Existence theorems for some nonlinear hyperbolic equations on a waveguide

by

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ABSTRACT

In this thesis, we prove existence theorems for nonlinear wave and Klein-Gordon equations with small initial data and quadratic nonlinearities in infinite homogeneous waveguides. We are able to show that solutions exist globally on waveguides with certain Robin boundary conditions and almost globally with Neumann boundary conditions. We need to assume a natural nonlinear condition on the quasilinear nonlinearity in order for standard energy estimates to hold.

Readers: Dr. Christopher Sogge (Advisor) and Dr. William Minicozzi

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Contents

1 Introduction

The purpose of this paper is to prove existence results for nonlinear wave and Klein-Gordon equations in infinite homogeneous waveguides with Robin and Neumann boundary conditions. The term waveguide refers to $\mathbb{R}^n \times \Omega$, where $\Omega \subset \mathbb{R}^d$ is a nonempty, bounded domain with smooth boundary. In the Robin case, we are able to show global existence of solutions to both wave and Klein-Gordon equations for dimensions $n \geq 3$ as long as the boundary condition itself meets certain criteria and the nonlinearity satisfies a necessary condition. In the Neumann boundary condition case, which is a collaborative result with Jason Metcalfe, we prove almost global existence of solutions for the wave equation in dimension $n = 3$. Both results are extensions of work done by Metcalfe, Sogge, and Stewart in [19], where the authors were able to show global existence of solutions in the Dirichlet case for both wave and Klein-Gordon equations and in the Neumann case for Klein-Gordon equations. In addition, almost global existence of solutions was shown for certain semilinear Neumann wave equations when $n = 3$, as well as global existence when $n \geq 4$.

1.1 Initial-Boundary Value Problems

Let us now describe the initial-boundary value problems that we will examine in more detail. We first consider equations on $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \Omega$ of the form

$$\begin{cases} (\Box + m^2)u = Q(u, u', u''), \\ u(0, x, y) = f(x, y), \quad \partial_t u(0, x, y) = g(x, y), \end{cases} \tag{1.1}$$

where $m \geq 0$ and where u satisfies the Robin boundary condition

$$(\alpha(y)u + \partial_\nu u)|_{\partial\Omega} = 0, \quad \text{with } \alpha(y) \geq 0 \text{ and } \int_{\partial\Omega} \alpha > 0. \tag{1.2}$$

Additionally, we require that $\alpha \in C^\infty(\partial\Omega)$. Here ∂_ν denotes the normal derivative on $\partial\Omega$, and $\Box = \partial_t^2 - (\Delta + \Delta_\Omega)$ is the d'Alembertian on $\mathbb{R}_+ \times \mathbb{R}^n \times \Omega$. Also,

$$\Delta = \Delta_{\mathbb{R}^n} = \sum_{j=1}^{n} \partial^2/\partial x_j^2$$

is the standard Laplacian on $\mathbb{R}^n$, and

$$\Delta_\Omega = \sum_{j=1}^{d} \partial^2/\partial y_j^2$$

denotes the Laplacian on $\Omega \subset \mathbb{R}^d$. For convenience, we will often use the notation $x_0 = t$, and $x_{n+j} = y_j$, $1 \leq j \leq d$.

Q is quadratic in its arguments, which are u, $u' = \nabla u$, and $u'' = \nabla^2 u$, where $\nabla = (\partial_t, \partial_x, \partial_y)$ is the full space-time gradient. We shall assume that our nonlinear equation is quasilinear, i.e. affine linear in u''. More specifically, we assume that Q can be expanded as

$$Q(u, u', u'') = \sum_{0 \leq j,k,l \leq n+d} A_l^{jk} \partial_l u \partial_j \partial_k u + u \sum_{0 \leq j,k \leq n+d} A^{jk} \partial_j \partial_k u + R(u, u') \tag{1.3}$$

where the A_l^{jk} and A^{jk} are real constants and R is a constant coefficient, quadratic form in u and u'. Since the existence of solutions depends on the energy method, we need the coefficients to be symmetric, i.e. $A_l^{jk} = A_l^{kj}$ and $A^{jk} = A^{kj}$. Finally, we must assume that the quasilinear terms satisfy a nonlinear compatibility condition which will be described in detail in the next subsection. This condition is necessary in order for energy estimates to hold for wave and Klein-Gordon equations with Robin boundary conditions.

We also examine wave equations on $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \Omega$ of the form

$$\begin{cases} \Box u = \bar{Q}(\partial_{t,x} u, \partial_{t,x}^2 u), \\ u(0, x, y) = f(x, y), \quad \partial_t u(0, x, y) = g(x, y), \end{cases} \tag{1.4}$$

where here u satisfies the Neumann boundary condition

$$\partial_\nu u(t, x, \,\cdot\,)|_{\partial\Omega} = 0. \tag{1.5}$$

We may expand $\bar{Q}$ similarly to (1.3) as

$$\bar{Q}(\partial_{t,x} u, \partial_{t,x}^2 u) = \sum_{0 \leq j,k,l \leq 3} A_l^{jk} \partial_l u \partial_j \partial_k u + R(\partial_{t,x} u, \partial_{t,x} u) \tag{1.6}$$

where again the A_l^{jk} are real constants satisfying the symmetry condition and R is a constant coefficient, quadratic form.

There are three important differences between the nonlinearities $\bar{Q}$ and Q that we are able to control in these two cases. First, note that our nonlinearity in the Neumann case does not depend on u. This is due to the fact that we do not have access to Poincaré's lemma, which we used in [19] in the Dirichlet case to give $\|u(t,\,\cdot)\|_{L^2(\Omega)} \leq C\|\nabla_y u(t,\,\cdot)\|_{L^2(\Omega)}$. In the Robin case, we are able to instead prove a similar bound using the Rayleigh Quotient; however this is not possible for the Neumann domain, where we have as a counterexample the constant functions. Also, for technical reasons which will be clear at a later point in the paper, we must restrict the derivatives in our nonlinearity $\bar{Q}$ to ∂_t and ∂_x. Finally, we note that our proof for the Neumann case is currently only valid when $n = 3$.

When Ω is a bounded interval (i.e. $d = 1$), however, we are in fact able to handle terms in our nonlinearity that depend on $\partial_y u$. This is because in the Neumann case, $\partial_\nu u = \partial_y u$ solves a Dirichlet wave equation, and so better estimates are available to control these terms than in the higher dimensional cases ($d > 1$). Thus we consider

$$\begin{cases} \Box u = \tilde{Q}(\partial u, \partial^2 u), \quad (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^3 \times [a, b], \\ u(0, x, y) = f(x, y), \quad \partial_t u(0, x, y) = g(x, y), \end{cases} \tag{1.7}$$

where u satisfies the Neumann boundary condition at the endpoints

$$\partial_y u(t, x, a) = 0, \quad \partial_y u(t, x, b) = 0, \tag{1.8}$$

and where $\partial = \partial_{t,x,y}$ denotes the full space-time gradient. Our nonlinearity is expanded as

$$\tilde{Q}(\partial u, \partial^2 u) = \sum_{0 \leq j,k,l \leq 4} B_l^{jk} \partial_l u \partial_j \partial_k u + \tilde{R}(\partial u, \partial u), \tag{1.9}$$

where the B_l^{jk} are real constants which satisfy the usual symmetry condition and $\tilde{R}$ is again a constant coefficient quadratic form. Note that we set $x_4 = y$ and $\partial_4 = \partial_y$. In order for energy estimates to hold for this problem, we must also assume a nonlinear compatibility condition as in the Robin case. This condition, which will be described fully in the next subsection, will automatically hold when the quasilinear terms only involve $\partial_j \partial_k u$, $0 \leq j, k \leq 3$, and so it is unnecessary to state the assumption explicitly when studying (1.4).

For simplicity, we shall assume that the initial data have compact support and are sufficiently small in the appropriate Sobolev norm. Specifically, we shall assume that there is a fixed constant $B > 0$ so that

$$f(x,y) = g(x,y) = 0, \quad |x| > B, \tag{1.10}$$

and

$$\|f\|_{H^N(\mathbb{R}^n\times\Omega)} + \|g\|_{H^{N-1}(\mathbb{R}^n\times\Omega)} \le \varepsilon, \tag{1.11}$$

where

$$\|f\|_{H^N(\mathbb{R}^n\times\Omega)} = \sum_{|\alpha|\le N} \|\partial^\alpha_{x,y} f\|_{L^2(\mathbb{R}^n\times\Omega)}.$$

Finally, to solve both (1.1) and (1.4) we must assume that the data satisfies certain compatibility conditions. Since these are well known (see [7]), we shall only give a brief description. Let $J_k u = \{\partial_x^\alpha u : 0 \le |\alpha| \le k\}$ denote the collection of all spatial derivatives of u of order up to k, using local coordinates in a small tubular neighborhood of $\partial\Omega$. Then for the Robin case, if N is fixed and if u is a formal H^N solution of (1.1) we can write $\partial_t^k(\alpha u(0,\cdot) + \partial_\nu u(0,\cdot)) = \psi_k(J_{k+1}f, J_k g)$, $0 \le k \le N$, for certain compatibility functions ψ_k which depend on the nonlinear term Q as well as $J_{k+1}f$ and $J_k g$. Thus, the compatibility condition for the Robin case (1.1), (1.2) with $(f,g) \in H^N \times H^{N-1}$ is just the requirement that the ψ_k vanish on $\mathbb{R}^n \times \partial\Omega$ when $0 \le k \le N-2$. If this condition holds for all N then we say that $(f,g) \in C^\infty$ satisfy the compatibility conditions to infinite order. For the Neumann case, if u is a formal H^N solution of (1.4) for some fixed N, then we can write $\partial_t^k \partial_\nu u(0,\,\cdot\,) = \Psi_k(J_{k+1}f, J_k g)$ where the Ψ_k depend on Q, $J_{k+1}f$ and $J_k g$. The compatibility condition for the Neumann case (1.4),(1.5) with $(f,g) \in H^N \times H^{N-1}$ simply requires that the Ψ_k vanish on $\mathbb{R}^3 \times \partial\Omega$ when $0 \le k \le N-2$. Again, $(f,g) \in C^\infty$ satisfy the compatibility conditions to infinite order if this condition holds for all N.

1.2 Nonlinear Compatibility Conditions

Energy arguments for the Robin boundary condition are similar to the Neumann condition studied in [19] in one important aspect. In both cases, it is necessary to put a natural condition on the quasilinear quadratic terms so that energy estimates can hold. This condition is similar in spirit to the symmetry condition for multispeed nonlinear hyperbolic systems (see, e.g., [9], [18], [22]). The issue in both the Neumann and Robin cases is a problematic boundary integral term involving the nonlinearity, which by contrast automatically vanishes in the Dirichlet case. The condition for the Robin waveguide is a generalized version of the condition on the quasilinear quadratic forms which was introduced in [19] for the Neumann case. In fact, it is this condition that we will impose upon our nonlinearity (1.9).

The natural nonlinear Neumann condition is that

$$\sum_{0\le j,k,l\le n+d} A_l^{jk}\xi_l\eta_j\theta_k = 0 \quad \text{if } (\theta,\xi,\eta)\in X, \tag{1.12}$$

where

$$X = \{\,(\theta,\xi,\eta) :\, \theta=(0,\dots,0,\nu_1(y),\dots,\nu_d(y)),\, \xi\cdot\theta=0,\, \eta\cdot\theta=0,\, y\in\partial\Omega\}.$$

Thus, we assume that θ is normal to $\mathbb{R}^{1+n}\times\partial\Omega$ and that ξ and η are both orthogonal to θ. This condition automatically holds if the quasilinear terms only involve $\partial_j\partial_k u$, $0\le j,k\le n$, i.e., $A_l^{jk}=0$ and $A^{jk}=0$ if $n+1\le j\le n+d$ or $n+1\le k\le n+d$. For this reason, we do not need to state the condition explicitly for (1.6). Also, since the quasilinear terms can also be viewed as a perturbation of the metric, we can interpret this condition geometrically as a further requirement on the metric.

For the Robin condition, we can state our condition similarly. First, however, note that (1.12) would actually work for the Robin condition as long as our quasilinear term is independent of ∇_y. This is because the vector $(0,...,0,\partial_{y_1}u,...\partial_{y_d}u)$ is no longer orthogonal to θ, as it is in the Neumann case. But, in order to avoid this restriction on our nonlinear term, we must generalize (1.12), and thus the natural nonlinear Robin condition is

$$
\begin{cases} \alpha^2 \sum\limits_{0\le j,k,l\le n+d} A_l^{jk}\theta_l\theta_j\theta_k = 0 \\ \alpha \sum\limits_{0\le j,k,l\le n+d} \big(A_l^{jk}\theta_l\eta_j\theta_k + A_l^{jk}\xi_l\theta_j\theta_k\big) = 0 \\ \sum\limits_{0\le j,k,l\le n+d} A_l^{jk}\xi_l\eta_j\theta_k = 0 \end{cases} \quad \text{and} \quad \begin{cases} \alpha \sum\limits_{0\le j,k\le n+d} A^{jk}\theta_j\theta_k = 0 \\ \sum\limits_{0\le j,k\le n+d} A^{jk}\xi_j\theta_k = 0 \end{cases} \tag{1.13}
$$

if $(\theta, \xi, \eta) \in X$, where

$$
X = \{\, (\theta, \xi, \eta) : \, \theta = (0, \dots, 0, \nu_1(y), \dots, \nu_d(y)),\, \xi \cdot \theta = 0,\, \eta \cdot \theta = 0,\, y \in \partial\Omega\}.
$$

As in the Neumann case, θ is assumed to be normal to $\mathbb{R}^{1+n} \times \partial\Omega$, and ξ and η are both assumed to be orthogonal to θ. Both conditions are derived by noting that ξ and η can be written in terms of their normal and tangential components relative to the boundary. The normal components can then be simplified here using the Robin boundary condition (1.2), hence the appearance of α in the first three terms. Note that we could use this version for the Neumann condition as well, in which case all but the last term would vanish to give us (1.12).

1.3 Results

The main results of this paper are the following three existence theorems. First, in the case of Robin boundary conditions (1.2) we always have small amplitude global existence for Klein-Gordon or wave equations with quadratic nonlinearities when $n \ge 3$:

Theorem 1.1. *Suppose that $m \ge 0$ in* (1.1)*, and assume that $n \ge 3$. Assume also that the Cauchy data $(f, g) \in C^\infty(\mathbb{R}^n \times \Omega)$ satisfies* (1.10) *and* (1.11) *as well as the appropriate infinite order compatibility conditions for the Robin boundary conditions* (1.2)*. Then, if Q satisfies the nonlinear Robin condition* (1.13)*, it follows that the corresponding nonlinear Klein-Gordon equation* (1.1), (1.2) *has a global smooth solution if N in* (1.11) *is a sufficiently large fixed integer and if $0 < \varepsilon < \varepsilon_0$ is sufficiently small.*

In the case of Neumann boundary conditions (1.5), we have small amplitude almost global existence for wave equations with quadratic nonlinearities when $n = 3$:

Theorem 1.2. *Assume that the Cauchy data* $(f, g) \in C^\infty(\mathbb{R}^3 \times \Omega)$ *satisfy* (1.10) *and* (1.11) *as well as the compatibility conditions to infinite order. Then there are constants* N, κ *and* $\varepsilon_0 > 0$ *so that if* $\varepsilon < \varepsilon_0$ *and* N *is sufficiently large in* (1.11)*, then* (1.4)*,* (1.5) *has a unique solution* $u \in C^\infty([0, T_\varepsilon) \times \mathbb{R}^3 \times \Omega)$ *where*

$$T_\varepsilon = \exp(\kappa/\varepsilon). \tag{1.14}$$

This lifespan (1.14) is sharp due to well-known counterexamples of John (see [5, 6]). John was able to show that solutions of $\Box u = (\partial_t u)^2$ have lifespan (1.14) in Minkowski space. Thus if we begin with initial data independent of y, then solutions of (1.4) on the waveguide are equivalent to solutions to $\Box u = Q(\partial_{t,x} u, \partial^2_{t,x} u)$ in Minkowski space.

When $d = 1$, we can obtain the following result for a more general nonlinearity:

Theorem 1.3. *Assume that the data* $(f, g) \in C^\infty(\mathbb{R}^3 \times [a, b])$ *satisfy* (1.10) *and* (1.11) *as well as the compatibility conditions to infinite order. Moreover, assume that* $\tilde{Q}$ *satisfies the nonlinear Neumann condition* (1.12)*. Then, there are constants* N, κ, *and* ε_0 *so that if* $\varepsilon < \varepsilon_0$ *and* N *is sufficiently large in* (1.11)*, then* (1.7),(1.8) *has a unique solution* $u \in C^\infty([0, T_\varepsilon) \times \mathbb{R}^3 \times [a, b])$.

1.4 Historical Background

This work expands on results by Lesky and Racke [14] for $n \geq 5$, which were then improved by Metcalfe, Sogge and Stewart [19] for $n \geq 3$. The key step in [14] was to use an eigenvalue expansion for Ω to convert a system of wave equations to Klein-Gordon equations. Their techniques involved proving estimates for solutions of Klein-Gordon equations $(\Box_{\mathbb{R}^{1+n}} + m^2)u = 0$ in $\mathbb{R} \times \mathbb{R}^n$ and largely relied on $L^p \to L^{p/(p-1)}$ decay estimates from Marshall, Strauss and Wainger [15] as well as arguments from Shibata and Tsutsumi [21]. By then using an eigenfunction expansion for the bounded part of the waveguide, Ω, they were able to obtain estimates on $\mathbb{R} \times (\mathbb{R}^n \times \Omega)$. This is helpful because one can exploit the $t^{-n/2}$ decay enjoyed by solutions to the

Klein-Gordon equation, rather than the usual $t^{-(n-1)/2}$ decay of solutions to the wave equation. This stronger decay is exactly what is needed to prove global existence when $n = 3$.

Using this idea in [19], global existence of solutions for both Klein-Gordon and the wave equation was proven for Dirichlet boundary conditions with the quasilinear term depending on u, u', and u'', but rather limited results were proven for the Neumann condition. In this case, the existence results are the same as for Dirichlet as long as $m > 0$ in the Klein-Gordon equation, and as long as the quasilinear terms satisfy a natural nonlinear Neumann condition. However, when $m = 0$, it was only possible to prove almost global results for $n = 3$, and global for $n \geq 4$, and only for semilinear terms that are quadratic in $\partial_x u$.

In [19], we also used the eigenfunction expansion method, but to establish bounds on $\mathbb{R} \times \mathbb{R}^n$ we used techniques related to recent work on nonlinear obstacle problems by Keel, Smith, and Sogge (see [7], [8], [9]). These include $L^\infty - L^2$ Sobolev-type inequalities and Klainerman's method of commuting vector fields [11]. This method relies on the favorable commutation properties of the Poincaré group with the d'Alembertian. However, unlike in obstacle problems or in Minkowski space, we were unable to use the scaling vector field $L = t\partial_t + r\partial_r$, since it doesn't preserve the equation $(\partial_t^2 - \triangle_{\mathbb{R}^n} + \mu^2)u = 0$ for $\mu \neq 0$. We instead used the generator of hyperbolic rotations, $\Omega_{0j} = x_j\partial_t + t\partial_j$ for $1 \leq j \leq n$. However, one drawback of using these vector fields is that it would then be difficult to extend our result to systems with multiple wave speeds, since in this case, each of these vector fields has an associated wave speed.

The additional difficulties in dealing with the Neumann wave equation are due to the fact that the Neumann Laplacian has a zero eigenmode. Because of this, the optimal $t^{-3/2}$ decay estimates for the Klein-Gordon equation are not available for controlling this zero eigenmode, and thus we were unable to use the eigenvalue expansion method in [19]. We were instead able to use KSS estimates from [8] to prove the almost global existence result for the semilinear case. However, these weighted mixed-norm estimates are even less favorable for $\partial_t u$, thus making necessary the restriction to the space gradient in the semilinear term.

The work in this thesis was inspired by unsuccessful attempts to extend the global Neumann condition results for the Klein-Gordon equation in [19] to the wave equation. During this study, it was noted that in the Robin case, certain restrictions can be placed on $\alpha(y)$ in order to ensure that there are no negative or zero eigenmodes. In this way, the estimates that exploit a partial eigenfunction projection will follow similarly to the Dirichlet case. Thus, it is possible to prove global existence.

The Neumann almost global existence results in this paper improve significantly on our semilinear result in [19], both because we can handle more general terms, and because we are able to use the eigenfunction expansion technique. The main innovation is an estimate for the zero eigenmode which is essentially the same type of estimate that was proven in [19] for the nonzero eigenmodes. In this case, the main building blocks are $L^\infty - L^2$ Sobolev-type inequalities, from Klainerman and Sideris (see [13],[22]) and Hidano [2], as well as estimates involving weighted norms. However, some modifications to these estimates and their proofs were necessary as we were again required to use the hyperbolic rotations rather than the scaling vector field. As in the previous results, we also use Klainerman's commuting vector field method. We will use the following notation for our full set of vector fields:

$$\{\tilde{\Gamma}\} = \{\Gamma\} \cup \{\partial_y\} = \{\partial_t, \partial_x, \Omega_{jk}, \partial_y \,:\, 0 \le j < k \le n\}, \tag{1.15}$$

where

$$\{\Gamma\} = \{\partial_t, \partial_x, \Omega_{jk} \,:\, 0 \le j < k \le n\}$$

with the spatial rotations

$$\Omega_{ij} = x_i \partial_j - x_j \partial_i, \quad 1 \le i < j \le n$$

and the hyperbolic rotations

$$\Omega_{0k} = x_k \partial_t + t\partial_k, \quad 1 \le k \le n.$$

We will also use

$$Z = \{\partial_t, \partial_x, \Omega_{jk} \,:\, 1 \le j < k \le n\}$$

when we wish to indicate that we have omitted the hyperbolic rotations.

2 Properties of Δ_Ω

2.1 Spectral Theory

To prove our main decay results we need to utilize a few basic facts from both the spectral theory and the elliptic regularity theory of Δ_Ω. This material was also presented in [19] and [20] for the Neumann case. For complete proofs of these results, we refer the reader to texts by Taylor and by Gilbarg and Trudinger ([25] and [1]). Recall that Δ_Ω denotes either the the Neumann Laplacian where

$$\partial_\nu u|_{\partial\Omega} = 0 \tag{2.1}$$

or the Robin Laplacian where we have

$$(\alpha(y)u + \partial_\nu u)|_{\partial\Omega} = 0. \tag{2.2}$$

We use ∂_ν to denote the outward normal derivative on the boundary of Ω and of course we have additional assumptions on $\alpha(y)$ that we will discuss in detail.

Since we are assuming that Ω is compact with smooth boundary, it is well-known that the spectrum of $-\Delta_\Omega$ is discrete. Furthermore, in the Neumann case the spectrum is nonnegative. If we let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ denote the eigenvalues counted with respect to multiplicity, then

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \quad \text{for the Neumann Laplacian.}$$

It is the presence of this zero eigenvalue when $j = 1$ that distinguishes the Neumann case from the Dirichlet case examined in [19].

The situation is a bit more complicated for the Robin boundary condition, since it is possible for the spectrum to include both negative and zero eigenvalues. But, if we require that $\alpha(y) \geq 0$ where $\int_{\partial\Omega} \alpha > 0$, then

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \quad \text{for the Robin Laplacian.}$$

This can be shown using Green's formula:

$$\int_\Omega (-\Delta u)u\,dy = \int_\Omega |\nabla u|^2\,dy - \int_{\partial\Omega} u\partial_\nu u\,d\sigma$$

and so we have

$$\begin{aligned}\int_\Omega \lambda u^2\,dy &= \int_\Omega |\nabla u|^2\,dy - \int_{\partial\Omega} u\partial_\nu u\,d\sigma\\ &= \int_\Omega |\nabla u|^2\,dy + \int_{\partial\Omega} \alpha(y)u^2\,d\sigma.\end{aligned}$$

Of course for an arbitrary function $\alpha(y)$, it is possible for the Robin problem to have a negative eigenvalue. If, however, we require $\alpha(y) \geq 0$, then we are guaranteed that the eigenvalues will be non-negative. We can also eliminate the possibility of a zero eigenvalue as long as $\int_{\partial\Omega}\alpha > 0$ using an argument by contradiction. If zero were an eigenvalue, then both integrals on the right-hand side would have to vanish. This would mean that u was a constant by the first integral, which would then mean that $\int_{\partial\Omega}\alpha = 0$.

In both cases, we let $E_j : L^2(\Omega) \to L^2(\Omega)$ denote the projection onto the jth eigenspace. For $h \in L^2(\Omega)$, we have that E_jh is smooth, we have the usual definition of eigenfunction

$$-\Delta_\Omega E_jh(x) = \lambda_j E_jh(x), \tag{2.3}$$

and Plancherel's theorem holds

$$\|h\|^2_{L^2(\Omega)} = \sum_{j=1}^\infty \|E_jh\|_2^2. \tag{2.4}$$

By Weyl's formula we have $\lambda_j \approx j^{2/d}$, $j = 2, 3, \dots$, and so if $h \in C^\infty(\overline{\Omega})$, we have

$$\begin{aligned}(1+j)^{2/d}\|E_jh\|_{L^2(\Omega)} &\leq C\|(I-\Delta_\Omega)E_jh\|_{L^2(\Omega)}\\ &= C\|E_j(I-\Delta_\Omega)h\|_{L^2(\Omega)}, \quad j = 1, 2, 3, \dots, \end{aligned}\tag{2.5}$$

assuming that either (2.1) or (2.2) holds.

We also require a simple lemma which demonstrates the important interaction between $(\Box + m^2)$ and the eigenfunction projection.

Proposition 2.1. *Let Δ_Ω denote the Neumann or Robin Laplacian on Ω. Then, if $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n \times \Omega)$ satisfies the relevant boundary condition* (2.1) *or* (2.2)*, it follows that*

$$E_j(\Box + m^2)u(t,x,y) = (\partial_t^2 - \Delta_{\mathbb{R}^n} + m^2 + \lambda_j)E_ju(t,x,y),$$

and so

$$\sum_j \|(\partial_t^2 - \Delta_{\mathbb{R}^n} + m^2 + \lambda_j)E_j u(t,x,\,\cdot\,)\|^2_{L^2(\Omega)} = \|(\Box + m^2)u(t,x,\,\cdot\,)\|^2_{L^2(\Omega)}.$$

Proof of Proposition 2.1 To prove this, it first suffices to look at the commutator of Δ_Ω with the eigenfunction projection. It is clear that $\partial_t^2 - \Delta_{\mathbb{R}^n}$ will commute with the eigenfunction projection, since it is only the partial projection onto the bounded domain. Let e_j be the jth eigenfunction. If we use integration by parts twice and the fact that both u and $\overline{e_j}$ satisfy the boundary conditions, we have for the Neumann case,

$$\begin{aligned}
E_j(-\Delta_\Omega u) &= \Big(\int_\Omega -\Delta_\Omega u\overline{e_j}\,dy\Big)e_j \\
&= \Big(-\int_{\partial\Omega} \partial_\nu u\overline{e_j}d\sigma + \int_\Omega \sum_k \partial_k u\overline{\partial_k e_j}\,dy\Big)e_j \\
&= \Big(\int_{\partial\Omega} u\overline{\partial_\nu e_j}d\sigma - \int_\Omega u\overline{\Delta_\Omega e_j}\,dy\Big)e_j \\
&= \Big(\int_\Omega u\lambda\overline{e_j}\,dy\Big)e_j \\
&= \Big(\int_\Omega u\overline{e_j}\,dy\Big)\lambda e_j \\
&= \Big(\int_\Omega u\overline{e_j}\,dy\Big)(-\Delta_\Omega e_j) \\
&= -\Delta_\Omega\Big(\int_\Omega u\overline{e_j}\,dy\Big)e_j \\
&= -\Delta_\Omega(E_j u).
\end{aligned}$$

The boundary integrals vanish due to the boundary condition. In the Robin case,

$$\begin{aligned}
E_j(-\Delta_\Omega u) &= \Big(\int_\Omega -\Delta_\Omega u\overline{e_j}\,dy\Big)e_j \\
&= \Big(-\int_{\partial\Omega} \partial_\nu u\overline{e_j}d\sigma + \int_\Omega \sum_k \partial_k u\overline{\partial_k e_j}\,dy\Big)e_j \\
&= \Big(-\int_{\partial\Omega} \partial_\nu u\overline{e_j}d\sigma + \int_{\partial\Omega} u\overline{\partial_\nu e_j}d\sigma - \int_\Omega u\overline{\Delta_\Omega e_j}\,dy\Big)e_j \\
&= \Big(\int_{\partial\Omega} \alpha u\overline{e_j}d\sigma + \int_{\partial\Omega} u(-\alpha\overline{e_j})d\sigma - \int_\Omega u\overline{\Delta_\Omega e_j}\,dy\Big)e_j \\
&= \Big(\int_\Omega u\lambda\overline{e_j}\,dy\Big)e_j \\
&= -\Delta_\Omega(E_j u).
\end{aligned}$$

where we use the boundary condition to allow the boundary integrals to cancel. For both cases, the equality of the L^2 norms follows immediately from (2.3) and (2.4).

2.2 Elliptic Regularity

We first present the definition of a Sobolev space on a bounded domain. We suppose that $\bar{\Omega} = \Omega \cup \partial\Omega$ is embedded as a submanifold of a compact manifold M, which is without boundary and of the same dimension d. $H^k(\Omega)$ is then the space of restrictions to Ω of elements in $H^k(M)$. A complete treatment can be found in Taylor, [25], p. 286, or Hörmander, [3], vol. 3, p. 478.

We will need the following standard elliptic regularity estimate:

Lemma 2.2. *Suppose that $h \in C^\infty(\overline{\Omega})$ and that we have the boundary condition* (2.1) *or* (2.2)*. Then for $N = 2, 3, \ldots$ there is a constant $C = C_{N,\Omega}$ so that*

$$\sum_{|\alpha| \le N} \|\partial_y^\alpha h\|_{L^2(\Omega)} \le C \sum_{|\alpha| \le N-2} \|\partial_y^\alpha \Delta_\Omega h\|_{L^2(\Omega)} + C\|h\|_{L^2(\Omega)}.$$

A thorough proof of the Neumann case can be found in Taylor [25], p. 349.

The Robin case is a result of the following estimate for the nonhomogeneous Neumann case, also from [25].

Lemma 2.3. *For $k = 0, 1, 2, \ldots$, given $f \in H^k(\Omega)$, $g \in H^{k+1/2}(\partial\Omega)$, there is a unique solution $u \in H^{k+2}(\Omega)$ to*

$$(-\Delta + 1)u = f \ \in \Omega, \quad \frac{\partial u}{\partial \nu} = g \text{ on } \partial\Omega$$

and

$$\|u\|^2_{H^{k+2}(\Omega)} \le C\|\Delta u\|^2_{H^k(\Omega)} + C\left\|\frac{\partial u}{\partial \nu}\right\|^2_{H^{k+1/2}(\partial\Omega)} + C\|u\|^2_{H^{k+1}(\Omega)}. \tag{2.6}$$

This is easily extended to the Robin case as long as $\alpha \in C^\infty(\partial\Omega)$. To eliminate the boundary term, one simply uses the boundary condition along with the trace theorem (see, e.g. Taylor [25], p. 287) to get that

$$\begin{aligned}\left\|\frac{\partial u}{\partial \nu}\right\|^2_{H^{k+1/2}(\partial\Omega)} &= \|\alpha u\|^2_{H^{k+1/2}(\partial\Omega)} \\ &\le \|\alpha u\|^2_{H^{k+1}(\Omega)} \\ &\le C\|u\|^2_{H^{k+1}(\Omega)}.\end{aligned}$$

Using our elliptic regularity result we can obtain the following

Lemma 2.4. *Suppose that* $u(t,x,y) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n \times \Omega)$ *and that either* (2.1) *or* (2.2) *holds for all* $y \in \partial\Omega$. *Then if* $m \geq 0$ *and* $N = 2,4,6,\dots$ *are fixed*

$$\sum_{|\alpha|\leq N} \|\partial_y^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)} \leq C \sum_{|\alpha|\leq N} \|\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)} + C \sum_{|\alpha|\leq N-2} \|\partial_{t,x,y}^\alpha (\Box + m^2) u(t,x,\,\cdot\,)\|_{L^2(\Omega)}. \quad (2.7)$$

Moreover, for any $N = 2,3,4,\dots,$

$$\sum_{|\alpha|\leq N} \|\partial_y^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)} \leq C \sum_{\substack{|\alpha|+|\beta|\leq N\\ |\beta|\leq 1}} \|\partial_{t,x}^\alpha \partial_y^\beta u(t,x,\,\cdot\,)\|_{L^2(\Omega)} + C \sum_{|\alpha|\leq N-2} \|\partial_{t,x,y}^\alpha (\Box + m^2) u(t,x,\,\cdot\,)\|_{L^2(\Omega)}. \quad (2.8)$$

Since $-\Delta_\Omega = (\Box + m^2) - \partial_t^2 + \Delta_{\mathbb{R}^n} - m^2$, and since ∂_t and ∇_x preserve the boundary conditions, the estimate follows easily from an induction argument using Lemma 2.2.

We then use this estimate to obtain the following pointwise result.

Proposition 2.5. *Let* u *be as above. Then if* $m \geq 0$ *is fixed*

$$|\partial_y^\beta u(t,x,y)| \leq C \sum_{|\alpha|\leq|\beta|+(d+4)/2} \|\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)} + C \sum_{|\alpha|\leq|\beta|+d/2} \|\partial_{t,x,y}^\alpha (\Box + m^2) u(t,x,\,\cdot\,)\|_{L^2(\Omega)}. \quad (2.9)$$

To show this, we first apply Sobolev's lemma for Ω to get

$$|\partial_y^\beta u(t,x,y)| \leq C \sum_{|\alpha|\leq|\beta|+(d+2)/2} \|\partial_y^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)}.$$

We can then use Lemma 2.4 (where we sum over indices of length one more when $|\beta| + (d+2)/2$ is odd) to obtain (2.9).

3 Estimates for wave equations in $\mathbb{R}_+ \times \mathbb{R}^3$

We now present the main decay estimates for wave equations in $\mathbb{R}_+ \times \mathbb{R}^3$. These will later be used to prove estimates on the waveguide to control the projection of u onto the eigenfunction with zero eigenvalue. Please note that throughout this section, we will use $\Box = \partial_t^2 - \Delta_{\mathbb{R}^3}$ to denote the d'Alembertian on Minkowski space rather than on the full waveguide. As stated in the introduction, we will use the following vector fields, which have favorable commutation properties with $\Box$. Here, as in (1.15), we let

$$\{\Gamma\} = \{\partial_t, \partial_x, \Omega_{jk} \,:\, 0 \le j < k \le 3\}.$$

3.1 Sobolev Estimates

We first present the following well-known weighted Sobolev inequalities.

Lemma 3.1. *Let $w \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$. Then,*

$$\langle r\rangle^{1/2}|w(t,x)| \le C \sum_{|\alpha|\le 1} \|Z^\alpha \partial w(t,\,\cdot\,)\|_2, \tag{3.1}$$

and

$$\langle r\rangle|\partial\Gamma^\alpha w(t,x)| \le C \sum_{|\beta|\le|\alpha|+2} \|\Gamma^\beta \partial w(t,\,\cdot\,)\|_2. \tag{3.2}$$

We use the standard notation $\langle x\rangle = \langle r\rangle = \sqrt{1+|x|^2}$. The first estimate appears in Sideris [22], as well as in Hidano [2], while the second estimate is due to Klainerman and Sideris [13]. By applying (3.1) to $\langle t-r\rangle\partial\Gamma^\alpha w(t,x)$, we obtain the following estimate, which also appeared in [2], as a corollary of (3.1).

Corollary 3.2. *Let $w \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$. Then,*

$$\langle r\rangle^{1/2}\langle t-r\rangle|\partial\Gamma^\alpha w(t,x)| \le C \sum_{|\beta|\le|\alpha|+1} \|\Gamma^\beta w'(t,\,\cdot\,)\|_2 + \sum_{|\beta|\le|\alpha|+1} \|\langle t-r\rangle\partial^2\Gamma^\beta w(t,\,\cdot\,)\|_2. \tag{3.3}$$

3.2 Decay Estimates

To control the last term in (3.3), we first need the following weighted pointwise bounds. These are analogous to estimates of Klainerman and Sideris [13]. However, in [13], the estimates were constructed using the vector fields Z and the scaling vector field $L = t\partial_t + r\partial_r$. Since we will use the hyperbolic rotations Ω_{0j} instead of L, we must instead derive estimates that only involve Γ.

Lemma 3.3. *Let $w \in C^2(\mathbb{R}_+ \times \mathbb{R}^3)$. Then,*

$$\langle t-r\rangle|\Delta w(t,x)| \le C \sum_{|\alpha|\le 1} |\partial\Gamma^\alpha w(t,x)| + (t+r)|\Box w(t,x)|, \tag{3.4}$$

$$\langle t-r\rangle|\partial_t^2 w(t,x)| \le C \sum_{|\alpha|\le 1} |\partial\Gamma^\alpha w(t,x)| + t|\Box w(t,x)|, \tag{3.5}$$

$$\langle t-r\rangle|\nabla_x\partial_t w(t,x)| \le C \sum_{|\alpha|\le 1} |\partial\Gamma^\alpha w(t,x)| + t|\Box w(t,x)|. \tag{3.6}$$

Proof of Lemma 3.3: We will break the proof into the cases $|x| \le t/2$ and $|x| \ge t/2$.
Case 1 ($|x| \le t/2$): Our main technique in this case will be to use substitutions involving $\Omega_{0i} = t\partial_i + x_i\partial_t$, as well as the commutation property $[\Gamma, \partial] = \partial$.

To obtain (3.4), we begin with

$$\begin{aligned}
\sum_{1\le i\le 3} \Omega_{0i}\partial_i w &= \sum_{1\le i\le 3} (t\partial_i + x_i\partial_t)\partial_i w \\
&= t\Delta w + \sum_{1\le i\le 3} x_i\partial_i\partial_t w \\
&= t\Delta w + \sum_{1\le i\le 3} \frac{x_i}{t}\Omega_{0i}\partial_t w - \frac{r^2}{t}\partial_t^2 w \\
&= \Big(t - \frac{r^2}{t}\Big)\Delta w + \sum_{1\le i\le 3} \frac{x_i}{t}\Omega_{0i}\partial_t w - \frac{r^2}{t}\Box w.
\end{aligned}$$

If we solve for the term involving Δ and use the fact that $t - \frac{r^2}{t} \ge t - r$ in this region, we easily see that

$$(t-r)|\Delta w| \le \sum_{1\le i\le 3} |\Omega_{0i}\partial_i w| + \sum_{1\le i\le 3} |\Omega_{0i}\partial_t w| + t|\Box w|,$$

from which (3.4) follows.

The proof of (3.5) begins similarly. Indeed, we have

$$\begin{aligned}
\sum_{1\le i\le 3} \Omega_{0i}\partial_i w &= t\Delta w + \sum_{1\le i\le 3} \frac{x_i}{t}\Omega_{0i}\partial_t w - \frac{r^2}{t}\partial_t^2 w \\
&= \Big(t - \frac{r^2}{t}\Big)\partial_t^2 w + \sum_{1\le i\le 3} \frac{x_i}{t}\Omega_{0i}\partial_t w - t\Box w,
\end{aligned}$$

and so (3.5) follows easily.

To prove (3.6), we begin with

$$(t-r)\partial_i\partial_t w = (t-r)\Big(\frac{1}{t}\Omega_{0i} - \frac{x_i}{t}\partial_t\Big)\partial_t w$$
$$= \Omega_{0i}\partial_t w - x_i\partial_t^2 w - \frac{r}{t}\Omega_{0i}\partial_t w + \frac{rx_i}{t}\partial_t^2 w.$$

Since we are in the region $r \le t/2$, which implies that $t \le \frac{1}{2}|t-r|$, this yields the bound

$$|(t-r)\partial_i\partial_t w| \le C|\Omega_{0i}\partial_t w| + |t-r||\partial_t^2 w|.$$

By applying (3.5), we see that (3.6) follows.

Case 2 ($|x| \ge t/2$): Here, we recall the radial derivative $\partial_r = \frac{x}{r}\cdot\nabla$ and introduce the vector field

$$\Omega_r = \frac{x^i}{|x|}\Omega_{0i} = t\partial_r + r\partial_t. \tag{3.7}$$

We also present the following lemma, which appeared in [13].

Lemma 3.4. *Let $w \in C^2(\mathbb{R}^3)$. Then,*

$$|\Delta w(x) - \partial_r^2 w(x)| \le C\frac{1}{r}\sum_{|\alpha|\le 1}|\nabla_x Z^\alpha w(x)|. \tag{3.8}$$

This follows immediately from the expansion of the Laplacian into its radial and angular parts

$$\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Omega\cdot\Omega,$$

where $\Omega = -\frac{x}{r^2}\wedge\nabla_x$ and $\wedge$ is the usual vector cross product on $\mathbb{R}^3$, as well as the fact that

$$|\partial_r w(\,\cdot\,)| \le |\nabla_x w(\,\cdot\,)|, \quad \Big|\frac{1}{r}\Omega w(\,\cdot\,)\Big| \le C|\nabla w(\,\cdot\,)|.$$

We begin our proof of the case $|x| \ge t/2$ by noticing that

$$\partial_t\Omega_r w - \partial_r w = t\partial_t\partial_r w + r\partial_t^2 w, \tag{3.9}$$

$$\partial_r\Omega_r w - \partial_t w = t\partial_r^2 w + r\partial_r\partial_t w, \tag{3.10}$$

which quickly give us

$$r(\partial_t\Omega_r w - \partial_r w) - t(\partial_r\Omega_r w - \partial_t w) = r^2\partial_t^2 w - t^2\partial_r^2 w. \tag{3.11}$$

In order to prove (3.4), we observe that (3.11) is equivalent to

$$\begin{aligned}r(\partial_t\Omega_r w - \partial_r w) - t(\partial_r\Omega_r w - \partial_t w) &= r^2\partial_t^2 w - t^2\partial_r^2 w + (t^2-r^2)\Delta w - (t^2-r^2)\Delta w\\ &= r^2\Box w + t^2(\Delta w - \partial_r^2 w) - (t^2-r^2)\Delta w.\end{aligned}$$

We can then rearrange terms to get

$$(t-r)\Delta w = \frac{1}{t+r}\Big[r^2\Box w + t^2(\Delta w - \partial_r^2 w) - r(\partial_t\Omega_r w - \partial_r w) + t(\partial_r\Omega_r w - \partial_t w)\Big],$$

which yields the bound

$$|(t-r)\Delta w| \le r|\Box w| + t|\Delta w - \partial_r^2 w| + |\partial_t\Omega_r w| + |\partial_r w| + |\partial_r\Omega_r w| + |\partial_t w|.$$

Estimate (3.4) follows if we use (3.8) and that we are in the region $t \le 2r$. Additionally, we use that

$$|\partial_t\Omega_r w| + |\partial_r\Omega_r w| \le \sum_{|\alpha|\le 1} |\partial\Gamma^\alpha w|, \tag{3.12}$$

which follows directly from (3.7).

To prove (3.5), we begin similarly by adding and subtracting $t^2\Box w$ to (3.11) to get

$$r(\partial_t\Omega_r w - \partial_r w) - t(\partial_r\Omega_r w - \partial_t w) = (r^2-t^2)\partial_t^2 w + t^2\Box w + t^2(\Delta w - \partial_r^2 w).$$

This, in turn, yields

$$(t-r)\partial_t^2 w = \frac{1}{t+r}\Big[t(\partial_r\Omega_r w - \partial_t w) - r(\partial_t\Omega_r w - \partial_r w) + t^2\Box w + t^2(\Delta w - \partial_r^2 w)\Big].$$

Using (3.8) and (3.12), we see that (3.5) follows.

To prove (3.6), we split the gradient into its radial and angular components. Recall that

$$\nabla_x = \frac{x}{r}\partial_r - \frac{x}{r^2}\wedge\Omega.$$

Beginning with the radial component, we see that

$$\begin{aligned}(t-r)\partial_t\partial_r &= \frac{t-r}{r}\Omega_r\partial_r - t\frac{t-r}{r}\partial_r^2\\ &= \frac{t-r}{r}\Omega_r\partial_r - t\frac{t-r}{r}\Big(\Delta - \frac{2}{r}\partial_r - \frac{1}{r^2}\Omega\cdot\Omega\Big).\end{aligned}$$

Using that $t \leq 2r$, we see that

$$\begin{aligned}|(t-r)\partial_t\partial_r w| &\leq |\Omega_r\partial_r w| + |(t-r)\Delta w| + |\partial_r w| + |\nabla\Omega w| \\ &\leq |\partial_r\Omega_r w| + |\partial_t w| + |(t-r)\Delta w| + |\nabla w| + |\nabla\Omega w|.\end{aligned}$$

By then using (3.12) and (3.4), we see that (3.6) holds when ∇_x is replaced by its radial component.

To prove (3.6) for the angular components of the gradient when $t \leq 2r$, the bound easily follows from

$$|(t-r)\frac{x}{r^2} \wedge \Omega\partial_t w| \leq C|\partial_t\Omega w|,$$

which completes the proof. □

We are now able to present the estimate which will be used to control the last term in (3.3).

Corollary 3.5. *Let $w \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$. Then,*

$$\|\langle t-r\rangle\partial_{t,x}^2\Gamma^\alpha w(t,\,\cdot\,)\|_2 \leq C\sum_{|\beta|\leq|\alpha|+1}\|\Gamma^\beta w'(t,\,\cdot\,)\|_2 + \sum_{|\beta|\leq|\alpha|}\|\langle t+r\rangle\Box\Gamma^\beta w(t,\,\cdot\,)\|_2. \quad (3.13)$$

The result clearly follows immediately from Lemma 3.3 except for the case where we have mixed second-order spatial derivatives. In this case we argue as in the proof of Gårding's inequality (see [13], Lemma 3.1). It suffices to consider $\alpha = 0$. Let $\sigma = \langle t-r\rangle$. Then integration by parts gives us

$$\begin{aligned}\sum_{|\alpha|=2}\|\sigma\nabla^\alpha w\|_{L^2(\mathbb{R}^3)}^2 &= \sum_{1\leq i,j\leq 3}\int_{\mathbb{R}^3}\sigma^2(\partial_i\partial_j w)(\partial_i\partial_j w)\,dx \\ &= \int\sigma^2(\triangle w)^2\,dx + \sum_{i,j}\int\sigma^2[(\partial_i\partial_j w)(\partial_i\partial_j w) + (\partial_j w)(\partial_i w)]\,dx \\ &\qquad - \sum_{i,j}\int\sigma^2[(\partial_j^2 w)(\partial_i^2 w) + (\partial_j w)(\partial_j\partial_i^2 w)]\,dx \\ &= \int\sigma^2(\triangle w)^2\,dx - \sum_{i,j}\int(\partial_i\sigma^2)(\partial_j w)(\partial_i\partial_j w)\,dx \\ &\qquad + \sum_{i,j}\int(\partial_i\sigma^2)(\partial_j w)(\partial_i^2 w)\,dx.\end{aligned}$$

The first integral is bounded by the right-hand side of (3.13) using (3.4). We can then use the Cauchy-Schwarz inequality and Young's inequality, as well as the fact that the derivatives of σ are uniformly bounded, to bound the last two integrals by

$$\frac{1}{4}\sum_{|\alpha|=2} \|\sigma\nabla^{\alpha} w\|^2 + C\|\nabla w\|^2.$$

If we then absorb the second derivatives into the left-hand side and bound the first derivatives by the energy, we have (3.13).

4 Linear Decay Estimates in Minkowski Space

In this section we present the decay estimates in the boundaryless case which will later be used to prove our main decay estimates on the waveguide. The first group are estimates for linear Klein-Gordon equations, which also appeared in [19] and are essentially from Hörmander [4]. We will use the following estimate when $n = 3$.

Proposition 4.1. *Suppose that $w \in C^\infty(\mathbb{R} \times \mathbb{R}^3)$ satisfies $w(t,x) = 0$, $t \leq 2B$, where B is a fixed positive constant. Suppose also that $(\Box_{\mathbb{R}^{1+3}} + \mu^2)w(t,x) = 0$ for $|x| > t - B$, Then there is a constant depending only on B so that when $\mu \geq 1$*

$$\sup_x t^{3/2}|w(t,x)| \leq C \sum_{|\alpha|\leq 5} \sum_k \sup_{\tau \in [2^{k-1}, 2^{k+1}] \cap [2B,t]} 2^k \|\Gamma^\alpha(\Box_{\mathbb{R}^{1+3}} + \mu^2)w(\tau, \,\cdot\,)\|_2. \tag{4.1}$$

This follows exactly from the proof of Proposition 7.3.6 in Hörmander [4] if we instead use a variation of Lemma 7.3.4 (in [4]). The appropriate version in this case is that if $v'' + \mu^2 v = h$ in $[a,b] \subset \mathbb{R}$, then

$$\sup_{a\leq\rho\leq b} |v(\rho)| \leq |v(a)| + |v'(a)| + \frac{1}{\mu}\int_a^b |h(\rho)|d\rho.$$

We will also use the following analog of Proposition 7.3.7 in [4] when $n \geq 4$. The proof follows similarly to the $n = 3$ case.

Proposition 4.2. *Suppose that $n \geq 4$ and that $w \in C^\infty(\mathbb{R}\times\mathbb{R}^n)$ satisfies $w(t,x) = 0$, $t \leq 2B$, where B is a fixed positive constant. Suppose also that $(\Box_{\mathbb{R}^{1+n}}+\mu^2)w(t,x) = 0$ for $|x| > t - B$. Then if $n = 4$ there is a constant depending only on B so that when $\mu \geq 1$*

$$\sup_x t^2|w(t,x)| \leq C \sum_{|\alpha|\leq 7} \sum_k \sup_{\tau \in [2^{k-1}, 2^{k+1}] \cap [2B,t]} (1+|k|)2^k \|\Gamma^\alpha(\Box_{\mathbb{R}^{1+4}}+\mu^2)w(\tau, \,\cdot\,)\|_2. \tag{4.2}$$

If $n \geq 5$ there is a constant depending only on B and n so that when $\mu \geq 1$

$$\sup_x t^{1+\frac{n}{4}}|w(t,x)| \leq C \sum_{|\alpha|\leq n+3} \sum_k \sup_{\tau \in [2^{k-1}, 2^{k+1}] \cap [2B,t]} 2^k \|\Gamma^\alpha(\Box_{\mathbb{R}^{1+n}} + \mu^2)w(\tau, \,\cdot\,)\|_2. \tag{4.3}$$

While these estimates can be applied to any eigenfunction projection of solutions to the Klein-Gordon equation ($m > 0$), they can only be used for the wave equation ($m = 0$) when the eigenfunctions have non-zero eigenvalues. Note that the estimate requires that μ be strictly positive. With this in mind, our technique is to use the non-vanishing eigenvalue as the mass term of a Klein-Gordon equation once we expand our solution in terms of the eigenfunction projection. If we apply our operator

$$\Box + m^2 = \Box_{\mathbb{R}^{1+n}} - \Delta_\Omega + m^2$$

to the eigenfunction projection $E_j u$, we have $\mu^2 = m^2 + \lambda_j$, which is zero when $m = 0$ and $j = 1$. Thus we are unable to use the estimate for $\lambda_1 = 0$ in the Neumann case.

We will instead use the following estimate for linear wave equations to handle the eigenfunctions with zero eigenvalue in the Neumann wave equation case. This result was first presented by Metcalfe and Stewart in [20] and is the main innovation in the Neumann case. We note here that this estimate is the reason for which we restrict our nonlinearity to derivatives only involving ∂_t and ∂_x. The proof follows quickly using the estimates presented in the previous section.

Proposition 4.3. *Suppose that $w \in C^\infty(\mathbb{R} \times \mathbb{R}^3)$ satisfies $w(t, x) = 0$, $t \leq 2B$ where B is a fixed positive constant. Suppose also that $\Box_{\mathbb{R}^{1+3}} w(t, x) = 0$ for $|x| > t - B$. Then,*

$$(1+t) \sup_x |\partial_{t,x} w(t,x)| \leq C \sum_{|\alpha| \leq 2} \sum_k \sup_{\tau \in [2^{k-1}, 2^{k+1}] \cap [2B, t]} 2^k \|\Gamma^\alpha \Box_{\mathbb{R}^{1+3}} w(\tau, \,\cdot\,)\|_2. \quad (4.4)$$

Proof of Proposition 4.3: As in the proof of our estimates from the previous section, we must treat the regions inside and outside the cone separately. When $|x| \leq t/2$, we use (3.3) and (3.13) to obtain

$$(1+t) \sup_{\{x\,:\,|x| \leq t/2\}} |\partial w(t,x)| \leq C \sum_{|\alpha| \leq 2} \|\Gamma^\alpha w'(t, \,\cdot\,)\|_2 + \sum_{|\alpha| \leq 1} \|\langle t + r \rangle \Gamma^\alpha \Box_{\mathbb{R}^{1+3}} w(t, \,\cdot\,)\|_2.$$

We can apply the standard energy inequality for Minkowski space (see, e.g. [23]) to the first term and the restriction on our domain to the second term to get that the right side is

$$\leq C \sum_{|\alpha| \leq 2} \int_{2B}^t \|\Gamma^\alpha \Box_{\mathbb{R}^{1+3}} w(\tau, \,\cdot\,)\|_2 \, d\tau + t \sum_{|\alpha| \leq 1} \|\Gamma^\alpha \Box_{\mathbb{R}^{1+3}} w(t, \,\cdot\,)\|_2.$$

If we then use a dyadic decomposition on $[2B, t]$ so that $\tau \approx 2^k$, then both of the above terms are

$$\leq C \sum_{|\alpha|\leq 2} \sum_k \sup_{\tau \in [2^{k-1}, 2^{k+1}] \cap [2B, t]} 2^k \|\Gamma^\alpha \Box_{\mathbb{R}^{1+3}} w(\tau, \,\cdot\,)\|_2.$$

When $|x| > t/2$, the proof is similar. If we instead use (3.2) along with the energy inequality, then we have

$$(1+t) \sup_{\{x\,:\,|x|>t/2\}} |\partial w(t,x)| \leq C \sum_{|\alpha|\leq 2} \int_{2B}^{t} \|\Gamma^\alpha \Box_{\mathbb{R}^{1+3}} w(\tau, \,\cdot\,)\|_2 \, d\tau.$$

We can again use a dyadic decomposition in time to get (4.4). $\square$

5 Linear Decay Estimates for Waveguides

We now present our main decay estimates for the waveguide. The main idea is that we will use the linear decay estimates from Minkowski space presented in the previous section, but in order to extend them to the waveguide, we must give up a few derivatives. The exact loss in regularity in each estimate depends on the dimensions n and d of the components of the waveguide.

5.1 Robin Waveguide

Our main decay estimate for the Robin case is the analog of Proposition 3.5 in [19] for the Dirichlet case. The proof follows exactly, but is included here for completeness as well as for comparison in the Neumann case. Recall that we use the vector fields as presented in (1.15).

Proposition 5.1. *Fix B and suppose that $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n \times \overline{\Omega})$ satisfies $u(t,x,y) = 0$, $t \le 2B$ and $(\Box + m^2)u(t,x,y) = 0$ if $|x| > t - B$. Suppose also that $m \ge 0$ and $(\alpha(y)u + \partial_\nu u)|_{\partial\Omega} = 0$, where $\alpha \ge 0$ and $\int_{\partial\Omega} \alpha > 0$. Then, if $n = 3$,*

$$t^{3/2}|\tilde{\Gamma}^\beta u(t,x,y)| \le C \sum_{|\alpha| \le |\beta| + (5d+14)/2} \sum_k \sup_{\tau \in [2^{k-1}, 2^{k+1}] \cap [2B, t]} 2^k \|\tilde{\Gamma}^\alpha (\Box + m^2) u(\tau, \cdot\,)\|_2 + C t^{3/2} \sum_{|\alpha| \le |\beta| + (5d+5)/2} \|\tilde{\Gamma}^\alpha (\Box + m^2) u(t, \cdot\,)\|_2. \quad (5.1)$$

If $n = 4$,

$$t^2 |\tilde{\Gamma}^\beta u(t,x,y)| \le C \sum_{|\alpha| \le |\beta| + (5d+18)/2} \sum_k \sup_{\tau \in [2^{k-1}, 2^{k+1}] \cap [2B, t]} (1 + |k|) 2^k \|\tilde{\Gamma}^\alpha (\Box + m^2) u(\tau, \cdot\,)\|_2 + C t^2 \sum_{|\alpha| \le |\beta| + (5d+n+2)/2} \|\tilde{\Gamma}^\alpha (\Box + m^2) u(t, \cdot\,)\|_2. \quad (5.2)$$

For $n \geq 5$

$$t^{1+\frac{n}{4}}|\tilde{\Gamma}^\beta u(t,x,y)| \leq C \sum_{|\alpha|\leq|\beta|+(5d+2n+10)/2} \sum_k \sup_{\tau\in[2^{k-1},2^{k+1}]\cap[2B,t]} 2^k \|\tilde{\Gamma}^\alpha(\Box+m^2)u(\tau,\,\cdot\,)\|_2 + Ct^{1+\frac{n}{4}} \sum_{|\alpha|\leq|\beta|+(5d+n+2)/2} \|\tilde{\Gamma}^\alpha(\Box+m^2)u(t,\,\cdot\,)\|_2. \quad (5.3)$$

Proof of Proposition 5.1: Since $\{\Gamma\}$ commute with $\Box+m^2$ and preserve the Robin boundary conditions, it is sufficient to prove the above estimates when $\tilde{\Gamma}^\beta = \partial_y^\beta$. Using Proposition 2.5 and the orthogonality of our eigenfunction projections (2.4), we see that

$$\begin{aligned} |\partial_y^\beta u(t,x,y)|^2 &\leq C \sum_{|\alpha|\leq|\beta|+(d+4)/2} \|\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)}^2 \\ &\quad + C \sum_{|\alpha|\leq|\beta|+d/2} \|\partial_{t,x,y}^\alpha(\Box+m^2)u(t,x,\,\cdot\,)\|_{L^2(\Omega)}^2 \\ &\leq C \sum_{|\alpha|\leq|\beta|+(d+4)/2} \sum_{j=1}^\infty \|E_j\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)}^2 \\ &\quad + C \sum_{|\alpha|\leq|\beta|+(d+n+2)/2} \|\partial_{t,x,y}^\alpha(\Box+m^2)u(t,\,\cdot\,)\|_{L^2(\mathbb{R}^n\times\Omega)}^2, \end{aligned} \quad (5.4)$$

where in the last step we applied Sobolev's lemma on $\mathbb{R}^n$ to the second term. We also note that even though our u depends on t, x, and y, it makes sense to use a partial eigenfunction projection since here both the t and x variables are fixed.

We must now control the first term on the right side of (5.4). If we apply (2.5), then

$$\begin{aligned} (1+j)^{2/d} \sum_{|\alpha|\leq|\beta|+(d+4)/2} \|E_j\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)} &\leq C \sum_{|\alpha|\leq|\beta|+(d+4)/2} \|E_j(I-\Delta_\Omega)\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)} \\ &\leq C \sum_{|\alpha|\leq 2+|\beta|+(d+4)/2} \|E_j\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)} \\ &\quad + C \sum_{|\alpha|\leq|\beta|+(d+4)/2} \|E_j\partial_{t,x}^\alpha(\Box+m^2)u(t,x,\,\cdot\,)\|_{L^2(\Omega)}. \end{aligned}$$

In the last step, we used a substitution for Δ_Ω in terms of the d'Alembertian. A recursive application of this estimate to the first term in the last step then gives us

$$\sum_{|\alpha|\le|\beta|+(d+4)/2} \|E_j\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)} \le \frac{C}{(1+j)^2} \sum_{|\alpha|\le|\beta|+(5d+4)/2} \|E_j\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)} + C\sum_{|\alpha|\le|\beta|+5d/2} \|E_j\partial_{t,x}^\alpha(\Box+m^2)u(t,x,\,\cdot\,)\|_{L^2(\Omega)}.$$

Therefore, by (2.4) and Sobolev's lemma for $\mathbb{R}^n$ it immediately follows that

$$\sum_{|\alpha|\le|\beta|+(d+4)/2}\sum_{j=1}^\infty \|E_j\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)}^2 \le C\sum_{|\alpha|\le|\beta|+(5d+n+2)/2} \|\partial_{t,x}^\alpha(\Box+m^2)u(t,\,\cdot\,)\|_{L^2(\mathbb{R}^n\times\Omega)}^2 + C\sum_{j=1}^\infty \frac{1}{(1+j)^4}\Big(\sum_{|\alpha|\le|\beta|+(5d+4)/2} \|E_j\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)}^2\Big). \quad (5.5)$$

Thus, if we combine (5.4) and (5.5), we obtain

$$t^n|\partial_y^\beta u(t,x,y)|^2 \le Ct^n\sum_{|\alpha|\le|\beta|+(5d+n+2)/2} \|\partial_{t,x}^\alpha(\Box+m^2)u(t,\,\cdot\,)\|_{L^2(\mathbb{R}^n\times\Omega)}^2 + Ct^n\sum_{j=1}^\infty \frac{1}{(1+j)^4}\Big(\sum_{|\alpha|\le|\beta|+(5d+4)/2} \|E_j\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)}^2\Big). \quad (5.6)$$

Finally, we need to estimate each of the summands in the last term of (5.6). Our assumptions on m and the boundary conditions guarantee that $m^2+\lambda_j\ge c$, for some constant $c>0$, and so we may apply Propositions 4.1 and 4.2. When $n=3$, Proposition 4.1 gives us that there is a constant C independent of j such that

$$\begin{aligned} t^{3/2}&\sum_{|\alpha|\le|\beta|+(5d+4)/2} \|E_j\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)} \\ &\le C\sum_{|\alpha|\le|\beta|+5+(5d+4)/2}\sum_k \sup_{\tau\in[2^{k-1},2^{k+1}]\cap[2B,t]} 2^k \\ &\qquad\qquad \times\|\Gamma^\alpha(\partial_t^2-\Delta_{\mathbb{R}^3}+m^2+\lambda_j)E_ju(\tau,\,\cdot\,)\|_{L^2(\mathbb{R}^3\times\Omega)} \\ &\le C\sum_{|\alpha|\le|\beta|+(5d+14)/2}\sum_k \sup_{\tau\in[2^{k-1},2^{k+1}]\cap[2B,t]} 2^k\|\Gamma^\alpha(\Box+m^2)u(\tau,\,\cdot\,)\|_{L^2(\mathbb{R}^3\times\Omega)}, \end{aligned}$$

where we also used Proposition 2.1 in the last step. This inequality and (5.6), as well as the fact that our sum over j in (5.6) is a convergent series, yield the estimate

(5.1). The estimates for (5.2) and (5.3) follow similarly, and so we will omit their proofs. □

5.2 Neumann Waveguide

Our main decay estimate for the Neumann waveguide first appeared in Metcalfe and Stewart [20].

Proposition 5.2. *Fix B and suppose that $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3 \times \bar{\Omega})$ satisfies $u(t,x,y) = 0$ for $t \le 2B$, $\Box u(t,x,y) = 0$ for $|x| > t - B$, and $\partial_\nu u(t,x,y)|_{y\in\partial\Omega} = 0$. Then,*

$$(1+t)|\tilde{\Gamma}^\beta \partial_{t,x} u(t,x,y)| \le C \sum_{|\alpha|\le|\beta|+(5d+16)/2} \sum_k \sup_{\tau\in[2^{k-1},2^{k+1}]\cap[2B,t]} 2^k \|\tilde{\Gamma}^\alpha \Box u(\tau,\,\cdot\,)\|_2 + (1+t) \sum_{|\alpha|\le|\beta|+(5d+6)/2} \|\tilde{\Gamma}^\alpha \Box u(t,\,\cdot\,)\|_2. \quad (5.7)$$

Proof of Proposition 5.2: The proof of this proposition follows that of Proposition 5.1 very closely except that we will make use of (4.4) to estimate those terms in our eigenfunction expansion with zero eigenvalues.

It again suffices to take $\tilde{\Gamma}^\beta = \partial_y^\beta$ since the Γ preserve the Neumann boundary conditions. Proceeding as before, we use (2.9), (2.4) and Sobolev's lemma to obtain

$$\begin{aligned}|\partial_y^\beta \partial_{t,x} u(t,x,y)|^2 \le C &\sum_{|\alpha|\le|\beta|+(d+4)/2} \|\partial_{t,x}^\alpha \partial_{t,x} u(t,x,\,\cdot\,)\|^2_{L^2(\Omega)} \\ &+ \sum_{|\alpha|\le|\beta|+(d+2)/2} \|\partial_{t,x,y}^\alpha \Box u(t,x,\,\cdot\,)\|^2_{L^2(\Omega)} \\ \le C &\sum_{|\alpha|\le|\beta|+(d+4)/2} \sum_{j=1}^\infty \|E_j \partial_{t,x}^\alpha \partial_{t,x} u(t,x,\,\cdot\,)\|^2_{L^2(\Omega)} \\ &+ \sum_{|\alpha|\le|\beta|+(d+6)/2} \|\partial_{t,x,y}^\alpha \Box u(t,\,\cdot\,)\|^2_{L^2(\mathbb{R}^3\times\Omega)}. \end{aligned} \quad (5.8)$$

Likewise, an application of (2.5) yields

$$(1+j)^{2/d} \sum_{|\alpha|\le|\beta|+(d+4)/2} \|E_j\partial_{t,x}^\alpha \partial_{t,x} u(t,x,\,\cdot\,)\|_{L^2(\Omega)}$$
$$\le C \sum_{|\alpha|\le|\beta|+(d+4)/2} \|E_j(I-\Delta_\Omega)\partial_{t,x}^\alpha \partial_{t,x} u(t,x,\,\cdot\,)\|_{L^2(\Omega)}$$
$$\le C \sum_{|\alpha|\le|\beta|+(d+8)/2} \|E_j\partial_{t,x}^\alpha \partial_{t,x} u(t,x,\,\cdot\,)\|_{L^2(\Omega)} + \sum_{|\alpha|\le|\beta|+(d+6)/2} \|E_j\partial_{t,x}^\alpha \Box u(t,x,\,\cdot\,)\|_{L^2(\Omega)},$$

and as before we apply this estimate recursively to see that

$$\sum_{|\alpha|\le|\beta|+(d+4)/2} \|E_j\partial_{t,x}^\alpha \partial_{t,x} u(t,x,\,\cdot\,)\|_{L^2(\Omega)}$$
$$\le C\frac{1}{(1+j)^2} \sum_{|\alpha|\le|\beta|+(5d+4)/2} \|E_j\partial_{t,x}^\alpha \partial_{t,x} u(t,x,\,\cdot\,)\|_{L^2(\Omega)} + \sum_{|\alpha|\le|\beta|+(5d+2)/2} \|E_j\partial_{t,x}^\alpha \Box u(t,x,\,\cdot\,)\|_{L^2(\Omega)}.$$

Plancherel's theorem and Sobolev's lemma on $\mathbb{R}^3$ on the last term above again yield

$$\sum_{|\alpha|\le|\beta|+(d+4)/2} \sum_{j=1}^\infty \|E_j\partial_{t,x}^\alpha \partial_{t,x} u(t,x,\,\cdot\,)\|^2_{L^2(\Omega)} \le C \sum_{|\alpha|\le|\beta|+(5d+6)/2} \|\partial_{t,x}^\alpha \Box u(t,\,\cdot\,)\|^2_{L^2(\mathbb{R}^3\times\Omega)} + \sum_{j=1}^\infty \frac{1}{(1+j)^4} \sum_{|\alpha|\le|\beta|+(5d+4)/2} \|E_j\partial_{t,x}^\alpha \partial_{t,x} u(t,x,\,\cdot\,)\|^2_{L^2(\Omega)}. \quad (5.9)$$

If we combine (5.8) and (5.9), we have

$$(1+t)^2|\partial_y^\beta \partial_{t,x} u(t,x,y)|^2 \le C(1+t)^2 \sum_{|\alpha|\le|\beta|+(5d+6)/2} \|\partial_{t,x}^\alpha \Box u(t,\,\cdot\,)\|^2_{L^2(\mathbb{R}^3\times\Omega)}$$
$$+ (1+t)^2 \sum_{|\alpha|\le|\beta|+(5d+4)/2} \|\partial_{t,x} E_1 \partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|^2_{L^2(\Omega)}$$
$$+ (1+t)^2 \sum_{j=2}^\infty \frac{1}{(1+j)^4} \sum_{|\alpha|\le|\beta|+(5d+6)/2} \|E_j\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|^2_{L^2(\Omega)}. \quad (5.10)$$

Note that we have isolated the projection onto the first eigenfunction, which has a vanishing eigenvalue, so that we can make use of Proposition 4.3. To this term we now apply (4.4), which gives us

$$
\begin{aligned}
(1+t)&\sum_{|\alpha|\le|\beta|+(5d+4)/2}\|\partial_{t,x}E_1\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)}\\
&\le C\sum_{|\alpha|\le|\beta|+(5d+8)/2}\sum_k\sup_{\tau\in[2^{k-1},2^{k+1}]\cap[2B,t]}2^k\|\Gamma^\alpha(\partial_t^2-\Delta)E_1u(\tau,\,\cdot\,)\|_{L^2(\mathbb{R}^3\times\Omega)}\\
&\le C\sum_{|\alpha|\le|\beta|+(5d+8)/2}\sum_k\sup_{\tau\in[2^{k-1},2^{k+1}]\cap[2B,t]}2^k\|\Gamma^\alpha\Box u(\tau,\,\cdot\,)\|_{L^2(\mathbb{R}^3\times\Omega)}
\end{aligned}
$$

The last inequality follows from Proposition 2.1.

As in the Robin case, we can use (4.1) to control each of the summands in the last term of (5.10), since $\lambda_j\ge c>0$ for each $j=2,3,\ldots$. In fact, (4.1) gives us stronger decay than we actually need in this case. Thus we have

$$
\begin{aligned}
(1+t)&\sum_{|\alpha|\le|\beta|+(5d+6)/2}\|E_j\partial_{t,x}^\alpha u(t,x,\,\cdot\,)\|_{L^2(\Omega)}\\
&\le C\sum_{|\alpha|\le|\beta|+(5d+16)/2}\sum_k\sup_{\tau\in[2^{k-1},2^{k+1}]\cap[2B,t]}2^k\|\Gamma^\alpha(\partial_t^2-\Delta+\lambda_j^2)E_ju(\tau,\,\cdot\,)\|_{L^2(\mathbb{R}^3\times\Omega)}\\
&\le C\sum_{|\alpha|\le|\beta|+(5d+16)/2}\sum_k\sup_{\tau\in[2^{k-1},2^{k+1}]\cap[2B,t]}2^k\|\Gamma^\alpha\Box u(\tau,\,\cdot\,)\|_{L^2(\mathbb{R}^3\times\Omega)}.
\end{aligned}
$$

where we once again apply 2.1 in the last step.

If we then combine these final two bounds with (5.10), the proof is complete. □

Finally, we present a decay estimate for $\partial_y u$ for use in the case where Ω is one-dimensional with Neumann boundary conditions. We use the simple fact that

$$v=\partial_y u=\partial_\nu u$$

satisfies the Dirichlet wave equation $\Box v=\partial_y\Box u$. With this in mind, we can use the following result which is a weaker version of Proposition 3.5 from [19].

Proposition 5.3. *Fix B and suppose that $u\in C^\infty(\mathbb{R}_+\times\mathbb{R}^3\times\bar{\Omega})$ satisfies $u(t,x,y)=0$ for $t\le 2B$, $\Box u(t,x,y)=0$ for $|x|>t-B$, and $u(t,x,y)|_{y\in\partial\Omega}=0$. Then,*

$$
\begin{aligned}
(1+t)|\tilde\Gamma^\beta u(t,x,y)|\le C&\sum_{|\alpha|\le|\beta|+(5d+14)/2}\sum_k\sup_{\tau\in[2^{k-1},2^{k+1}]\cap[2B,t]}2^k\|\tilde\Gamma^\alpha\Box u(\tau,\,\cdot\,)\|_2\\
&+(1+t)\sum_{|\alpha|\le|\beta|+(5d+5)/2}\|\tilde\Gamma^\alpha\Box u(t,\,\cdot\,)\|_2. \quad (5.11)
\end{aligned}
$$

To prove this, one simply uses a weaker version of the Minkowski decay estimate in Proposition 4.1, so that the term on the left is $(1+t)$. Thus the modification to the proof in [19] is minor and follows the proof presented for Proposition 5.1 in this paper almost exactly. This version is necessary for the proof of our existence theorem, however, since without the modification, we would be unable to handle the $(1+t)^{3/2}$ factor that would appear in the last term on the right side, as in Proposition 3.5 of [19].

We can then apply (5.11) to $\partial_y u$ to obtain

Corollary 5.4. *Fix B and suppose that $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3 \times [a,b])$ for two fixed constants $a < b$. Suppose further that u satisfies $u(t,x,y) = 0$ for $t \le 2B$, $\Box u(t,x,y) = 0$ for $|x| > t - B$, and $\partial_y u(t,x,a) = \partial_y u(t,x,b) = 0$. Then,*

$$(1+t)|\tilde{\Gamma}^\beta \partial_y u(t,x,y)| \le C \sum_{|\alpha|\le|\beta|+10} \sum_k \sup_{\tau\in[2^{k-1},2^{k+1}]\cap[2B,t]} 2^k \|\tilde{\Gamma}^\alpha \Box u(\tau,\,\cdot\,)\|_2 + (1+t) \sum_{|\alpha|\le|\beta|+6} \|\tilde{\Gamma}^\alpha \Box u(t,\,\cdot\,)\|_2. \quad (5.12)$$

6 Energy Estimates

In this section, we present the well-known energy estimates for wave and Klein-Gordon equations with variable coefficients. These are essential to the construction of our arguments in the final two sections of the paper. Let $\gamma^{jk} \in C^\infty$ satisfy

$$\sum_{j,k=0}^{3+d} |\gamma^{jk}| \leq 1/2, \qquad \sum_{i,j,k=0}^{3+d} \|\partial_i \gamma^{jk}\|_{L^1_t L^\infty_{x,y}([0,T]\times\mathbb{R}^3\times\Omega)} \leq C_0. \tag{6.1}$$

We first consider the Neumann wave equation

$$\begin{cases} \Box u + \sum_{j,k=0}^{3+d} \gamma^{jk}(t,x,y)\partial_j\partial_k u = F, \quad 2B \leq t \leq T, \\ \partial_\nu u(t,x,y) = 0, \quad y \in \partial\Omega, \\ u(t,x,y) = 0, \quad t \leq 2B. \end{cases} \tag{6.2}$$

Notice that we now include the quasilinear portion of our nonlinearity as part of our operator and thus as a perturbation of the metric. In the Neumann case, we have either

$$\gamma^{jk}(t,x,y) = -\sum_{l=0}^{3} A_l^{jk}\partial_l u \quad \text{or} \quad \gamma^{jk}(t,x,y) = -\sum_{l=0}^{4} A_l^{jk}\partial_l u$$

from our expansions of our nonlinearities $\bar{Q}$ and $\tilde{Q}$ in (1.6) and (1.9) respectively. In order to establish energy estimates with Neumann boundary conditions, we must assume that the γ^{jk} satisfy the following nonlinear compatibility condition, so that the requirement described for $\tilde{Q}$ in (1.12) is satisfied. Recall that we do not need to state this explicitly for $\bar{Q}$, since it automatically holds. In the $d = 1$ case, the required condition is that

$$\sum_{j,k=0}^{4} \gamma^{jk}(t,x,y)\xi_j\theta_k = 0, \quad \text{if } y \in \partial\Omega,\ \theta = (0,\dots,0,\nu(y)),\ \text{and } \xi\cdot\theta = 0. \tag{6.3}$$

With this condition as well as the assumption that u vanishes for large $|x|$, the standard proof of the energy inequality for (6.2) (see, e.g., Proposition 2.1 from Chapter 1 of [23]) yields

$$\|\partial_{t,x,y} u(t,\,\cdot\,)\|_2 \leq C \int_0^t \|F(s,\,\cdot\,)\|_2\, ds. \tag{6.4}$$

For our energy method arguments we also need an analog of (6.4) for $\tilde{\Gamma}^\beta u$. Since Γ preserves the boundary conditions, (2.8) immediately gives us that

$$\sum_{|\alpha|+|\beta|\le N} \|\Gamma^\alpha \partial_y^\beta \partial_{t,x,y} u(t,\,\cdot\,)\|_2 \le C \sum_{|\alpha|\le N} \|\Gamma^\alpha \partial_{t,x,y} u(t,\,\cdot\,)\|_2 + \sum_{|\alpha|\le N-1} \|\tilde{\Gamma}^\alpha \Box u(t,\,\cdot\,)\|_2.$$

We can then use the fact that

$$\Box \Gamma^\alpha u + \sum_{j,k=0}^{3+d} \gamma^{jk} \partial_j \partial_k \Gamma^\alpha u = \Gamma^\alpha F + \sum_{j,k=0}^{3+d} [\gamma^{jk}, \Gamma^\alpha] \partial_j \partial_k w + \sum_{j,k=0}^{3+d} \gamma^{jk} [\partial_j \partial_k, \Gamma^\alpha] w$$

to show that for $N = 0, 1, 2, \dots$ we have

$$\begin{aligned}\sum_{|\alpha|\le N} \|\tilde{\Gamma}^\alpha \partial_{t,x,y} u(t,\,\cdot\,)\|_2 &\le C \sum_{|\alpha|\le N} \int_0^t \|\Gamma^\alpha F(s,\,\cdot\,)\|_2\, ds \\ &+ \sum_{|\alpha|\le N} \sum_{j,k=0}^{3+d} \int_0^t \|[\gamma^{jk}, \Gamma^\alpha] \partial_j \partial_k u(s,\,\cdot\,)\|_2\, ds + \sum_{|\alpha|\le N} \sum_{j,k=0}^{3+d} \int_0^t \|\gamma^{jk} [\Gamma^\alpha, \partial_j \partial_k] u(s,\,\cdot\,)\|_2\, ds \\ &+ \sum_{|\beta|\le N-1} \|\tilde{\Gamma}^\beta \Box u(t,\,\cdot\,)\|_2. \qquad (6.5)\end{aligned}$$

Next we examine the Robin Klein-Gordon equation ($m \ge 0$) where

$$\begin{cases} \Box u + \sum_{j,k=0}^{3+d} \gamma^{jk}(t,x,y) \partial_j \partial_k u + m^2 u = F, \quad 2B \le t \le T, \\ \alpha(y) u + \partial_\nu u = 0, \quad y \in \partial\Omega, \quad \text{where } \alpha(y) \ge 0 \text{ with } \int_{\partial\Omega} \alpha > 0 \\ u(t,x,y) = 0, \quad t \le 2B. \end{cases} \qquad (6.6)$$

In this case we have

$$\gamma^{jk}(t,x,y) = -\sum_{l=0}^{n+d} A_l^{jk} \partial_l u - u A^{jk}$$

from our expansion of our nonlinearity Q in (1.3). Here we assume the following nonlinear compatibility condition on the γ^{jk}, as first described for Q by (1.13). The necessary assumption is that

$$\begin{cases} \sum_{0\le j,k\le n+d} \alpha \gamma^{jk}(t,x,y) \theta_j \theta_k = 0 \\ \sum_{0\le j,k\le n+d} \gamma^{jk}(t,x,y) \xi_j \theta_k = 0 \end{cases}$$

$$\text{if } y \in \partial\Omega,\ \theta = (0, \dots, 0, \nu_1(y), \dots, \nu_d(y)), \text{ and } \xi \cdot \theta = 0. \qquad (6.7)$$

Before we state the variable coefficient energy inequality for the Robin Klein-Gordon equation, we will briefly note a few simple facts (see, e.g. [24]). The standard energy for the wave equation with Robin boundary conditions is defined as

$$\int_{\mathbb{R}^n\times\Omega}\Big(|\partial_t u(t,x,y)|^2+|\partial_x u(t,x,y)|^2+|\partial_y u(t,x,y)|^2\Big)\,dxdy \\ +\int_{\mathbb{R}^n\times\partial\Omega}\alpha(y)(u)^2\,dxd\sigma.$$

Unlike in the Dirichlet and Neumann cases, where the energy is simply $\|u'\|_{L^2}$, here we have a boundary term which depends on u. This is due to the fact that Robin boundary conditions model exchange of energy between the interior and the boundary. Thus while it is true that the total energy is conserved for the free Robin wave equation, the interior and the boundary may actually exchange energy back and forth. For example, the case $a(y)>0$ corresponds to the case where energy is radiated from the interior to the boundary over the entire boundary. This boundary integral term is exactly the term needed to cancel the usual boundary integral that arises during the proof of the energy inequality from integration by parts (see [23]) and that vanishes due to the boundary conditions in both the Dirichlet and Neumann cases.

With these modifications, the proof of Lemma 7.4.1 for Klein-Gordon equations in [4] shows that the following variable coefficient energy inequality holds for (6.6). If u vanishes for large $|x|$ we have

$$\|\nabla_{t,x,y}u(t,\,\cdot\,)\|_2+m\|u(t,\,\cdot\,)\|_2+\left(\int_{\mathbb{R}^n\times\partial\Omega}\alpha(y)(u)^2\,dxd\sigma\right)^{1/2} \\ \le C\int_0^t\|F(s,\,\cdot\,)\|_2\,ds. \quad (6.8)$$

As in the Neumann case, we also need an estimate similar to (6.8) which holds for $\tilde{\Gamma}^\beta u$. Proceeding as in the proof of (6.5), we have for $N=0,1,2,\dots$

$$\sum_{|\alpha|\le N}\|\tilde{\Gamma}^\alpha\partial_{t,x,y}u(t,\,\cdot\,)\|_2+m\sum_{|\alpha|\le N}\|\tilde{\Gamma}^\alpha w(t,\,\cdot\,)\|_2+\sum_{|\alpha|\le N}\left(\int_{\mathbb{R}^n\times\partial\Omega}\alpha(y)(\tilde{\Gamma}^\alpha w)^2\,dxd\sigma\right)^{1/2} \\ \le C\sum_{|\alpha|\le N}\int_0^t\|\Gamma^\alpha F(s,\,\cdot\,)\|_2\,ds+\sum_{|\alpha|\le N}\sum_{j,k=0}^{3+d}\int_0^t\|[\gamma^{jk},\Gamma^\alpha]\partial_j\partial_k u(s,\,\cdot\,)\|_2\,ds \\ +\sum_{|\alpha|\le N}\sum_{j,k=0}^{3+d}\int_0^t\|\gamma^{jk}[\Gamma^\alpha,\partial_j\partial_k]u(s,\,\cdot\,)\|_2\,ds+\sum_{|\beta|\le N-1}\|\tilde{\Gamma}^\beta\Box u(t,\,\cdot\,)\|_2. \quad (6.9)$$

7 Global Existence for Robin Waveguides

We will first prove Theorem 1.1. We restrict our proof to the $n = 3$ case for simplicity of notation. To modify the proof for $n \geq 4$, we would simply increase the number of vector fields by the appropriate amounts in the definitions of (7.4) and (7.11) below.

To begin, we must shift the time variable so that our initial conditions occur at time $t = 2B$. This change is necessary so that we may apply our decay estimates. For $\varepsilon > 0$ sufficiently small and N sufficiently large in (1.11), local existence theory gives us that there is a solution to (1.1) on the interval $[2B, 2B + 1]$ (see [7]). We will use this solution to reduce (1.1) to the case of vanishing data (as in [8]). Fix $\eta \in C^\infty(\mathbb{R})$ with $\eta(t) \equiv 1$ for $t \leq 2B + \frac{1}{2}$ and $\eta(t) \equiv 0$ for $t \geq 2B + 1$, and let

$$u_0(t, x, y) = \eta(t)u(t, x, y).$$

This new function solves

$$\Box u_0 = \eta Q(u, u', u'') + [\Box, \eta]u$$

with Robin boundary condition

$$(\alpha(y)u_0 + \partial_\nu u_0)|_{\partial\Omega} = 0, \quad \text{with } \alpha(y) \geq 0 \text{ and } \int_{\partial\Omega} \alpha > 0.$$

Local existence results also give that

$$\sup_t \sum_{|\alpha| \leq N} \|\tilde{\Gamma}^\alpha u_0(t, \,\cdot\,)\|_{L^2(\mathbb{R}^3 \times \Omega)} \leq C_0 \varepsilon. \tag{7.1}$$

We have thus constructed an equivalent problem, where u solves (1.1) for $2B < t < T$ if and only if $w = u - u_0$ solves

$$\begin{cases} \Box w = (1 - \eta)Q(u_0 + w, u_0' + w', u_0'' + w'') - [\Box, \eta](u_0 + w), \\ \alpha(y)w + \partial_\nu w(t, x, y) = 0, \quad y \in \partial\Omega, \\ w(t, x, y) = 0, \quad t \leq 2B \end{cases} \tag{7.2}$$

where the conditions on α remain as above.

Our next step is to solve (7.2) on $[0, T_\varepsilon)$ using a Picard iteration. We begin with $w_0 \equiv 0$. Our w_k is then defined recursively as the solution of

$$\begin{cases} \Box w_k = (1-\eta)Q(u_0 + w_{k-1}, u_0' + w_{k-1}', u_0'' + w_k'') - [\Box, \eta](u_0 + w_k), \\ \alpha(y) w_k + \partial_\nu w_k(t, x, y) = 0, \quad y \in \partial\Omega, \\ w_k(t, x, y) = 0, \quad t \le 2B, \end{cases} \tag{7.3}$$

where $k = 1, 2, 3, \dots$.

We define

$$M_k(T) = \sup_{2B \le t \le T} \Big(\sum_{|\alpha| \le 5d+24} \|\tilde{\Gamma}^\alpha \partial w_k(t, \cdot\,)\|_2 + m \sum_{|\alpha| \le 5d+24} \|\tilde{\Gamma}^\alpha w_k(t, \cdot\,)\|_2 \tag{7.4}$$
$$+ \sum_{|\beta| \le \frac{5d+26}{2}} (1+t) \sup_{x,y} |\tilde{\Gamma}^\beta w_k(t, x, y)| \Big)$$
$$= I_k(T) + II_k(T) + III_k(T). \tag{7.5}$$

We first use induction to show that

$$M_k(T_\varepsilon) \le 4C_1 \varepsilon \tag{7.6}$$

for ε sufficiently small. C_1 is a uniform constant which can be thought of as 10 times greater than $(1 + C_0)$ times the constants from (5.1) and (6.9). When $k = 1$, (7.6) follows from (1.11), the well-known local estimates, (5.1), (6.9), and Gronwall's inequality.

We assume that (7.6) holds in the $k - 1$ case, and try to show that it holds for k.

We first apply (6.9) to $I_k + II_k$, and let $\delta \leq 1$, to obtain

$$
\begin{aligned}
&I_k(T_\varepsilon) + II_k(T_\varepsilon) \\
&\quad \leq C \int_0^{T_\varepsilon} \sum_{|\beta| \leq \frac{5d+24}{2}} \|\tilde{\Gamma}^\beta \partial^\delta (u_0 + w_{k-1})(t,\,\cdot\,)\|_\infty \sum_{|\alpha| \leq 5d+24} \|\tilde{\Gamma}^\alpha \partial^\delta (u_0 + w_k)(t,\,\cdot\,)\|_2 \, dt \\
&\quad + C \int_0^{T_\varepsilon} \sum_{|\beta| \leq \frac{5d+24}{2}} \|\tilde{\Gamma}^\beta \partial^\delta (u_0 + w_k)(t,\,\cdot\,)\|_\infty \sum_{|\alpha| \leq 5d+24} \|\tilde{\Gamma}^\alpha \partial^\delta (u_0 + w_{k-1})(t,\,\cdot\,)\|_2 \, dt \\
&\quad + C \int_0^{T_\varepsilon} \sum_{|\beta| \leq \frac{5d+24}{2}} \|\tilde{\Gamma}^\beta \partial^\delta (u_0 + w_{k-1})(t,\,\cdot\,)\|_\infty \sum_{|\alpha| \leq 5d+24} \|\tilde{\Gamma}^\alpha \partial^\delta (u_0 + w_{k-1})(t,\,\cdot\,)\|_2 \, dt \\
&\quad + C \sup_{0 \leq t \leq T_\varepsilon} \sum_{|\beta| \leq \frac{5d+24}{2}} \|\tilde{\Gamma}^\beta \partial^\delta (u_0 + w_{k-1})(t,\,\cdot\,)\|_\infty \sum_{|\alpha| \leq 5d+24} \|\tilde{\Gamma}^\alpha \partial^\delta (u_0 + w_k)(t,\,\cdot\,)\|_2 \\
&\quad + C \sup_{0 \leq t \leq T_\varepsilon} \sum_{|\beta| \leq \frac{5d+24}{2}} \|\tilde{\Gamma}^\beta \partial^\delta (u_0 + w_k)(t,\,\cdot\,)\|_\infty \sum_{|\alpha| \leq 5d+24} \|\tilde{\Gamma}^\alpha \partial^\delta (u_0 + w_{k-1})(t,\,\cdot\,)\|_2 \\
&\quad + C \sup_{0 \leq t \leq T_\varepsilon} \sum_{|\beta| \leq \frac{5d+24}{2}} \|\tilde{\Gamma}^\beta \partial^\delta (u_0 + w_{k-1})(t,\,\cdot\,)\|_\infty \sum_{|\alpha| \leq 5d+24} \|\tilde{\Gamma}^\alpha \partial^\delta (u_0 + w_{k-1})(t,\,\cdot\,)\|_2 \\
&\qquad\qquad + C_1 \varepsilon + \sum_{|\alpha| \leq 5d+24} \int_{2B+(1/2)}^{2B+1} \|\Gamma^\alpha \partial_t w_k(t,\,\cdot\,)\|_2 \, dt. \qquad (7.7)
\end{aligned}
$$

Here we let

$$\gamma^{jk}(t,x,y) = -\sum_{l=0}^{n+d} A_l^{jk} \partial_l w_{k-1}$$

where we satisfy (6.7) by referring to (1.13).

Before we proceed, we must mention one more result which we will use to gain further control of (7.7). Recall that our nonlinearity Q depends on u. When $m = 0$, $\delta = 0$ and $\tilde{\Gamma} = \Gamma$, we must also use that $\|u(t,\cdot)\|_{L^2}$ is bounded by the left side of (6.8) to control the second factor of the first six terms of (7.7). When $m > 0$, this factor is part of the definition of (7.4), but this is not the case for $m = 0$. In the Dirichlet case in [19], the bound is true due to the Poincaré lemma; however, the proof of this lemma does not hold for Robin boundary conditions. In this case, we can use the Rayleigh Quotient. Recall that for the Robin boundary condition,

$$\lambda_1 = \min \Big\{ \frac{\int_\Omega |\nabla h|^2 \, dy + \int_{\partial\Omega} \alpha h^2 \, dS}{\int_\Omega h^2 \, dy} \Big\},$$

for any test function h. In fact, the minimum is achieved by the first eigenfunction of Δ_Ω. We can thus attain the required bound, since we have restricted ourselves to the case where $\lambda_1 > 0$.

Recalling that u_0 is supported only on $[2B, 2B+1]$, we can then apply (8.1), (7.4), the inductive hypothesis, and Gronwall's inequality to see that

$$\begin{aligned} I_k(T_\varepsilon) + II_k(T_\varepsilon) &\le C_1\varepsilon + C\varepsilon^2 + C\varepsilon M_{k-1}(T_\varepsilon) + C\varepsilon M_k(T_\varepsilon) \\ &\quad + C[(M_{k-1}(T_\varepsilon))^2 + M_{k-1}(T_\varepsilon)M_k(T_\varepsilon)]\Big(1 + \int_{2B}^{T_\varepsilon} \frac{1}{(1+t)^{3/2}}\,dt\Big) \\ &\le C_1\varepsilon + C\varepsilon^2 + 16CC_1^2\varepsilon^2 + C\varepsilon M_k(T_\varepsilon) + 4CC_1\varepsilon M_k(T_\varepsilon). \end{aligned} \tag{7.8}$$

To control $III_k(T_\varepsilon)$, we first use (5.1). Let $A_j = [2^{j-1}, 2^{j+1}] \cap [2B, T_\varepsilon]$ and $\delta \le 1$. We have

$$\begin{aligned} &III_k(T_\varepsilon) \\ &\le \sum_j \sup_{A_j} 2^j \sum_{|\beta|\le \frac{5d+20}{2}} \|\tilde\Gamma^\beta \partial^\delta (u_0 + w_{k-1})(\tau,\,\cdot\,)\|_\infty \sum_{|\alpha|\le 5d+20} \|\tilde\Gamma^\alpha \partial^\delta (u_0 + w_k)(\tau,\,\cdot\,)\|_2 \\ &+ \sum_j \sup_{A_j} 2^j \sum_{|\beta|\le \frac{5d+20}{2}} \|\tilde\Gamma^\beta \partial^\delta (u_0 + w_k)(\tau,\,\cdot\,)\|_\infty \sum_{|\alpha|\le 5d+20} \|\tilde\Gamma^\alpha \partial^\delta (u_0 + w_{k-1})(\tau,\,\cdot\,)\|_2 \\ &+ \sum_j \sup_{A_j} 2^j \sum_{|\beta|\le \frac{5d+20}{2}} \|\tilde\Gamma^\beta \partial^\delta (u_0 + w_{k-1})(\tau,\,\cdot\,)\|_\infty \sum_{|\alpha|\le 5d+20} \|\tilde\Gamma^\alpha \partial^\delta (u_0 + w_{k-1})(\tau,\,\cdot\,)\|_2 \\ &+ Ct^{3/2} \sum_{|\beta|\le \frac{5d+20}{2}} \|\tilde\Gamma^\beta \partial^\delta (u_0 + w_{k-1})(\tau,\,\cdot\,)\|_\infty \sum_{|\alpha|\le 5d+20} \|\tilde\Gamma^\alpha \partial^\delta (u_0 + w_k)(\tau,\,\cdot\,)\|_2 \\ &+ Ct^{3/2} \sum_{|\beta|\le \frac{5d+20}{2}} \|\tilde\Gamma^\beta \partial^\delta (u_0 + w_k)(\tau,\,\cdot\,)\|_\infty \sum_{|\alpha|\le 5d+20} \|\tilde\Gamma^\alpha \partial^\delta (u_0 + w_{k-1})(\tau,\,\cdot\,)\|_2 \\ &+ Ct^{3/2} \sum_{|\beta|\le \frac{5d+20}{2}} \|\tilde\Gamma^\beta \partial^\delta (u_0 + w_{k-1})(\tau,\,\cdot\,)\|_\infty \sum_{|\alpha|\le 5d+20} \|\tilde\Gamma^\alpha \partial^\delta (u_0 + w_{k-1})(\tau,\,\cdot\,)\|_2 \\ &\qquad + C_1\varepsilon + C \sup_{2B+(1/2)\le t\le 2B+1} \sum_{|\alpha|\le 5d+20} \|\tilde\Gamma^\alpha \partial_t w_k(t,\,\cdot\,)\|_2. \end{aligned} \tag{7.9}$$

We then use (8.1) and (7.4) as well as the control gained in (7.8) to obtain

$$\begin{aligned} III_k(T_\varepsilon) &\le 2C_1\varepsilon + C\varepsilon^2 + C\varepsilon M_{k-1}(T_\varepsilon) + C\varepsilon M_k(T_\varepsilon) \\ &\qquad + C[M_k(T_\varepsilon)M_{k-1}(T_\varepsilon) + (M_{k-1}(T_\varepsilon))^2]. \end{aligned}$$

The inductive hypothesis then gives us

$$III_k(T_\varepsilon) \le 2C_1\varepsilon + C\varepsilon^2 + 16CC_1^2\varepsilon^2 + C\varepsilon M_k(T_\varepsilon) + 4CC_1\varepsilon M_k(T_\varepsilon). \tag{7.10}$$

Both (7.8) and (7.10) give us (7.6) for ε sufficiently small and for T_ε as in (1.14) where κ must be sufficiently small as well.

Our proof of Theorem 1.1 is complete if we show that our sequence w_k converges for $t \in [2B, T_\varepsilon)$. We define

$$\begin{aligned} &A_k(T) \\ &= \sup_{2B\le t\le T}\Big(\sum_{|\alpha|\le 5d+24} \|\tilde\Gamma^\alpha \partial(w_k - w_{k-1})(t,\,\cdot\,)\|_2 + m \sum_{|\alpha|\le 5d+24} \|\tilde\Gamma^\alpha (w_k - w_{k-1})(t,\,\cdot\,)\|_2 \\ &\qquad + \sum_{|\beta|\le \frac{5d+26}{2}} (1+t)\sup_{x,y} |\tilde\Gamma^\beta (w_k - w_{k-1})(t,x,y)|\Big). \end{aligned} \tag{7.11}$$

If we argue exactly as in the proof of (7.6), it follows that

$$A_k(T_\varepsilon) \le \frac{1}{2} A_{k-1}(T_\varepsilon).$$

This and the fact that we have a bounded sequence is enough to show that the sequence is Cauchy, and thus is convergent. □

8 Almost Global Existence for Neumann Waveguides

We will now prove Theorem 1.2 and Theorem 1.3. However, for the one-dimensional case, we will simply point out the minor changes that would be necessary to adapt the proof of the general case.

As in the previous section, we shift the time variable by $2B$ so that our initial conditions occur at $t = 2B$. We will again use the existence of a local solution to (1.4) on $[2B, 2B+1]$ to reduce to an equivalent problem with vanishing data. Fix $\eta \in C^\infty(\mathbb{R})$ with $\eta(t) \equiv 1$ for $t \le 2B + \frac{1}{2}$ and $\eta(t) \equiv 0$ for $t \ge 2B+1$, and let

$$u_0(t,x,y) = \eta(t)u(t,x,y).$$

This new function solves

$$\Box u_0 = \eta Q(\partial_{t,x} u, \partial^2_{t,x} u) + [\Box, \eta]u$$

with Neumann boundary condition

$$\partial_\nu u_0(t,x,y)|_{y \in \partial\Omega} = 0.$$

Local existence results also give that

$$\sup_t \sum_{|\alpha| \le N} \|\tilde{\Gamma}^\alpha u_0(t,\,\cdot\,)\|_{L^2(\mathbb{R}^3 \times \Omega)} \le C_0 \varepsilon. \tag{8.1}$$

As in the previous section, we have constructed an equivalent problem, where u solves (1.4) for $2B < t < T$ if and only if $w = u - u_0$ solves

$$\begin{cases} \Box w = (1-\eta)\bar{Q}(\partial_{t,x}(u_0+w), \partial^2_{t,x}(u_0+w)) - [\Box, \eta](u_0+w), \\ \partial_\nu w(t,x,y) = 0, \quad y \in \partial\Omega, \\ w(t,x,y) = 0, \quad t \le 2B. \end{cases} \tag{8.2}$$

We of course instead use $\tilde{Q}$ in (8.2) for Theorem 1.3.

We can now solve (8.2) on $[0, T_\varepsilon)$ using a Picard iteration, where $w_0 \equiv 0$. Our w_k is then defined recursively as the solution of

$$\begin{cases} \Box w_k = (1-\eta)\bar{Q}(\partial_{t,x}(u_0 + w_{k-1}), \partial^2_{t,x}(u_0 + w_k)) - [\Box, \eta](u_0 + w_k), \\ \partial_\nu w_k(t,x,y) = 0, \quad y \in \partial\Omega, \\ w_k(t,x,y) = 0, \quad t \le 2B, \end{cases} \tag{8.3}$$

where $k = 1, 2, 3, \ldots$.

We let

$$M_k(T) = \sup_{2B \le t \le T} \Big(\sum_{|\alpha| \le 5d+24} \|\tilde{\Gamma}^\alpha \partial_{t,x,y} w_k(t,\,\cdot\,)\|_2 + \sum_{|\beta| \le \frac{5d+26}{2}} (1+t) \sup_{x,y} |\tilde{\Gamma}^\beta \partial_{t,x} w_k(t,x,y)| \Big) \tag{8.4}$$

$$= I_k(T) + II_k(T).$$

For the proof of Theorem 1.3, we modify (8.4) to include $\partial_y w_k$ in $II_k(T)$.

We first use induction to show that

$$M_k(T_\varepsilon) \le 4C_1\varepsilon \tag{8.5}$$

for ε sufficiently small. As in the previous section, C_1 is a uniform constant which can be thought of as 10 times greater than $(1 + C_0)$ times the constants from (5.7) and (6.5). When $k = 1$, (8.5) follows from (1.11), the well-known local estimates, (5.7), (6.5), and Gronwall's inequality.

We now assume that (8.5) holds for $k - 1$, and try to show that it holds for k. We first apply (6.5) to I_k to obtain

$$\begin{aligned}
&I_k(T_\varepsilon)\\
&\le C\int_0^{T_\varepsilon}\sum_{|\beta|\le\frac{5d+24}{2}}\|\tilde\Gamma^\beta\partial_{t,x}(u_0+w_{k-1})(t,\,\cdot\,)\|_\infty\sum_{|\alpha|\le 5d+24}\|\tilde\Gamma^\alpha\partial_{t,x}(u_0+w_k)(t,\,\cdot\,)\|_2\,dt\\
&\quad+C\int_0^{T_\varepsilon}\sum_{|\beta|\le\frac{5d+24}{2}}\|\tilde\Gamma^\beta\partial_{t,x}(u_0+w_k)(t,\,\cdot\,)\|_\infty\sum_{|\alpha|\le 5d+24}\|\tilde\Gamma^\alpha\partial_{t,x}(u_0+w_{k-1})(t,\,\cdot\,)\|_2\,dt\\
&\quad+C\int_0^{T_\varepsilon}\sum_{|\beta|\le\frac{5d+24}{2}}\|\tilde\Gamma^\beta\partial_{t,x}(u_0+w_{k-1})(t,\,\cdot\,)\|_\infty\sum_{|\alpha|\le 5d+24}\|\tilde\Gamma^\alpha\partial_{t,x}(u_0+w_{k-1})(t,\,\cdot\,)\|_2\,dt\\
&\quad+C\sup_{0\le t\le T_\varepsilon}\sum_{|\beta|\le\frac{5d+24}{2}}\|\tilde\Gamma^\beta\partial_{t,x}(u_0+w_{k-1})(t,\,\cdot\,)\|_\infty\sum_{|\alpha|\le 5d+24}\|\tilde\Gamma^\alpha\partial_{t,x}(u_0+w_k)(t,\,\cdot\,)\|_2\\
&\quad+C\sup_{0\le t\le T_\varepsilon}\sum_{|\beta|\le\frac{5d+24}{2}}\|\tilde\Gamma^\beta\partial_{t,x}(u_0+w_k)(t,\,\cdot\,)\|_\infty\sum_{|\alpha|\le 5d+24}\|\tilde\Gamma^\alpha\partial_{t,x}(u_0+w_{k-1})(t,\,\cdot\,)\|_2\\
&\quad+C\sup_{0\le t\le T_\varepsilon}\sum_{|\beta|\le\frac{5d+24}{2}}\|\tilde\Gamma^\beta\partial_{t,x}(u_0+w_{k-1})(t,\,\cdot\,)\|_\infty\sum_{|\alpha|\le 5d+24}\|\tilde\Gamma^\alpha\partial_{t,x}(u_0+w_{k-1})(t,\,\cdot\,)\|_2\\
&\qquad\qquad+C_1\varepsilon+\sum_{|\alpha|\le 5d+24}\int_{2B+(1/2)}^{2B+1}\|\Gamma^\alpha\partial_t w_k(t,\,\cdot\,)\|_2\,dt. \qquad (8.6)
\end{aligned}$$

Here we let

$$\gamma^{jk}(t,x,y)=-\sum_{l=0}^{3}A_l^{jk}\partial_l w_{k-1}$$

for Theorem 1.2 or

$$\gamma^{jk}(t,x,y)=-\sum_{l=0}^{4}A_l^{jk}\partial_l w_{k-1}$$

for Theorem 1.3. In the latter (6.3) is satisfied by referring to (1.12). In the former, (6.3) is not needed explicitly since the A_l^{jk} vanish for $l\ge 3+d$.

Recalling that u_0 is supported only on $[2B,2B+1]$, we can then apply (8.1), (8.4), the inductive hypothesis, and Gronwall's inequality to see that

$$\begin{aligned}
I_k(T_\varepsilon)&\le C_1\varepsilon+C\varepsilon^2+C\varepsilon M_{k-1}(T_\varepsilon)+C\varepsilon M_k(T_\varepsilon)\\
&\qquad+C[(M_{k-1}(T_\varepsilon))^2+M_{k-1}(T_\varepsilon)M_k(T_\varepsilon)]\Big(1+\int_{2B}^{T_\varepsilon}\frac{1}{1+t}\,dt\Big)\\
&\le C_1\varepsilon+C\varepsilon^2+16CC_1^2\varepsilon^2(1+\log(1+T_\varepsilon))+C\varepsilon M_k(T_\varepsilon)\\
&\qquad+4CC_1\varepsilon(1+\log(1+T_\varepsilon))M_k(T_\varepsilon).
\end{aligned} \qquad (8.7)$$

To control $II_k(T_\varepsilon)$, we first use (5.7) (where the second term in the right side of (5.7) is in fact controlled by the preceding term). For the proof of Theorem 1.3, we also apply (5.12). Let $A_j = [2^{j-1}, 2^{j+1}] \cap [2B, T_\varepsilon]$. We then have

$$\begin{aligned}
&II_k(T_\varepsilon)\\
&\le \sum_j \sup_{\tau\in A_j} 2^j \sum_{|\beta|\le\frac{5d+21}{2}} \|\tilde{\Gamma}^\beta \partial_{t,x}(u_0+w_{k-1})(\tau,\,\cdot\,)\|_\infty \sum_{|\alpha|\le 5d+21} \|\tilde{\Gamma}^\alpha \partial_{t,x}(u_0+w_k)(\tau,\,\cdot\,)\|_2\\
&+ \sum_j \sup_{\tau\in A_j} 2^j \sum_{|\beta|\le\frac{5d+21}{2}} \|\tilde{\Gamma}^\beta \partial_{t,x}(u_0+w_k)(\tau,\,\cdot\,)\|_\infty \sum_{|\alpha|\le 5d+21} \|\tilde{\Gamma}^\alpha \partial_{t,x}(u_0+w_{k-1})(\tau,\,\cdot\,)\|_2\\
&+ \sum_j \sup_{\tau\in A_j} 2^j \sum_{|\beta|\le\frac{5d+21}{2}} \|\tilde{\Gamma}^\beta \partial_{t,x}(u_0+w_{k-1})(\tau,\,\cdot\,)\|_\infty \sum_{|\alpha|\le 5d+21} \|\tilde{\Gamma}^\alpha \partial_{t,x}(u_0+w_{k-1})(\tau,\,\cdot\,)\|_2\\
&\qquad + C_1\varepsilon + C \sup_{2B+(1/2)\le t\le 2B+1} \sum_{|\alpha|\le 5d+21} \|\tilde{\Gamma}^\alpha \partial_t w_k(t,\,\cdot\,)\|_2. \quad (8.8)
\end{aligned}$$

where for Theorem 1.3 we have $\partial_{t,x,y}$ in place of $\partial_{t,x}$ in the appropriate places on the right-hand side. We then use (8.1) and (8.4) as well as the control gained in (8.7) to obtain

$$\begin{aligned}
II_k(T_\varepsilon) &\le 2C_1\varepsilon + C\varepsilon^2 + C\varepsilon M_{k-1}(T_\varepsilon) + C\varepsilon M_k(T_\varepsilon)\\
&\qquad + C\Big(1+\log(2+T_\varepsilon)\Big)[M_k(T_\varepsilon)M_{k-1}(T_\varepsilon) + (M_{k-1}(T_\varepsilon))^2].
\end{aligned}$$

The $\log(2+T_\varepsilon)$ term appears both from (8.7) as well as the fact that our sum is over $O(\log(2+T_\varepsilon))$ choices of j where $2^{j-1}\le T_\varepsilon$. The inductive hypothesis then gives us

$$\begin{aligned}
II_k(T_\varepsilon) &\le 2C_1\varepsilon + C\varepsilon^2 + 16CC_1^2\varepsilon^2(1+\log(2+T_\varepsilon)) + C\varepsilon M_k(T_\varepsilon)\\
&\qquad + 4CC_1\varepsilon(1+\log(2+T_\varepsilon))M_k(T_\varepsilon). \quad (8.9)
\end{aligned}$$

Both (8.7) and (8.9) give us (8.5) for ε sufficiently small and for T_ε as in (1.14) where κ must be sufficiently small as well.

Our proofs of Theorems 1.2 and 1.3 will be complete if we can show that our sequence w_k converges for $t\in[2B, T_\varepsilon)$. We define

$$\begin{aligned}
A_k(T) = \sup_{2B\le t\le T}\Big(&\sum_{|\alpha|\le 5d+24} \|\tilde{\Gamma}^\alpha \partial_{t,x,y}(w_k-w_{k-1})(t,\,\cdot\,)\|_2\\
&+ \sum_{|\beta|\le(5d+26)/2} (1+t)\sup_{x,y} |\tilde{\Gamma}^\beta \partial_{t,x}(w_k-w_{k-1})(t,x,y)|\Big). \quad (8.10)
\end{aligned}$$

If we argue exactly as in the proof of (8.5), it follows that

$$A_k(T_\varepsilon) \leq \frac{1}{2} A_{k-1}(T_\varepsilon).$$

This and the fact that we have a bounded sequence is enough to show that the sequence is Cauchy, and thus is convergent. $\square$

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Vita

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