} .$$

Using the above estimate together with the KSS estimates and (3.22), we get

$$\begin{aligned} \|\langle x\rangle^{-(1/2)-\epsilon}\partial_x^\alpha u\|_{L^2_t \dot{H}^{-1}} &\lesssim \|\langle x\rangle^{-(5/2)-\epsilon} x\partial_x^{\alpha-1} u\|_{L^2_t \dot{H}^{-1}} + \|\langle x\rangle^{-(1/2)-\epsilon}\partial_x^{\alpha-1} u\|_{L^2_t L^2_x} \\ &\lesssim \|\langle x\rangle^{-(1/2)-\epsilon}\partial_x^{\alpha-1} u\|_{L^2_t L^2_x} \\ &\lesssim \|f\|_{\dot{H}^{N-1}} + \|g\|_{\dot{H}^{N-2}\cap \dot{H}^{-1}} . \end{aligned}$$

Now (3.21) follows from an interpolation between $s = 0$ and $s = 1$.

The next two lemmas are to develop relation between P and ∂^2.

Lemma 3.2.5. For $0 < \mu \leqslant 3/2$ and $k \geqslant 2$, we have

$$\left\|\langle x\rangle^{-\mu}\widetilde{\partial}_{j_1}\cdots\widetilde{\partial}_{j_k}u\right\|_{L^2_x} \lesssim \sum_{j=0}^{[\frac{k-1}{2}]}\left\|\langle x\rangle^{-\mu}\widetilde{\partial}P^j u\right\|_{L^2_x} + \sum_{j=1}^{[\frac{k}{2}]}\left\|\langle x\rangle^{-\mu}P^j u\right\|_{L^2_x}, \tag{3.23}$$

where $[a]$ denotes the integer part of a ($\max\{k \in \mathbb{Z}, k \leqslant a\}$).

Lemma 3.2.6. For $f \in \dot{H}^s(\mathbb{R}^n) \cap \dot{H}^{s+2}(\mathbb{R}^n)$ with $n \geqslant 3$ and $s \in [0,1]$, we have

$$\|\partial_x^2 f\|_{\dot{H}^s} \lesssim \|Pf\|_{\dot{H}^s} + \|f\|_{\dot{H}^s}. \tag{3.24}$$

On the other hand,

$$\|Pf\|_{\dot{H}^s} \lesssim \sum_{|\alpha|\leqslant 2}\|\partial_x^\alpha f\|_{\dot{H}^s}. \tag{3.25}$$

The first lemma is just Lemma 4.8 in [2]. The second lemma is technical and will need the following fractional Lebniz rule.

Fractional Leibniz rule. Let $0 \leqslant s < n/2$, $2 \leqslant p_i < \infty$ and $1/2 = 1/p_i + 1/q_i$ ($i = 1, 2$). Then

$$\|fg\|_{\dot{H}^s} \lesssim \|f\|_{L^{q_1}}\|g\|_{\dot{H}^{s,p_1}} + \|f\|_{\dot{H}^{s,p_2}}\|g\|_{L^{q_2}}.$$

Moreover, for any $s \in (-n/2, 0) \cup (0, n/2)$,

$$\|fg\|_{\dot{H}^s} \lesssim \|f\|_{L^\infty \cap \dot{H}^{|s|,n/|s|}}\|g\|_{\dot{H}^s}.$$

The first inequality above is well known, see, e.g., [16]. The second inequality with $s \geqslant 0$ is an easy consequence of the first inequality together with Sobolev embedding. Then the result for negative s follows by duality. Now we present the proof of Lemma 3.2.6.

First, we give the proof for the estimate (3.25). When $s = 0$, notice that $Pf = \mathfrak{g}^{ij}\partial_i\partial_j f + r_1\partial_x f + r_2 f$, we have

$$\|Pf\|_{L^2_x} \lesssim \|\partial_x^2 f\|_{L^2_x} + \|\partial_x f\|_{L^2_x} + \|f\|_{L^2_x} \lesssim \|f\|_{\dot{H}^2 \cap L^2_x} .$$

When $s = 1$, recalling that $\partial^j r_i = O(\langle x\rangle^{-\rho-i-j})$, by Hardy's inequality,

$$\|\partial_x(r_2 f)\|_{L^2_x} \lesssim \|\partial_x(r_2) f\|_{L^2_x} + \|r_2\partial_x f\|_{L^2_x} \lesssim \|\partial_x f\|_{L^2_x}.$$

Thus

$$\begin{aligned} \|Pf\|_{\dot{H}^1_x} = \|\partial_x Pf\|_{L^2_x} &\leqslant \|\partial_x(\mathfrak{g}^{ij}\partial_i\partial_j f)\|_{L^2_x} + \|\partial_x(r_1\partial_x f)\|_{L^2_x} + \|\partial_x(r_2 f)\|_{L^2_x} \\ &\lesssim \|\partial_x^3 f\|_{L^2_x} + \|\partial_x f\|_{L^2_x} \\ &\lesssim \|f\|_{\dot{H}^3 \cap \dot{H}^1} . \end{aligned}$$

Our estimate (3.25) is obtained by an interpolation between the above two estimates on Pf.

Now we turn to the proof of the estimate (3.24). First, when $s = 0$, by elliptic property of P, we have

$$\|\partial_x^2 f\|_{L^2_x} \lesssim \|Pf\|_{L^2_x} + \|f\|_{L^2_x} . \tag{3.26}$$

Second, for $s = 1$, using (3.26),

$$\begin{aligned}
\|\partial_x^3 f\|_{L^2_x} &\lesssim \|P\partial_x f\|_{L^2_x} + \|\partial_x f\|_{L^2_x} \\
&\lesssim \|[P,\partial_x]f\|_{L^2_x} + \|\partial_x P f\|_{L^2_x} + \|\partial_x f\|_{L^2_x} \\
&\lesssim \|\sum_{|\alpha|\leqslant 2} r_{3-|\alpha|}\partial_x^\alpha f\|_{L^2_x} + \|Pf\|_{\dot H^1} + \|f\|_{\dot H^1} \\
&\lesssim \|Pf\|_{\dot H^1} + \|f\|_{\dot H^1} + \|f\|_{\dot H^2} \\
&\lesssim \|Pf\|_{\dot H^1} + \|f\|_{\dot H^1} + \epsilon\|f\|_{\dot H^3} + (1/\epsilon)\|f\|_{\dot H^1}, \quad \forall \epsilon > 0.
\end{aligned}$$

Here we have used Hardy's inequality and the fact that $\dot H^3 \cap \dot H^1 \subset \dot H^2$. Now if we choose $\epsilon > 0$ small enough and use (3.25) with $s = 0$, we have

$$\|\partial_x^2 f\|_{\dot H^1} \lesssim \|Pf\|_{\dot H^1} + \|f\|_{\dot H^1} \lesssim \|PP^{1/2}f\|_{L^2_x} + \|f\|_{\dot H^1} \lesssim \|P^{1/2}f\|_{\dot H^2} + \|P^{1/2}f\|_{L^2_x} \tag{3.27}$$

On the basis of (3.26) and (3.27), by an interpolation for the operator $\partial^2 P^{-1/2}$ and making use of Lemma 3.1.4, we have,

$$\begin{aligned}
\|\partial_x^2 f\|_{\dot H^s} &\lesssim \|P^{1/2}f\|_{\dot H^{1+s}} + \|P^{1/2}f\|_{\dot H^{s-1}} \\
&\lesssim \|P^{1/2}f\|_{\dot H^{1+s}} + \|P^{1/2+(s-1)/2}f\|_{L^2_x} \lesssim \|P^{1/2}f\|_{\dot H^{1+s}} + \|f\|_{\dot H^s}.
\end{aligned} \tag{3.28}$$

We need only to deal with the term $\|P^{1/2}f\|_{\dot H^{1+s}}$. Note that for $s \in [0,1]$, we have

$$\|P^{-1/2}v\|_{\dot H^{1+s}} \lesssim \|v\|_{\dot H^s} + \|v\|_{\dot H^{-s}},$$

which is true for $s = 0$ (see (3.1.4)) and $s = 1$ (see (3.26)). Recalling that $P - \mathfrak{g}^{ij}\partial_i\partial_j =$

$r_1\partial_x + r_2$, and by Sobolev embedding, we have,

$$\begin{aligned}
\|P^{1/2}f\|_{\dot{H}^{1+s}} &\lesssim \|Pf\|_{\dot{H}^s} + \|Pf\|_{\dot{H}^{-s}} \\
&\lesssim \|Pf\|_{\dot{H}^s} + \|f\|_{\dot{H}^{2-s}} + \|r_1\partial_x f\|_{\dot{H}^{-s}} + \|r_2 f\|_{\dot{H}^{-s}} \\
&\lesssim \|Pf\|_{\dot{H}^s} + \|f\|_{\dot{H}^{2-s}} + \|f\|_{\dot{H}^1} \\
&\lesssim \|Pf\|_{\dot{H}^s} + \|f\|_{\dot{H}^s}^{\theta_1}\|f\|_{\dot{H}^{2+s}}^{1-\theta_1} + \|f\|_{\dot{H}^s}^{\theta_2}\|f\|_{\dot{H}^{2+s}}^{1-\theta_2}, \quad \text{where } \theta_i \in (0,1]. \\
&\lesssim \|Pf\|_{\dot{H}^s} + \|f\|_{\dot{H}^s}^{\theta_1}\|\partial_x^2 f\|_{\dot{H}^s}^{1-\theta_1} + \|f\|_{\dot{H}^s}^{\theta_2}\|\partial_x^2 f\|_{\dot{H}^s}^{1-\theta_2}\ , \qquad (3.29)
\end{aligned}$$

where in the third inequality we used duality of Sobolev embedding and hölder's inequality, and in fourth inequality we used the fact that $s \leqslant 1 < 2+s$ and $s \leqslant 2-s < 2+s$(so that $\theta_i > 0$) for $s \in (0,1]$. Now our estimate (3.24) (for $s > 0$) follows from (3.28) and (3.29).

Now we have obtained all the ingredients needed to show Theorem 3.2.1. The proof consists of the following four estimates.

Proposition 3.2.7 (Generalized Morawetz estimates). Let $n \geqslant 3$, $s \in [0,1)$ and $\rho > 2$. Then for the solution u of the equation (1.4) with $F = 0$, we have

$$\sum_{|\alpha|\leqslant 2} \|Z^\alpha u\|_{L_t^2 Y_{s,\epsilon}} \lesssim \sum_{|\alpha|\leqslant 2} (\|Z^\alpha f\|_{\dot{H}^s} + \|Z^\alpha g\|_{\dot{H}^{s-1}}) \qquad (3.30)$$

Moreover, if we assume only $\rho > 1$ and $s \in [0,1]$, the estimate still holds with $|\alpha| \leqslant 1$.

Proof. Since $Z^\alpha = \partial_x$ is already proven in Section 3.1, we first check with $Z^\alpha = \Omega$. Recall that by the interpolation of (3.19) and the duality of (3.19), we have

$$\|u\|_{L_t^2 Y_{s,\epsilon}} \leqslant \|F\|_{L_t^2 Y'_{1-s,\epsilon}}, \qquad (3.31)$$

if u is a solution of (1.4) with vanishing initial data. Since $[P,\Omega]u = \sum_{|\alpha|\leqslant 2} r_{2-|\alpha|}\partial_x^\alpha u$, by using a combination of the estimate with order 0 and Lemma 3.2.2 for Ωu, we have

$$\begin{aligned}\|\Omega u\|_{L_t^2 Y_{s,\epsilon}} &\lesssim \|\Omega f\|_{\dot H^s} + \|\Omega g\|_{\dot H^{s-1}} \\ &\quad + \sum_{|\alpha|\leqslant 1} \|\langle x\rangle^{3/2-s+\epsilon} r_{2-|\alpha|}\partial_x^\alpha u\|_{L^2_{t,x}} + \|r_0\langle x\rangle^{1/2+\epsilon}\partial_x^2 u\|_{L_t^2\dot H^{s-1}} \end{aligned} \tag{3.32}$$

Now since $\rho > 1$, by (3.12) and Lemma 3.2.2,

$$\begin{aligned}\sum_{|\alpha|\leqslant 1} \|\langle x\rangle^{3/2-s+\epsilon} r_{2-|\alpha|}\partial_x^\alpha u\|_{L^2_{t,x}} &\lesssim \sum_{|\alpha|\leqslant 1} \|\langle x\rangle^{-1/2-s-\epsilon'}\partial_x^\alpha u\|_{L^2_{t,x}} \\ &\lesssim \sum_{|\alpha|\leqslant 1} \|\langle x\rangle^{-1/2-s-\epsilon'}\tilde\partial_x^\alpha u\|_{L^2_{t,x}} \\ &\lesssim \sum_{i\leqslant 1} \|\langle x\rangle^{-1/2-s-\epsilon'/2} P^{i/2} u\|_{L^2_{t,x}} \\ &\lesssim \sum_{i\leqslant 1} \left(\|P^{i/2} f\|_{\dot H^s} + \|P^{i/2} g\|_{\dot H^{s-1}}\right) \\ &\lesssim \sum_{|\alpha|\leqslant 1} \|\partial_x^\alpha f\|_{\dot H^s} + \|\partial_x^\alpha g\|_{\dot H^{s-1}} \end{aligned} \tag{3.33}$$

where in the last inequality we have used fractional Leibniz rule and Lemma 3.1.4.

Let $r(x) = r_0\langle x\rangle^{1/2+\epsilon} = O(\langle x\rangle^{-\rho+1/2+\epsilon})$. Then $r'(x) = O(\langle x\rangle^{-\rho-1/2+\epsilon})$. Since $n \geqslant 3$, by Hardy's inequality with duality, the KSS estimates (3.6) and interpolation,

$$\begin{aligned}\|r\partial_x^2 u\|_{L_t^2\dot H^{s-1}} &\leqslant \|\partial_x(r\partial_x u)\|_{L_t^2\dot H^{s-1}} + \|r'\partial_x u\|_{L_t^2\dot H^{s-1}} \\ &\lesssim \|r\partial_x u\|_{L_t^2\dot H^s} + \|\langle x\rangle\, r'\partial_x u\|_{L_t^2\dot H^s} \\ &\lesssim \sum_{|\alpha|\leqslant 1}\|\partial_x^\alpha f\|_{\dot H^s} + \sum_{|\alpha|\leqslant 1}\|\partial_x^\alpha g\|_{\dot H^{s-1}}\ . \end{aligned} \tag{3.34}$$

On the basis of (3.32), (3.33) and (3.34), we are done with $Z^\alpha = \Omega$. This completes the proof of the first order estimates under the condition $\rho > 1$.

For the second order part, we first consider the case $Z^\alpha = \partial_x^2$. Since $s \in [0, 1)$, we can always find $\epsilon > 0$ such that $1/2 + s + \epsilon \leqslant 3/2$. By Lemma 3.2.5, the proof for $Z^\alpha = \partial_x$, Lemma 3.1.4 and Lemma 3.2.6, we have

$$\begin{aligned}
\|\partial_x^2 u\|_{L_t^2 Y_{s,\epsilon}} &\lesssim \sum_{|\alpha|\leqslant 2} \|\tilde{\partial}_x^\alpha u\|_{L_t^2 Y_{s,\epsilon}} \\
&\lesssim \sum_{|\alpha|\leqslant 1} \|\tilde{\partial}^\alpha u\|_{L_t^2 Y_{s,\epsilon}} + \|Pu\|_{L_t^2 Y_{s,\epsilon}} \\
&\lesssim \sum_{|\alpha|\leqslant 1} (\|\partial_x^\alpha f\|_{\dot{H}^s} + \|\partial_x^\alpha g\|_{\dot{H}^{s-1}}) + \|Pf\|_{\dot{H}^s} + \|Pg\|_{\dot{H}^{s-1}} \\
&\approx \sum_{|\alpha|\leqslant 1} (\|\partial_x^\alpha f\|_{\dot{H}^s} + \|\partial_x^\alpha g\|_{\dot{H}^{s-1}}) + \|Pf\|_{\dot{H}^s} + \|P^{1/2} g\|_{\dot{H}^s} \\
&\lesssim \sum_{|\alpha|\leqslant 1} (\|\partial_x^\alpha f\|_{\dot{H}^s} + \|\partial_x^\alpha g\|_{\dot{H}^{s-1}}) + \sum_{|\alpha|\leqslant 2} \|\partial_x^\alpha f\|_{\dot{H}^s} + \sum_{|\alpha|\leqslant 1} \|\tilde{\partial}^\alpha g\|_{\dot{H}^s} \\
&\lesssim \sum_{|\alpha|\leqslant 2} (\|\partial_x^\alpha f\|_{\dot{H}^s} + \|\partial_x^\alpha g\|_{\dot{H}^{s-1}}) \ ,
\end{aligned}$$

where the fractional Leibniz rule is used in the last inequality. Next, we consider the case $Z^\alpha = \Omega^2$. Since $[P, \Omega^2]u = \sum_{|\alpha|\leqslant 3} \left(r_{2-|\alpha|} \partial_x^\alpha u\right)$, and $\Omega^2 u$ solves the wave equation with initial data $(\Omega^2 f, \Omega^2 g)$ and forcing term $[P, \Omega^2]u$, by (3.31), Lemma 3.2.2, Lemma 3.2.4

and the higher order estimates we have proved,

$$\begin{aligned}
\|\Omega^2 u\|_{L_t^2 Y_{s,\epsilon}} &\lesssim \|\Omega^2 f\|_{\dot H^s} + \|\Omega^2 g\|_{\dot H^{s-1}} \\
&\quad + \sum_{|\alpha|\leqslant 2} \|\langle x\rangle^{3/2-s+\epsilon} r_{2-|\alpha|}\partial_x^\alpha u\|_{L^2_{t,x}} + \sum_{|\alpha|=3} \|\langle x\rangle^{1/2+\epsilon} r_{2-|\alpha|}\partial_x^\alpha u\|_{L_t^2 \dot H^{s-1}} \\
&\lesssim \|\Omega^2 f\|_{\dot H^s} + \|\Omega^2 g\|_{\dot H^{s-1}} \\
&\quad + \sum_{|\alpha|\leqslant 2} \|\partial_x^\alpha u\|_{L_t^2 Y_{s,\epsilon}} + \sum_{|\alpha|=3} \|\langle x\rangle^{1/2+\epsilon} r_{-1}\partial_x^\alpha u\|_{L_t^2 \dot H^{s-1}} \\
&\lesssim \sum_{|\alpha|\leqslant 2} (\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}) + \sum_{|\alpha|=3} \|\langle x\rangle^{1/2+\epsilon} r_{-1}\partial_x^\alpha u\|_{L_t^2 \dot H^{s-1}} \\
&\lesssim \sum_{|\alpha|\leqslant 2} (\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}) \\
&\quad + \|\langle x\rangle^{1+2\epsilon} r_{-1}\|_{L^\infty\cap \dot W^{1,n}} \|\langle x\rangle^{-1/2-\epsilon}\partial_x^3 u\|_{L_t^2 \dot H^{s-1}} \\
&\lesssim \sum_{|\alpha|\leqslant 2} (\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}) + \sum_{|\alpha|\leqslant 2} (\|\partial_x^\alpha f\|_{\dot H^s} + \|\partial_x^\alpha g\|_{\dot H^{s-1}}) \\
&\lesssim \sum_{|\alpha|\leqslant 2} (\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}})
\end{aligned}$$

where we have used the fact that $\rho > 2$.

Since the commutator term $[P, \partial\Omega]u = [P, \Omega\partial]u = \sum_{|\alpha|\leqslant 3} \left(r_{3-|\alpha|}\partial_x^\alpha u\right)$ corresponds to an even better case than what for Ω^2, the proof proceeds in the same way. This completes the proof of the higher order estimates under the conditions $\rho > 2$ and $s \in [0, 1)$. ☐

Proposition 3.2.8 (Higher order energy estimates). Let $n \geqslant 3$, $s \in [0, 1]$ and $\rho > 2$. Then for the solution u of the equation (1.4) with $F = 0$, we have

$$\sum_{|\alpha|\leqslant 2} \|Z^\alpha u(t,x)\|_{L_t^\infty \dot H^s} \lesssim \sum_{|\alpha|\leqslant 2} (\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}) . \tag{3.35}$$

Moreover, if we assume only $\rho > 1$, the estimate still holds with $|\alpha| \leqslant 1$.

Proof. By Lemma 3.1.4 and elliptic regularity for P, we know

$$\|\partial_x u\|_{\dot{H}^1} \lesssim \|\partial_x^2 u\|_{L^2} \lesssim \|Pu\|_{L^2} + \|u\|_{L^2} \lesssim \|P^{1/2}u\|_{\dot{H}^1} + \|P^{1/2}u\|_{\dot{H}^{-1}}.$$

Interpolating this estimate with (3.1.4) with $s = 1$, $\|\partial_x u\|_{L^2} \simeq \|P^{1/2}u\|_{L^2}$, we get that for $s \in [0, 1]$,

$$\begin{aligned} \|\partial_x u\|_{\dot{H}^s} &\lesssim \|P^{1/2}u\|_{\dot{H}^s} + \|P^{1/2}u\|_{\dot{H}^{-s}} \\ &\lesssim \|P^{1/2}u\|_{\dot{H}^s} + \|u\|_{\dot{H}^{1-s}}. \end{aligned} \tag{3.36}$$

Thus by Lemma 3.1.4 we have for $s \in [0, 1/2]$ (such that $s \leqslant 1-s$ and $\dot{H}^s \cap \dot{H}^{1+s} \subset \dot{H}^{1-s}$),

$$\begin{aligned} \sum_{|\alpha|\leqslant 1} \|\partial_x^\alpha u\|_{L_t^\infty \dot{H}^s} &\lesssim \sum_{j\leqslant 1} \|P^{j/2}u\|_{L_t^\infty \dot{H}^s} + \|u\|_{L_t^\infty \dot{H}^{1-s}} \\ &\lesssim \sum_{|\alpha|\leqslant 1} (\|\partial_x^\alpha f\|_{\dot{H}^s} + \|\partial_x^\alpha g\|_{\dot{H}^{s-1}}) + \|f\|_{\dot{H}^{1-s}} + \|g\|_{\dot{H}^{-s}} \\ &\lesssim \sum_{|\alpha|\leqslant 1} (\|\partial_x^\alpha f\|_{\dot{H}^s} + \|\partial_x^\alpha g\|_{\dot{H}^{s-1}}) \ . \end{aligned} \tag{3.37}$$

Now we can deal with Ωu. Noticing that

$$\tilde{\Omega}_{ij} f = g^{-1}\Omega_{ij} f + (x_i \partial_j g^{-1} - x_j \partial_i g^{-1}) f \ ,$$

by the fractional Leibniz rule, we have

$$\|\tilde{\Omega} f\|_{\dot{H}^s} \lesssim \sum_{|\alpha|\leqslant 1} \|\Omega^\alpha f\|_{\dot{H}^s} \ , \ |s| < n/2 \ .$$

We have similar relationship between $\partial_x u$ and $\tilde{\partial}_x u$. By the Sobolev embedding, for any

$h \in L^n$, we have

$$\begin{aligned}
\|\langle x\rangle^{-1/2-\epsilon} hu\|_{\dot H^{s-1}} &\lesssim \|\langle x\rangle^{-1/2-\epsilon} hu\|_{L^{2n/(n+2(1-s))}} \\
&\lesssim \|h\|_{L^n}\|\langle x\rangle^{-1/2-\epsilon} u\|_{L^{2n/(n-2s)}} \\
&\lesssim \|\langle x\rangle^{-1/2-\epsilon} u\|_{\dot H^s}. \qquad (3.38)
\end{aligned}$$

Thus by the energy estimate, Lemma 3.2.3, fractional Leibniz rule and 3.2.4:

$$\begin{aligned}
\|\tilde\Omega u\|_{L^\infty_t \dot H^s} &\lesssim \|\tilde\Omega f\|_{\dot H^s} + \|\tilde\Omega g\|_{\dot H^{s-1}} + \|\langle x\rangle^{1/2+\epsilon}[P,\tilde\Omega]u\|_{L^2_t \dot H^{s-1}} \\
&\lesssim \|\tilde\Omega f\|_{\dot H^s} + \|\tilde\Omega g\|_{\dot H^{s-1}} + \sum_{1\leqslant|\alpha|\leqslant 2} \|r_{2-|\alpha|}\langle x\rangle^{1/2+\epsilon}\tilde\partial_x^\alpha u\|_{L^2_t\dot H^{s-1}} \\
&\lesssim \sum_{|\alpha|\leqslant 1}\left(\|\Omega^\alpha f\|_{\dot H^s} + \|\Omega^\alpha g\|_{\dot H^{s-1}}\right) \\
&\quad + \sum_{1\leqslant|\alpha|\leqslant 2} \|r_{2-|\alpha|}\langle x\rangle^{1+2\epsilon}\|_{L^\infty\cap \dot H^{1-s,n/(1-s)}}\|\langle x\rangle^{-1/2-\epsilon}\tilde\partial_x^\alpha u\|_{L^2_t\dot H^{s-1}} \\
&\lesssim \sum_{|\alpha|\leqslant 1}\left(\|\Omega^\alpha f\|_{\dot H^s} + \|\Omega^\alpha g\|_{\dot H^{s-1}}\right) + \sum_{1\leqslant|\alpha|\leqslant 2}\|\langle x\rangle^{-1/2-\epsilon}\partial_x^\alpha u\|_{L^2_t\dot H^{s-1}} \\
&\quad + \|\langle x\rangle^{-1/2-\epsilon}(\partial g^{-1})u\|_{L^2_t\dot H^{s-1}} + \|\langle x\rangle^{-1/2-\epsilon}[\partial(g^{-1}\partial g^{-1})]u\|_{L^2_t\dot H^{s-1}} \\
&\lesssim \sum_{|\alpha|\leqslant 1}\left(\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}\right)\ ,
\end{aligned}$$

where we have used the fact that $\rho > 1$ and (3.38) with $h = \partial g^{-1}$, ∂g^{-2} and $h = \partial(g^{-1}\partial g^{-1})$ (the condition $h \in L^n$ is satisfied since the condition (H1) on the metric g). Noticing that $\Omega u = g\tilde\Omega u - g(\Omega g^{-1})u$, we hence have

$$\|\Omega u\|_{L^\infty_t\dot H^s} \lesssim \sum_{|\alpha|\leqslant 1}\|\tilde\Omega^\alpha u\|_{L^\infty_t\dot H^s} \lesssim \sum_{|\alpha|\leqslant 1}\left(\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}\right)\ . \qquad (3.39)$$

On the basis of (3.37) and (3.39), we complete the proof of the energy estimates of order one, under the conditions $s \in [0, 1/2]$ and $\rho > 1$.

For the part with second order derivatives, we need only to deal with ∂_x^2 and Ω^2 as before.

By Lemma 3.2.6 and Lemma 3.1.4, we have

$$\begin{aligned}\|\partial_x^2 u\|_{L_t^\infty \dot H^s} &\lesssim \|Pu\|_{L_t^\infty \dot H^s} + \|u\|_{L_t^\infty \dot H^s}\\ &\lesssim \|Pf\|_{\dot H^s} + \|Pg\|_{\dot H^{s-1}} + \|f\|_{\dot H^s} + \|g\|_{\dot H^{s-1}}\\ &\lesssim \sum_{|\alpha|\leqslant 2} \left(\|\partial_x^\alpha f\|_{\dot H^s} + \|\partial_x^\alpha g\|_{\dot H^{s-1}}\right) \ . \end{aligned} \tag{3.40}$$

Here we remark that we can control $\sum_{|\alpha|=1}\|\partial_x^\alpha u\|_{L_t^\infty \dot H^s}$ for $s\in[0,1]$ instead of the restriction $s\in[0,1/2]$ in (3.37), by (3.40) and the estimate with order 0, which enables us to relax the condition to $s\in[0,1]$ in the estimates of order one.

By Lemma 3.2.3, Lemma 3.2.4, and what we have gained in previous steps, if $\rho>2$,

$$\begin{aligned}\|\Omega^2 u\|_{L_t^\infty \dot H^s} &\lesssim \sum_{|\alpha|\leqslant 2} \|\tilde\Omega^\alpha u\|_{L_t^\infty \dot H^s}\\ &\lesssim \sum_{|\alpha|\leqslant 2}\left(\|\tilde\Omega^2 f\|_{\dot H^s} + \|\tilde\Omega^2 g\|_{\dot H^{s-1}}\right) + \sum_{1\leqslant|\alpha|\leqslant 3} \|r_{2-|\alpha|}\langle x\rangle^{1/2+\epsilon}\tilde\partial_x^\alpha u\|_{L_t^2 \dot H^{s-1}}\\ &\lesssim \sum_{|\alpha|\leqslant 2}\left(\|\tilde\Omega^2 f\|_{\dot H^s} + \|\tilde\Omega^2 g\|_{\dot H^{s-1}}\right) + \sum_{1\leqslant|\alpha|\leqslant 3} \|\langle x\rangle^{-1/2-\epsilon}\partial_x^\alpha u\|_{L_t^2 \dot H^{s-1}}\\ &\lesssim \sum_{|\alpha|\leqslant 2}\left(\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}\right) \ . \end{aligned} \tag{3.41}$$

We are done with the second order estimates based on (3.40) and (3.41). □

Proposition 3.2.9 (Sobolev inequality with angular smoothing). Let u be a solution of (1.4) with $F=0$ and $n\geqslant 3$. Then for any $s\in(1/2,1]$ and $\rho>1$, there exists a suitable $\eta>0$ (determined by (3.9) and interpolation) so that we have:

$$\sum_{|\alpha|\leqslant 1} \||x|^{n/2-s} Z^\alpha u(t,x)\|_{L^\infty_{t,|x|} L^{2+\eta}_\omega} \lesssim \sum_{|\alpha|\leqslant 1}\left(\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}\right). \tag{3.42}$$

Furthermore, if we assume $\rho > 2$, then we have

$$\sum_{|\alpha|\leqslant 2} \||x|^{n/2-s} Z^\alpha u(t,x)\|_{L^\infty_{t,|x|}L^{2+\eta}_\omega} \lesssim \sum_{|\alpha|\leqslant 2} \left(\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}\right). \tag{3.43}$$

Proof. This is a direct consequence of the energy estimates Proposition 3.2.8 and the inequality (3.9). ☐

Proposition 3.2.10 (Local energy estimates). Assume $n \geqslant 3$, let $s \in [0,1]$, $p \geqslant 2$, $k = 0, 1, 2$, $\rho > k$ and u be a solution of (1.4) with $F = 0$. We have

$$\sum_{|\alpha|\leqslant k} \|\phi Z^\alpha u\|_{L^p_t \dot H^s} \lesssim \sum_{|\alpha|\leqslant k} \left(\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}\right) \ , \tag{3.44}$$

where $\phi \in C_0^\infty(\mathbb{R}^n)$.

Proof. For the higher order estimates with $|\alpha| = k \geqslant 1$, by the higher order KSS estimates (3.6),

$$\begin{aligned}
\|\phi Z^\alpha u\|_{L^2_t \dot H^1} &\lesssim \|\phi\, \partial_x Z^\alpha u\|_{L^2_{t,x}} + \|\phi'\, Z^\alpha u\|_{L^2_{t,x}} \\
&\lesssim \|\langle x\rangle^{-1/2-\epsilon} \partial_x Z^\alpha u\|_{L^2_{t,x}} + \|\langle x\rangle^{-3/2-\epsilon} Z^\alpha u\|_{L^2_{t,x}} \\
&\lesssim \sum_{|\alpha|\leqslant k} \left(\|Z^\alpha f\|_{\dot H^1} + \|Z^\alpha g\|_{L^2}\right).
\end{aligned}$$

For $s = 0$, note that $\phi\ \Omega = r_0 \partial_x$,

$$\begin{aligned}
\|\phi Z^\alpha u\|_{L^2_{t,x}} &\lesssim \|\langle x\rangle^{-1/2-\epsilon} \partial_x Z^{\alpha-1} u\|_{L^2_{t,x}} \\
&\lesssim \sum_{|\alpha|\leqslant k-1} \left(\|Z^\alpha f\|_{\dot H^1} + \|Z^\alpha g\|_{L^2}\right) \\
&\lesssim \sum_{|\alpha|\leqslant k} \left(\|Z^\alpha f\|_{L^2} + \|Z^\alpha g\|_{\dot H^{-1}}\right) \ .
\end{aligned}$$

By interpolation between the above two estimates, we get (3.44) with $p = 2$. This will complete the proof if we combine it with the energy estimates in Proposition 3.2.8. ☐

With the above four propositions, we have proved the required higher order estimates (3.3) and (3.4) as in Section 3.1. ☐

3.3 Local in time Strichartz Estimates

In this section we provide the finite time weighted Strichartz estimates, which will be used to give a life span for the almost global solution when $p < p_c$ and $n = 3$.

Theorem 3.3.1. *Let u be the solution of* (1.4) *with $F = 0$. Assume that* (H1) *and* (H2) *hold with $\rho > 2$, $n \geqslant 3$, $0 < a < 1/p$, $2 \leqslant p < \infty$ and $s = s_d$. Then we have*

$$\text{(3.45)} \quad \sum_{|\alpha|\leqslant 2} \|\langle x\rangle^{-a}|x|^{(n-1)s} Z^\alpha u\|_{L_t^p L_{|x|}^p L_\omega^2([0,T]\times\mathbb{R}^n)} \lesssim (1+T)^{(1/p)-a+\epsilon} \sum_{|\alpha|\leqslant 2} \left(\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}\right).$$

We will adapt adapt the arguments in [21] to obtain the above estimates, which rely on the local energy decay (3.44) and the KSS estimates for wave equations with variable coefficients. We first present the corresponding KSS estimates for pertubed wave equations (see Theorem 2.1 of [7] or Theorem 5.1 in [17]).

Proposition 3.3.2. Assume that (H1) and (H2) hold with $\rho > 1$. Let $N \geqslant 0$, $0 < \mu < 1/2$. Then the solution of (1.4) satisfies

$$\text{(3.46)} \quad \sum_{|\alpha|\leqslant N} (1+T)^{\mu-1/2} \Big\|\langle x\rangle^{-\mu}\left(|(\Gamma^\alpha u)'| + \frac{|\Gamma^\alpha u|}{\langle x\rangle}\right)\Big\|_{L_T^2 L_x^2} \lesssim \sum_{|\alpha|\leqslant N} \big\|(Z^\alpha u)'(0,\cdot)\big\|_{L_x^2} + \sum_{|\alpha|\leqslant N} \big\|\Gamma^\alpha F(s,\cdot)\big\|_{L_T^1 L_x^2},$$

where $L_T^q L_x^r = L^q([0,T]; L^r(\mathbb{R}^n))$.

As a consequence of this KSS estimate, similarly to the previous proof of Proposition 3.2.7, we can have the following estimates.

Corollary 3.3.3. Assume that (H1) and (H2) hold with $\rho > 2$. Let $0 < \mu \leqslant 1/2$ and

$$A_\mu(T) = \begin{cases} (\log(2+T))^{-1/2} & \mu = 1/2, \\ (1+T)^{\mu-1/2} & 0 < \mu < 1/2. \end{cases}$$

We have

$$\|\langle x\rangle^{-\mu} e^{itP^{1/2}} f\|_{L^2_T L^2_x} \lesssim A_\mu(T)^{-1}\|f\|_{L^2}. \tag{3.47}$$

Moreover, if $0 < \mu < 1/2$, for the solution u of the equation (1.4) with $F = 0$, we have

$$\sum_{|\alpha|\leqslant 2} \|\langle x\rangle^{-\mu} Z^\alpha u\|_{L^2_T L^2_x} \lesssim T^{1/2-\mu+\epsilon} \sum_{|\alpha|\leqslant 2} \left(\|Z^\alpha f\|_{L^2} + \|Z^\alpha g\|_{\dot H^{-1}}\right). \tag{3.48}$$

And, if we assume $\rho > 1$ instead of $\rho > 2$, we have the same estimates of first order ($|\alpha| \leqslant 1$).

Proof. (3.47) is a direct consequence if we employ (3.46) with $\alpha = 0$ for $u' = \partial_t u$. To obtain (3.48), we basically follow the argument as in Proposition 3.2.7 with some modifications. For the second order part, we first consider the case $Z^\alpha = \partial_x^2$. We claim that we have the following inequality

$$\|\langle x\rangle^{-\mu}\partial_x u\|_{L^2_x} \leqslant \epsilon\|\langle x\rangle^{-\mu}\partial_x^2 u\|_{L^2_x} + C(\epsilon)\|\langle x\rangle^{-\mu} u\|_{L^2_x}\ . \tag{3.49}$$

By Lemma 3.2.5, Lemma 3.1.4 and Lemma 3.2.6, we have

$$\begin{aligned}
A_\mu(T)\|\langle x\rangle^{-\mu}\partial_x^2 u\|_{L^2_T L^2_x} &\lesssim A_\mu(T)\sum_{|\alpha|\leqslant 2}\|\langle x\rangle^{-\mu}\tilde{\partial}_x^\alpha u\|_{L^2_T L^2_x}\\
&\lesssim A_\mu(T)\sum_{|\alpha|\leqslant 1}\|\langle x\rangle^{-\mu}\tilde{\partial}_x^\alpha u\|_{L^2_T L^2_x}+A_\mu(T)\|\langle x\rangle^{-\mu}Pu\|_{L^2_T L^2_x}\\
&\lesssim A_\mu(T)\sum_{|\alpha|\leqslant 1}\|\langle x\rangle^{-\mu}\partial_x^\alpha u\|_{L^2_T L^2_x}+A_\mu(T)\|\langle x\rangle^{-\mu}Pu\|_{L^2_T L^2_x}\\
&\lesssim \epsilon A_\mu(T)\|\langle x\rangle^{-\mu}\partial_x^2 u\|_{L^2_T L^2_x}+C(\epsilon)A_\mu(T)\|\langle x\rangle^{-\mu}u\|_{L^2_T L^2_x}\\
&\quad +A_\mu(T)\|\langle x\rangle^{-\mu}Pu\|_{L^2_T L^2_x}\\
&\lesssim \epsilon A_\mu(T)\|\langle x\rangle^{-\mu}\partial_x^2 u\|_{L^2_T L^2_x}\\
&\quad +C(\epsilon)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^{-1}}\right)+\|Pf\|_{L^2}+\|Pg\|_{\dot{H}^{-1}},
\end{aligned}$$

where we have used (3.47) and (3.49). Hence we have

$$\begin{aligned}
A_\mu(T)\|\langle x\rangle^{-\mu}\partial_x^2 u\|_{L^2_T L^2_x} &\lesssim \|f\|_{L^2}+\|g\|_{\dot{H}^{-1}}+\|Pf\|_{L^2}+\|Pg\|_{\dot{H}^{-1}}\\
&\lesssim \|f\|_{L^2}+\|g\|_{\dot{H}^{-1}}+\|Pf\|_{L^2}+\|P^{1/2}g\|_{L^2}\\
&\lesssim \|g\|_{\dot{H}^{-1}}+\sum_{|\alpha|\leqslant 2}\|\partial_x^\alpha f\|_{L^2}+\|\tilde{\partial} g\|_{L^2}\\
&\lesssim \sum_{|\alpha|\leqslant 2}\left(\|\partial_x^\alpha f\|_{L^2}+\|\partial_x^\alpha g\|_{\dot{H}^{-1}}\right)\ .
\end{aligned}$$

Now we are left with the norm for $Z=\Omega,\Omega^2$, but from the proof of Proposition 3.2.7, we know it suffices to prove the following estimates

$$\|\langle x\rangle^{-\mu}w\|_{L^2_{t,x}([0,T]\times\mathbb{R}^n)}\lesssim T^{1/2-\mu+\epsilon}\|\langle x\rangle^{1/2+\epsilon}F\|_{L^2_t\dot{H}^{-1}([0,T]\times\mathbb{R}^n)}, \tag{3.50}$$

if w is the solution of (1.4) with vanishing initial data. Recall that we have proved in

Lemma 3.2.2 that

$$\|\langle x\rangle^{-1/2-\epsilon}w\|_{L^2_{t,x}([0,T]\times\mathbb{R}^n)}\lesssim\|\langle x\rangle^{1/2+\epsilon}F\|_{L^2_t\dot{H}^{-1}([0,T]\times\mathbb{R}^n)}. \tag{3.51}$$

Also if we restrict the time t in $[0,T]$, it is easy to verify that Lemma 3.2.3 still holds, i.e.

$$\|w\|_{L^2_tL^2_x([0,T]\times\mathbb{R}^n)}\lesssim T^{1/2}\|w\|_{L^\infty_tL^2_x([0,T]\times\mathbb{R}^n)}\lesssim T^{1/2}\|\langle x\rangle^{1/2+\epsilon}F\|_{L^2_t\dot{H}^{-1}([0,T]\times\mathbb{R}^n)}. \tag{3.52}$$

Now (3.50) just follows from the interpolation between (3.51) and (3.52). To conclude the proof of (3.48), it remains to prove the claim (3.49).

Proof of (3.49)**.** This inequality is true for $\mu=0$. For general $\mu\geqslant 0$, we apply the estimate for $\mu=0$ to $v=\phi u$ with $\phi=\psi(x/R),\psi\in C^\infty,0\leqslant\psi\leqslant 1,\mathrm{supp}\psi\subset\{1/4<|x|<2\}$, $\psi=1$ in $B_1\backslash B_{1/2}$ and $R\geqslant 1$. Because of $\{x:\phi(x)=1\}\subset\{|x|>R/4\}$ and $\mathrm{supp}\phi\subset\{R/4<|x|<2R\}$, we get

$$\begin{aligned}\|\langle x\rangle^{-\mu}\partial_x u\|_{L^2(\{x:\phi(x)=1\})} &= \|\langle x\rangle^{-\mu}\partial_x(\phi u)\|_{L^2(\{x:\phi(x)=1\})}\\ &\leqslant CR^{-\mu}\|\partial_x(\phi u)\|_{L^2(\mathbb{R}^n)}\\ &\leqslant CR^{-\mu}(\epsilon\|\partial_x^2(\phi u)\|_{L^2(\mathbb{R}^n)}+C(\epsilon)\|\phi u\|_{L^2(\mathbb{R}^n)})\\ &\leqslant C(\epsilon\|\langle x\rangle^{-\mu}\partial_x^2(\phi u)\|_{L^2(\mathbb{R}^n)}+C(\epsilon)\|\langle x\rangle^{-\mu}\phi u\|_{L^2(\mathbb{R}^n)})\\ &\leqslant C\left(\epsilon\|\langle x\rangle^{-\mu}\partial_x^2 u\|_{L^2(\mathrm{supp}\phi)}+C\epsilon R^{-1}\|\langle x\rangle^{-\mu}\partial_x u\|_{L^2(\mathrm{supp}\phi')}+\right.\\ &\qquad\left.(C(\epsilon)+C\epsilon R^{-2})\|\langle x\rangle^{-\mu}u\|_{L^2(\mathrm{supp}\phi)}\right)\ .\end{aligned}$$

If we choose instead $\psi=1$ in B_1 and 0 for $|x|\geqslant 2$, then

$$\begin{aligned}\|\langle x\rangle^{-\mu}\partial_x u\|_{L^2(\{x:|x|\leqslant 1\})}\leqslant C\epsilon\|\langle x\rangle^{-\mu}\partial_x^2 u\|_{L^2(\{x:|x|\leqslant 2\})}+&\\ C\epsilon\|\langle x\rangle^{-\mu}\partial_x u\|_{L^2(\{x:|x|\leqslant 2\})}+(C(\epsilon)+C\epsilon)\|\langle x\rangle^{-\mu}u\|_{L^2(\{x:|x|\leqslant 2\})}&\ .\end{aligned}$$

Combining the above two inequalities, we see

$$\|\langle x\rangle^{-\mu}\partial_x u\|_{L^2(\mathbb{R}^n)} \leqslant C\epsilon\|\langle x\rangle^{-\mu}\partial_x^2 u\|_{L^2(\mathbb{R}^n)} + C\epsilon\|\langle x\rangle^{-\mu-1}\partial_x u\|_{L^2(\mathbb{R}^n)} + C(C(\epsilon)+\epsilon)\|\langle x\rangle^{-\mu}u\|_{L^2(\mathbb{R}^n)} ,$$

which implies (3.49), by choosing small enough $\epsilon > 0$. □

The next estimate is based on the endpoint trace lemma.

Proposition 3.3.4. Let $\dot{B}^s_{pq}$ denote the homogeneous Besov space. Then we have

$$\||x|^{(n-1)/2}e^{itP^{1/2}}f\|_{L^\infty_t L^\infty_r L^2_\omega} \lesssim \|f\|_{\dot{B}^{1/2}_{2,1}} . \tag{3.53}$$

Proof. Recall that we have the endpoint Trace lemma (see (1.7) in [3]):

$$r^{(n-1)/2}\|f(r\cdot)\|_{L^2_\omega} \lesssim \|f\|_{\dot{B}^{1/2}_{2,1}} , \tag{3.54}$$

which gives that

$$\||x|^{(n-1)/2}e^{itP^{1/2}}f\|_{L^\infty_r L^2_\omega} \lesssim \|e^{itP^{1/2}}f\|_{\dot{B}^{1/2}_{2,1}} . \tag{3.55}$$

On the other hand, by Lemma 3.1.4 we have

$$\|e^{itP^{1/2}}f\|_{\dot{H}^1} \lesssim \|P^{1/2}e^{itP^{1/2}}f\|_{L^2_x} \lesssim \|P^{1/2}f\|_{L^2_x} \lesssim \|f\|_{\dot{H}^1}.$$

Noticing that $\|f\|_{\dot{B}^s_{2,2}} = \|f\|_{\dot{H}^s}$, we can rewrite the above estimate as

$$\|e^{itP^{1/2}}f\|_{\dot{B}^1_{2,2}} \lesssim \|f\|_{\dot{B}^1_{2,2}} .$$

Interpolating this estimate with the energy estimate

$$\|e^{itP^{1/2}}f\|_{\dot{B}^0_{2,2}}\lesssim\|f\|_{\dot{B}^0_{2,2}}$$

gives

$$\|e^{itP^{1/2}}f\|_{\dot{B}^{1/2}_{2,1}} = \|e^{itP^{1/2}}f\|_{(\dot{B}^1_{2,2},\dot{B}^0_{2,2})_{1/2,1}}\lesssim\|f\|_{(\dot{B}^1_{2,2},\dot{B}^0_{2,2})_{1/2,1}} = \|f\|_{\dot{B}^{1/2}_{2,1}}\ , \tag{3.56}$$

where we have used the fact that (Theorem 6.4.5 in [1])

$$(\dot{B}^{s_0}_{pq_0}, \dot{B}^{s_1}_{pq_1})_{\theta,r} = \dot{B}^{s^*}_{pr},\ \text{if } s_0 \neq s_1,\ 0<\theta<1,\ r, q_0, q_1 \geqslant 1 \text{ and } s^* = (1-\theta)s_0 + \theta s_1.$$

Now our estimate (3.53) follows from (3.55) and (3.56). □

Now we are ready to obtain the local in time Strichartz estimates as follows.

Proposition 3.3.5. Let $2 \leqslant p < \infty$ and $a \in (0, 1/p)$. Then we have

$$\|\langle x\rangle^{-a}|x|^{(n-1)(1/2-1/p)}e^{itP^{1/2}}f\|_{L^p_T L^p_r L^2_\omega}\lesssim(1+T)^{1/p-a}\|f\|_{\dot{H}^{1/2-1/p}}. \tag{3.57}$$

Proof. This estimate follows from the real interpolation between (3.47) and (3.53) with $\theta = 2/p$ (for similar arguments, see, e.g., [5], [27]). □

Finally we give the proof of Theorem 3.3.1.

Proof of Theorem 3.3.1: Since the estimates in Theorem 3.3.1 with order 0 are just obtained in Proposition 3.3.5, we are left with the higher order estimates. Similarly to the proof of Proposition 3.3.5, we need only to show the higher order estimates that correspond to (3.47) and (3.53).

The higher order estimates corresponding to (3.47) are known from Corollary 3.3.3.

For the higher order estimates of (3.53), by (3.54) we have

$$\sum_{|\alpha|\leqslant 2} \||x|^{(n-1)/2} Z^\alpha u(t,\cdot)\|_{L^\infty_r L^2_\omega} \lesssim \sum_{|\alpha|\leqslant 2} \|Z^\alpha u(t,\cdot)\|_{\dot B^{1/2}_{2,1}} \ . \tag{3.58}$$

On the other hand, from the energy estimates in Proposition 3.2.8, we have for any $s \in [0,1]$

$$\sum_{|\alpha|\leqslant 2} \|Z^\alpha u(t,\cdot)\|_{\dot H^s} \lesssim \sum_{|\alpha|\leqslant 2} \left(\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}}\right) \ .$$

Now the real interpolation between the above two estimates with $s = 0$ and $s = 1$ gives

$$\sum_{|\alpha|\leqslant 2} \|Z^\alpha u(t,\cdot)\|_{\dot B^{1/2}_{2,1}} \lesssim \sum_{|\alpha|\leqslant 2} \left(\|Z^\alpha f\|_{\dot B^{1/2}_{2,1}} + \|Z^\alpha g\|_{\dot B^{-1/2}_{2,1}}\right) \ .$$

Combining this estimate with (3.58), we get the second order estimates of (3.53), which completes the proof of Theorem 3.3.1 for $\rho > 2$. ☐

3.4 Strauss Conjecture when $p < p_c, \rho > 2, n = 3$

With the estimates proved in Section 3.1 and Section 3.2, we can use arguments as in [6] to get the existence results stated in Main Theorem 2.2.2. For the reader's convenience, we provide the proof when $n = 3$ and $p < 1 + \sqrt{2}$ here.

Define $s = s_d = 1/2 - 1/p$, and a be the number such that

$$p\left[(n-1)(1/2 - 1/p) - a\right] = 1 - s - n/2 \ ,$$

i.e., $a = -1/p^2 - (n-1)/(2p) + (n-1)/2$. Since $2 \leqslant p < p_c$, we have $a \in (0, 1/p)$. By the estimates (3.4), (3.45) and Duhamel's principle, we have for $T \geqslant 1$

$$\begin{aligned}\sum_{|\alpha|\leqslant 2} &\left(\||x|^{(n-1)(1/2-1/p)-a} Z^\alpha u\|_{L^p_t L^p_r L^2_\omega([0,T]\times\{|x|>1\})} + \|Z^\alpha u\|_{L^p_t L^{q_s}_x([0,T]\times\{|x|<1\})}\right) \\ &\lesssim T^{1/p-a+\epsilon} \sum_{|\alpha|\leqslant 2} \left(\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}} + \|Z^\alpha F\|_{L^1_t \dot H^{s-1}}\right) \\ &\lesssim T^{1/p-a+\epsilon} \sum_{|\alpha|\leqslant 2} \left(\|Z^\alpha f\|_{\dot H^s} + \|Z^\alpha g\|_{\dot H^{s-1}} + \|Z^\alpha F\|_{L^1_t X'_{1-s,0,\infty}}\right). \end{aligned} \tag{3.59}$$

Now if we set

$$\begin{aligned} M_k = &\sum_{|\alpha|\leqslant 2} \left(\|Z^\alpha u^{(k)}\|_{L^\infty_t \dot H^s} + \|\partial_t Z^\alpha u^{(k)}\|_{L^\infty_t \dot H^{s-1}}\right) \\ &+ T^{a-1/p-\epsilon} \sum_{|\alpha|\leqslant 2} \left(\||x|^{(-1/2-s)/p} Z^\alpha u\|_{L^p_t L^p_r L^2_\omega([0,T]\times\{|x|>1\})} + \|Z^\alpha u\|_{L^p_t L^{q_s}_x([0,T]\times\{|x|<1\})}\right), \end{aligned} \tag{3.60}$$

then on the basis of (3.4) and (3.59), we can use the iteration method (with $\eta = 0$) as in Section 4.1 to get the existence result for $2 \leqslant p < p_c$ and $\rho > 2$ in Theorem 2.2.2.

Heuristically, the life span is given when we have

$$M_k \sim \left(T_\delta^{1/p-a+\epsilon} M_k\right)^p \sim \delta \ ,$$

which yields that

$$T_\delta \sim \delta^{(p(p-1))/(p^2-2p-1)+\epsilon'}, \ \forall \epsilon' > 0 \ .$$

Bibliography

[1] J. Bergh and J. Löfström, Interpolation spaces. An introduction Grundlehren der Mathematischen Wissenschaften, No. **223**. Springer-Verlag, Berlin-New York, 1976. x+207 pp.

[2] J.-F. Bony and D. Häfner, *The semilinear wave equation on asymptotically Euclidean manifolds*, Comm. Partial Differential Equations **35** (2010), no. 1 , 23–67.

[3] D. Fang and C. Wang *Weighted Strichartz Estimates with Angular Regularity and their Applications*, Forum Math. **23** (2011), no. 1, 181-205.

[4] V. Georgiev, H. Lindblad and C. D. Sogge, *Weighted Strichartz estimates and global existence for semilinear wave equations*, Amer. J. Math. **1**19 (1997), no. 6, 1291–1319.

[5] K. Hidano, *Morawetz-Strichartz estimates for spherically symmetric solutions to wave equations and applications to semilinear Cauchy problems*, Differential Integral Equations **20** (2007), no. 7, 735–754.

[6] K.Hidano, J.Metcalfe, H.Smith, C.Sogge and Y.Zhou, *On Abstract Strichartz Estimates and the Strauss Conjecture for Nontrapping Obstacles*, Trans. Amer. Math. Soc., **362** (2010), 2789–2809.

[7] K. Hidano, C. Wang and K. Yokoyama, *On almost global existence and local well-posedness for some 3-D quasi-linear wave equations*, arXiv: 1004.3349.

[8] M. Ikawa, *Decay of solutions of the wave equation in the exterior of two convex bodies*, Osaka J.Math.,**19** (1982), 459-509.

[9] M. Ikawa, *Decay of solutions of the wave equation in the exterior of several convex bodies*, Ann. Inst. Fourier(Grenoble),**38** (1998), 113–146.

[10] J.-C. Jiang, C. Wang and X. Yu, *Generalized and Weighted Strichartz Estimates*, arXiv: 1008.5397.

[11] F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math. **28** (1979), 235–265.

[12] H. Lindblad and C. D. Sogge, *Long-time existence for small amplitude semilinear wave equations*, Amer. J. Math. **118** (1996), no. 5, 1047–1135.

[13] M. Keel, H. Smith and C. D. Sogge, *Almost global existence for some semilinear wave equations*, Dedicated to the memory of Thomas H. Wolff. J. Anal. Math. **87** (2002), 265–279.

[14] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), 955-980.

[15] M. Keel, H. F. Smith, and C. D. Sogge, *Almost global existence for some semilinear wave equations*, J. Anal. Math. **87** (2002), 265-279.

[16] M. Kon and A. Gulisashvili, *Exact Smoothing Properties of Schrodinger Semigroups*, Amer. J. Math. **118** (1996), no. 6, 1215–1248.

[17] J. Metcalfe and C. Sogge, *Long-time existence of quasilinear wave equations exterior to star-shaped obstacles via energy methods*, SIAM J. Math. Anal. **38** (2006), no. 1, pp.188–209.

[18] T. C. Sideris, *Nonexistence of global solutions to semilinear wave equations in high dimensions*, J. Differential Equations, **52** (1984), 378–406.

[19] H. F. Smith and C. D. Sogge, *Global Strichartz estimates for nontrapping perturbations of the Laplacian*, Comm. Partial Differential Equations **25** (2000), 2171–2183.

[20] C. D. Sogge, Lectures on nonlinear wave equations, 2nd edition, International Press, Boston, MA, 2008.

[21] C. D. Sogge, C. Wang, *Concerning the wave equation on asymptotically Euclidean manifolds*, J. Anal. Math., **112** (2010), no. 1, 1–32.

[22] W. A. Strauss, *Nonlinear scattering theory at low energy: sequel*, J. Funct. Anal. **43** (1981), no. 3, 281-293.

[23] D. Tataru, *Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation*, Trans. Amer. Math. Soc. **353** (2001), no. 2, 795–807.

[24] H. Triebel, Interpolation theory, function spaces, differential operators, North-Holland Mathematical Library, **18**. North-Holland Publishing Co., Amsterdam-New York (1978), 528 pp.

[25] B. Yordanov and Q. S. Zhang, *Finite time blow up for critical wave equations in high dimensions*, J. Funct. Anal., **231** (2006),361–374.

[26] C. Wang and X. Yu *Concerning the Strauss Conjecture on Asymptotically Euclidean Manifolds*, DOI 10.1016/j.jmaa.2011.01.053 , J.Math.Anal.Appl, to appear.

[27] X. Yu, *Generalized Strichartz Estimates on Perturbed Wave Equation and Applications on Strauss Conjecture*, arXiv:0905.0038, Differential Integral Equations, to appear.

[28] Y. Zhou, *Blow up of classical solutions to* $\Box u = |u|^{1+\alpha}$ *in three space dimensions*, J. Partial Differential Equations **5**, (1992), 21–32.

[29] Y. Zhou, *Blow up of solutions to semilinear wave equations with critical exponent in high dimensions*, Chin. Ann. Math. Ser. B **28** (2007), 205–212.

[30] Y. Zhou and W. Han, *Blow up of Solutions to Semilinear Wave Equations with variable coefficients and boundary*, J. Math. Anal. Appl. **374** (2011), 585–601.

Vitae

Xin Yu was born on November 25, 1984 in Chongqing, China. She received her Bachelor of Science in Mathematics and Applied Mathematics from University of Science and Technology of China in June 2006. After college, she was accepted into a Doctoral Program at Johns Hopkins University in the fall of 2006. Her dissertation was completed under the guidance of Dr. Christopher Sogge and this dissertation was defended on 03/15/2011.