REAL JOHNSON-WILSON THEORIES AND NON-IMMERSIONS OF PROJECTIVE SPACES

by

Romie Banerjee

A dissertation submitted to the Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy

Baltimore, Maryland

April, 2010

Abstract

In this paper we compute the second real Johnson-Wilson cohomology $ER(2)$ of the odd-dimensional real projective space RP^{16K+9}. This enables us to solve certain non-immersion problems of projective spaces using obstructions in $ER(2)$-cohomology.

Advisors: J. Michael Boardman and W. Stephen Wilson

Acknowledgement

I would like to thank my advisor Dr. W. Stephen Wilson for introducing me to this problem and for his help and guidance during the work on this project. I would also like to thank my other advisor, Dr. Michael J. Boardman for his helpful advice and reading through my thesis. This project would never have completed without their assistance.

I would also like to thank Dr. Jack Morava for giving several helpful suggestions. I owe many thanks to Andrew Salch for helping me learn the algebro-geometric point of view of stable homotopy theory and giving valuable guidance during my research. I also thank all the topology postdocs, Michael Ching, Andrew Salch, Rekha Santhanam and Snigdhayan Mahanta. Our regular discussions contributed immensely to my understanding of homotopy theory.

Additionally, I thank all my fellow graduate students. I owe special thanks John Baber, Laarni Lucero, Joel Kramer and Thomas Wright.

Finally, I would like to thank my family for their love and support.

Contents

1 Introduction **1**

2 G-spectra and G-cohomology theories **3**

2.1 The Tate Spectrum 4

2.2 Filtrations 7

2.3 The Borel and Tate spectral sequences 8

2.4 A morphism of spectral sequences 9

3 Real-oriented spectra **12**

3.1 $E\mathbb{R}(n)$ 15

3.2 The elements $\sigma^{\pm 1}$ and a 17

3.3 The Tate spectral sequence for $E\mathbb{R}(n)$ 18

3.4 The Borel spectral sequence for $E\mathbb{R}(n)$ 19

3.5 Strong completion and cofibrations 21

3.6 The boundary map 24

4 Cohomology of projective spaces **24**

5 The Bockstein spectral sequence **26**

5.1 The spectral sequence for $ER(2)^*$ 29

5.2 The Bockstein Spectral Sequence for $ER(2)^*(RP^\infty)$ 32

6 $ER(2)^*(RP^{16K+9}, *)$ **35**

6.1 The Bockstein spectral sequence for $ER(2)^*(RP^{16K+9})$ 37

7 Non-Immersions **44**

7.1 Products with an odd space 46

References **48**

1 Introduction

Atiyah [2] developed K-theory with Reality KR, which is in a sense a mixture of the K-theory of real vector bundles KO, the K-theory of self-conjugate bundles KSC, and the K-theory of G-vector bundles over G-spaces K_G. A *Real* vector bundle over a space X with involution($x \mapsto \overline{x}$) is a complex vector bundle E over X which also has an involution such that the projection $E \to X$ commutes with the involution and the map of fibers $E_x \to E_{\overline{x}}$ is anti-linear.

A special class of $\mathbb{Z}/2$ equivariant cohomology theories are called Real-oriented theories. These are Real-oriented in the sense that any Real bundle is orientable with respect to such a theory. The theory of real complex cobordism was developed by Landweber and Araki ([1],[15]) by taking homotopy fixed points of complex cobordism under complex conjugation. The language of equivariant topology has changed quite substantially ever since then. The equivariant stable homotopy category developed by May-Lewis-Steinberger [14] has provided the basic framework for equivariant topology and cohomology theories for all current work in the field.

The $\mathbb{Z}/2$-equivariant Johnson-Wilson spectrum $E\mathbb{R}(n)$ was first constructed by Hu and Kriz in [9]. Kitchloo and Wilson [12] have used the homotopy fixed point spectrum of this to solve certain non-immersion problems of real projective spaces. The homotopy fixed point spectrum $ER(n)$ is $2^{n+2}(2^n-1)$-periodic compared to the $2(2^n-1)$-periodic $E(n)$. $ER(1)$ is $KO_{(2)}$ and $E(1)$ is $KU_{(2)}$.

Kitchloo and Wilson have demonstrated the existence of a stable cofibration connecting $E(n)$ and $ER(n)$,

$$\Sigma^{\lambda(n)} ER(n) \xrightarrow{\ x\ } ER(n) \longrightarrow E(n) \tag{1}$$

This leads to a Bockstein spectral sequence for x-torsion. It is known that $x^{2^{n+1}-1} = 0$ so there can be only $2^{n+1}-1$ differentials. For the case of our interest $n = 2$ there are only 7 differentials.

From [10] we know that if there is an immersion of RP^b to $\mathbb{R}^c$ then there is an axial map

$$RP^b \times RP^{2^L-c-2} \to RP^{2^L-b-2}. \tag{2}$$

For $b = 2n$ and $c = 2k$ Don Davis shows in [4] that there is no such map when $n = m+\alpha(m)-1$ and $k = 2m-\alpha(m)$, where $\alpha(m)$ is the number of ones in the binary expression of m by finding an obstruction to James's map (2) in $E(2)$-cohomology. Kitchloo and Wilson get new non-immersion results by computing obstructions in $ER(2)$-cohomology. In this paper we extend Kitchloo-Wilson's results by computing the $ER(2)$-cohomology of the odd projective space RP^{16K+9}. This will give us newer non-immersion results.

In the first section we introduce the preliminary constructions of equivariant stable homotopy theory from [14] and [6]. We introduce the Tate and Borel spectra associated to a G-equivariant spectrum and introduce the filtrations necessary to construct the Tate and Borel spectral sequences.

In the second section define a Real-oriented spectrum following [9] and construct the Real complex cobordism spectrum $M\mathbb{R}$. This is an E_∞-$\mathbb{Z}/2$-ring spectrum. We use the language of [5] to construct Real versions of various complex oriented spectra. In particular, we construct $E\mathbb{R}(n)$ as the Real-oriented spectrum corresponding to the complex-oriented $E(n)$. Next we compute the Borel and Tate spectral sequences for $E\mathbb{R}(n)$ and reproduce the cofibration (1).

In section 3 we use calculations from section 2 to compute the $E\mathbb{R}(n)$-cohomology of $\mathbb{C}P^\infty$. In section 4 we begin calculations for the Bockstein spectral sequence induced by (1). We focus on the cohomology theory $ER(2)$. Finally we recall essential results from [12] and [13] and embark on the computation of the Bockstein spectral sequence for $ER(2)^*(RP^{16K+9})$.

In section 5 we use results from the previous section to obtain obstructions to the axial map of James (2) for different values of b and c, thus resulting in new non-immersions. The main results are the following.

Theorem 1.1. *A 2-adic basis of* $ER(2)^{8*}(RP^{16K+9}, *)$ *is given by the elements*

$$\alpha^k u^j, \ (k \geq 0, 1 \leq j \leq 8K+4)$$

$$v_2^4 \alpha^k u^j, \ (k \geq 1, 1 \leq j \leq 8K+4)$$

$$v_2^4 u^j, \ (4 \leq j \leq 8K+4)$$

$$x\alpha^k i_{16K+9}, \ x v_2^4 \alpha^k i_{16K+9}, \ (k \geq 0)$$

Theorem 1.2. *If* $(m, \alpha(m)) \equiv$ *(6,2) or (1,0) mod 8,*

$RP^{2(m+\alpha(m)-1)}$ *does not immerse in* $\mathbb{R}^{2(2m-\alpha(m))+1}$.

2 G-spectra and G-cohomology theories

We will be working in the equivariant stable homotopy category of Lewis-May-Steinberger. Let us begin by recalling some definitions. Let G be a finite group.

A complete universe U is an infinite dimensional real inner product space with G acting through isometries such that U contains a countably infinite sum of all irreducible representations of G as subspace.

A G-spectrum k_G indexed on U associates a based G-space $k_G(V)$ to each finite dimensional G-subspace $V \subset U$ such that for any two G subspaces V and W of U with $V \subset W$ the structure maps $k_G(V) \to \Omega^{W-V} k_G(W)$ are G-homeomorphisms. Here, $W - V$ denotes the orthogonal complement of V in W. For any V, $S(V)$ denotes the unit sphere in V and S^V denotes the one-point compactification $V \cup \infty$ of V, with ∞ as basepoint. Then for a G-space Y, $\Omega^V Y$ denotes the G-space of all based maps $S^V \to Y$.

Let $U = \oplus(V_i)^\infty$ for a complete set of distinct irreducible representations V_i. Then $RO(G,U)$ is the free abelian group generated by the V_i. We define the $RO(G,U)$-graded homology and cohomology theory associated to any G-spectrum k_G. For any virtual representation $a = V - W$ there are sphere G-spectra $S^a = \Sigma^{-W} S^V$, and we let

$$k_G^a(X) = [X \wedge S^{-a}, k_G]_G$$

and

$$k_a^G(X) = [S^a, X \wedge k_G]_G$$

for any G-spectrum X. For a G-space Y let $k_G^a(Y) = k_G^a(\Sigma^\infty Y)$ and similarly for homology; here Σ^∞ is the functor from G-spaces to G-spectra that is left adjoint to the $0th$ space functor [14].

A map of G-spectra $f : E \to F$ is a stable equivalence when $f_*^H : \pi_n(E^H) \to \pi_n(F^H)$ is an isomorphism for all n and all $H \leq G$.

2.1 The Tate Spectrum

We recall necessary definitions and facts from [6]. The various equivariant spectra associated to any given G-spectrum k_G are displayed in the Tate diagram given below. Let X_+ be the disjoint union of the G-space X with a fixed base point and let EG be a contractible free G-space. Let $\widetilde{EG}$ be the unreduced suspension of EG. Then there is a cofibering

$$EG_+ \to S^0 \to \widetilde{EG} \to \Sigma EG_+. \tag{3}$$

Let $F(EG_+, k_G)$ be the function G-spectrum of maps from EG_+ to k_G with diagonal G action. The projection $EG_+ \to S^0$ induces a map of G-spectra

$$\epsilon : k_G = F(S^0, k_G) \to F(EG_+, k_G).$$

Smashing ϵ with the previous cofibering gives the following map of cofiberings of G-spectra, known as the Tate diagram.

$$\begin{array}{ccccc} k_G \wedge EG_+ & \longrightarrow & k_G & \longrightarrow & k_G \wedge \widetilde{EG} \\ \simeq \big\downarrow & & \big\downarrow & & \big\downarrow \\ F(EG_+, k_G) \wedge EG_+ & \longrightarrow & F(EG_+, k_G) & \longrightarrow & F(EG_+, k_G) \wedge \widetilde{EG} \end{array}$$

Notice that the leftmost vertical arrow is always an equivalence. We define the Borel, geometric and Tate spectra associated to k_G respectively as follows:

$$c(k_G) = F(EG_+, k_G)$$

$$g(k_G) = k_G \wedge \widetilde{EG}$$

$$t(k_G) = F(EG_+, k_G) \wedge \widetilde{EG}$$

$$f(k_G) = k_G \wedge EG_+$$

It follows that, up to equivalence, $t(k_G)$ is the cofiber of the composite map

$$k_G \wedge EG_+ \to k_G \to F(EG_+, k_G).$$

The canonical smash product pairing

$$F(X, Y) \wedge F(X', Y') \to F(X \wedge X', Y \wedge Y')$$

and the equivalences

$$EG_+ \wedge EG_+ \simeq EG_+ \qquad \widetilde{EG} \wedge \widetilde{EG} \simeq \widetilde{EG}$$

give the following proposition.

Proposition 2.1. *If k_G is a ring G-spectrum then $c(k_G)$ and $t(k_G)$ are ring G-spectra and the following part of the Tate diagram is a commutative diagram of ring G-spectra:*

$$\begin{array}{ccc} k_G & \longrightarrow & k_G \wedge \widetilde{EG} \\ \downarrow & & \downarrow \\ c(k_G) & \longrightarrow & t(k_G) \end{array}$$

The unit of $t(k_G)$ is the smash product of the unit of $c(k_G)$ and the canonical map $S^0 \to \widetilde{EG}$.

Proposition 2.2.

$$t(k_G)^n(X) \simeq f(k_G)^n(\Sigma^{-1}\widetilde{EG} \wedge X)$$

We may identify $c(k_G) \to t(k_G)$ with

$$F(\Sigma^{-1}(\widetilde{EG}/S^0), k_G \wedge EG_+) \to F(\Sigma^{-1}\widetilde{EG}, k_G \wedge EG_+)$$

These are maps of ring spectra.

This will give a morphism of the spectral sequences.

Proof From the exact triangle (3) we construct a diagram whose rows are exact triangles.

$$\begin{array}{ccccc} F(S^0, k_G \wedge EG_+) & \longrightarrow & F(\Sigma^{-1}(\widetilde{EG}/S^0), k_G \wedge EG_+) & \longrightarrow & F(\Sigma^{-1}\widetilde{EG}, k_G \wedge EG_+) \\ \| & & \| & & \| \\ k_G \wedge EG_+ & \longrightarrow & F(EG_+, k_G \wedge EG_+) & \longrightarrow & F(\Sigma^{-1}\widetilde{EG}, k_G \wedge EG_+) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ F(EG_+, k_G) \wedge EG_+ & \longrightarrow & F(EG_+, k_G) & \longrightarrow & F(EG_+, k_G) \wedge \widetilde{EG} \end{array}$$

It commutes by naturality of $F(X,Y) \wedge Z \to F(X, Y \wedge Z)$ in X and Z. As the left and center vertical maps are equivalences, so is the one on the right.

2.2 Filtrations

Filter EG by E_kG, EG_+ by E_kG_+ for $k \geq 0$. In particular, $E_0G = G$, $E_{-1}G = \emptyset$, so $E_0G_+ = G_+, E_{-1}G_+ =$ point. The quotients $R_k = E_kG/E_{k-1}G$ have the form $(\bigvee_{(G-e)^{\times k}} S^k) \wedge G_+$.

Filter $\widetilde{EG} = S(EG)$ (unreduced suspension) by $F_k = \widetilde{E_{k-1}G} = S(E_{k-1}G)$ for $k \geq 0$, so $F_0 = S(\emptyset) = S^0$. Then

$$F_k/F_{k-1} \simeq \Sigma R_{k-1} \simeq \left(\bigvee_{(G-e)^{\times k}} S^k \right) \wedge G_+$$

Extend to $F_{-k} = DF_k$, the Spanier-Whitehead dual of F_k, for $k < 0$. Then

$$F_{-k}/F_{-k-1} = DF_k/DF_{k+1} = \Sigma DR_{k+1} = \left(\bigvee_{(G-e)^{\times(k+1)}} S^{-k} \right) \wedge G_+.$$

The case $G = \mathbb{Z}/2$. Then $RO(G) = \mathbb{Z} \oplus \mathbb{Z}\alpha$ where α denotes the sign representation of $\mathbb{Z}/2$. Take $EG = S(\mathbb{R}^{\infty\alpha})$, $E_kG = S(\mathbb{R}^{(k+1)\alpha})$, a k-sphere. $E_{k-1}G \subset E_kG$ is $S(\mathbb{R}^{k\alpha}) \subset S(\mathbb{R}^{(k+1)\alpha})$, induced by $\mathbb{R}^{k\alpha} \subset \mathbb{R}^{(k+1)\alpha}$.

So $E_kG_+/E_{k-1}G_+ \simeq S^{k\alpha} \wedge \mathbb{Z}/2_+ \simeq S^k \wedge \mathbb{Z}/2_+$. For $k \geq 0$, $F_k = S(E_{k-1}G) = SS(\mathbb{R}^{k\alpha}) \simeq S^{k\alpha}$ (generally $SS(V) = S^V$). $F_{k-1} \subset F_k$ is induced by $\mathbb{R}^{(k-1)\alpha} \subset \mathbb{R}^{k\alpha}$, i.e

$$S^{(k-1)\alpha} = S^{(k-1)\alpha} \wedge S^0 \xrightarrow{1 \wedge a} S^{(k-1)\alpha} \wedge S^\alpha = S^{k\alpha}$$

where $a : S^0 \subset S^\alpha$ is induced by $\mathbb{R}^0 \subset \mathbb{R}^\alpha$.

2.3 The Borel and Tate spectral sequences

The Borel spectral sequence for $F(EG_+, k_G)_\star = [EG_+, k_G]_\star$ is constructed from the unrolled exact couple, for $p \geq 0$, with

$$\begin{array}{c} D_1^p(c) = [EG_+/E_{p-1}G_+, k_G]_\star \\ \downarrow \\ E_1^p(c) = [E_pG_+/E_{p-1}G_+, k_G]_\star \simeq [(\bigvee S^p) \wedge G_+, k_G]_\star = \oplus[G_+, k_G]_{\star+p} \end{array}$$

Here, the $\star$ denotes the $RO(G)$-grading.

Use $E_pG_+ \simeq \Sigma^{-1}F_{p+1}$ and Proposition 2.2 to rewrite this as

$$\begin{array}{c} D_1^p = [(EG_+/E_{p-1}G_+), k_G]_\star = [\Sigma^{-1}(\widetilde{EG}/F_p), k_G \wedge EG_+]_\star \\ \downarrow \\ E_1^p = [\Sigma^{-1}(F_{p+1}/F_p), k_G]_\star \simeq \oplus[G_+, k_G]_{\star+p} = [\Sigma^{-1}(F_{p+1}/F_p), k_G \wedge EG_+]_\star \end{array}$$

For $p < 0$ we take $D_1^p = D_1^0, E_1^p = 0$.

This is a half plane spectral sequence, conditionally convergent to $[EG_+, k_G]_\star = k_G^\star(EG_+)$. It is also multiplicative, as EG_+ is filtered by skeletons.

Theorem 2.1. *$E_2^{p,q}(c) = H^p(G; k_G^q)$, for $p \geq 0$ and $q \in RO(G)$.*

Proof $E_1^{p,q}(c) = [R_p, k_G]^{p+q} = \mathrm{Hom}_{\mathbb{Z}[G]}(H_p(R_p), k_G^q)$ for $p \geq 0$, since R_p is a wedge of p-spheres. The differential d_1 is induced by homomorphisms $H_{p+1}(R_{p+1}) \to H_p(R_p)$, which form a resolution of $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules.

For the Tate spectral sequence we filter $\Sigma^{-1}\widetilde{EG}$ by $\Sigma^{-1}F_{p+1}$, ($p \in \mathbb{Z}$), for

$$t(k_G)_\star = (F(EG_+, k_G) \wedge \widetilde{EG})_\star = [\Sigma^{-1}\widetilde{EG}, k_G \wedge EG_+]_\star.$$

It is defined by the unrolled exact couple (all $p \in \mathbb{Z}$) with

$$\begin{array}{c} D_1^p(t) = [\Sigma^{-1}(\widetilde{EG}/F_p), k_G \wedge EG_+]_\star = [\widetilde{EG}/F_p, k_G \wedge EG_+]_{\star-1} \\ \downarrow \\ E_1^p(t) = [\Sigma^{-1}(F_{p+1}/F_p), k_G \wedge EG_+]_\star = [\Sigma^{-1}(F_{p+1}/F_p), k_G]_\star = \oplus[G_+, k_G]_{\star+p} \end{array}$$

This gives a whole plane spectral sequence, conditionally convergent to $\mathrm{colim}_p D_1^p$; it is a multiplicative spectral sequence, since $\Sigma^{-1}\widetilde{EG}$ is filtered by the skeletons. We extend the above resolution of $\mathbb{Z}$ by defining $R_p = \Sigma^{-1}(F_{p+1}/F_p)$ for all $p \in \mathbb{Z}$. Given any $\mathbb{Z}[G]$-module M, the cohomology groups of the chain complex of groups $\mathrm{Hom}_{\mathbb{Z}[G]}(H_p(R_p), M)$ are known as the *Tate cohomology* groups $\widehat{H}^p(G; M)$ of G with coefficients in M.

Theorem 2.2. *$E_2^{p,q}(t) = \widehat{H}^p(G; k_G^q)$, for $p \in \mathbb{Z}$ and $q \in RO(G)$.*

2.4 A morphism of spectral sequences

The filtered map $\Sigma^{-1}\widetilde{EG} \to \Sigma^{-1}(\widetilde{EG}/S^0)$ induces a morphism of spectral sequences from the Borel spectral sequence to the Tate spectral sequence. In more detail, $E_1^p(c) = E_1^p(t)$ for $p \geq 0$, $E_1^p(c) = 0$ for $p < 0$ and

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & D_1^2(c) & \longrightarrow & D_1^1(c) & \longrightarrow & D_1^0(c) & \xrightarrow{=} & D_1^{-1}(c) & \xrightarrow{=} & D_1^{-2}(c) & \longrightarrow & \cdots \\ & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & D_1^2(t) & \longrightarrow & D_1^1(t) & \longrightarrow & D_1^0(t) & \longrightarrow & D_1^{-1}(t) & \longrightarrow & D_1^{-2}(t) & \longrightarrow & \cdots \end{array}$$

The ring structures of $F(\Sigma^{-1}\widetilde{EG}, k_G \wedge EG_+)$ and $F(\Sigma^{-1}(\widetilde{EG}/S^0), k_G \wedge EG_+)$ are inherited from $c(k_G)$ and $t(k_G)$ from Proposition 2.1.

Our third spectral sequence is again obtained by filtering EG_+. We use the unrolled exact couple with

$$\begin{array}{ccc} D^1_p(f) & = & [S, k_G \wedge E_pG_+]_\star \\ & & \downarrow \\ E^1_p(f) & = & [S, k_G \wedge R_p]_\star \end{array}$$

for $p \geq 0$, also $D^1_p(f) = D^1_0(f)$ and $E^1_p(f) = 0$ for $p < 0$.

Since R_p is a wedge of p-spheres, we may write

$$E^1_{p,q}(f) = [S, k_G \wedge R_p]_{p+q} = H_p(R_p) \otimes (k_G)_q$$

for $p \geq 0$.

Theorem 2.3. *$E^2_{p,q}(f) = H_p(G; k^G_q)$ for $p \geq 0$ and $q \in RO(G)$.*

There is a map, using Spanier-Whitehead duality for $p < 0$, from the Tate spectral sequence to this spectral sequence, induced by the following morphism of unrolled exact couples.

$$
\begin{array}{ccccc}
D_1^{p,q}(t) & \longrightarrow & E_1^{p,q}(t) & \longrightarrow & D_1^{p+1,q}(t) \\
\Vert & & \Vert & & \Vert \\
[\Sigma^{-1}\frac{\widetilde{EG}}{F_p}, k_G \wedge EG_+]^{p+q} & \longrightarrow & [\Sigma^{-1}\frac{F_{p+1}}{F_p}, k_G \wedge EG_+]^{p+q} & \xrightarrow{\delta^*} & [\Sigma^{-1}\frac{\widetilde{EG}}{F_{p+1}}, k_G \wedge EG_+]^{p+q+1} \\
\downarrow & & \simeq\downarrow & & \downarrow \\
[\Sigma^{-1}\frac{S^0}{F_p}, k_G]^{p+q} & \longrightarrow & [\Sigma^{-1}\frac{F_{p+1}}{F_p}, k_G]^{p+q} & \xrightarrow{\delta^*} & [\Sigma^{-1}\frac{S^0}{F_{p+1}}, k_G]^{p+q+1} \\
\Vert & & \Vert & & \Vert \\
k_G^{p+q}\left(\Sigma^{-1}\frac{S^0}{F_p}\right) & \longrightarrow & k_G^{p+q}\left(\Sigma^{-1}\frac{F_{p+1}}{F_p}\right) & \xrightarrow{\delta^*} & k_G^{p+q+1}\left(\Sigma^{-1}\frac{S^0}{F_{p+1}}\right) \\
=\downarrow & & =\downarrow & & =\downarrow \\
k^G_{-p-q}\left(\frac{F_{-p}}{S^0}\right) & \longrightarrow & k^G_{-p-q}\left(\frac{F_{-p}}{F_{-p-1}}\right) & \xrightarrow{\delta_*} & k^G_{-p-q-1}\left(\frac{F_{-p-1}}{S^0}\right) \\
=\downarrow & & =\downarrow & & =\downarrow \\
k^G_{-p-q}(\Sigma E_{-p-1}G_+) & \longrightarrow & k^G_{-p-q}\left(\Sigma\frac{E_{-p-1}G_+}{E_{-p-2}G_+}\right) & \xrightarrow{\delta_*} & k^G_{-p-q-1}(\Sigma E_{-p-2}G_+) \\
\Vert & & \Vert & & \Vert \\
k^G_{-p-q-1}(E_{-p-1}G_+) & \longrightarrow & k^G_{-p-q-1}\left(\frac{E_{-p-1}G_+}{E_{-p-2}G_+}\right) & \xrightarrow{\delta_*} & k^G_{-p-q-2}(E_{-p-2}G_+) \\
\Vert & & \Vert & & \Vert \\
D^1_{-p-1,-q}(f) & \longrightarrow & E^1_{-p-1,-q}(f) & \longrightarrow & D^1_{-p-2,-q}(f)
\end{array}
$$

For $p \geq 0$, use $D_1^p(t) \to D_1^{-1}(t) \to D_0^1(f) = D^1_{-p-1}(f)$, $E^1_{-p-1}(f) = 0$

There is a short exact sequence

$$0 \to E_1^p(c) \to E_1^p(t) \to E^1_{-p-1}(f) \to 0$$

for all p, where $E_1^p(c) = 0$ for $p < 0$ and $E^1_{-p-1}(f) = 0$ for $p \geq 0$.

We summarize the relation between Tate cohomology and group (co)homology.

Corollary 2.1. *For any $\mathbb{Z}[G]$-module M,*

$$\widehat{H}^p(G;M) = \begin{cases} H^p(G;M) & \textit{for } p \geq 1 \\ H_{-p-1}(G;M) & \textit{for } p \leq -2 \end{cases}$$

3 Real-oriented spectra

We shall review the theory of Real cobordism and Real-oriented spectra discovered by Landweber and Araki ([1],[15]). All our spectra will be $\mathbb{Z}/2$-equivariant.

Let $\mathbb{S}^1$ denote the unit circle group $S^1 \subset \mathbb{C}^*$ with complex conjugation as involution and a basepoint 1. A *Real space* in the sense of Atiyah [2] is space with an involution. Let $B\mathbb{S}^1$ be the classifying space of S^1, which is the space of all complex lines in $\mathbb{C}^\infty$ with complex conjugation as involution. We have $\Omega B\mathbb{S}^1 \simeq \mathbb{S}^1$ in the category of based $\mathbb{Z}/2$-spaces. Thus, by adjunction we have a canonical equivariant based map

$$\eta : S^{1+\alpha} \to B\mathbb{S}^1$$

where α is the non-trivial one-dimensional representation of $\mathbb{Z}/2$.

Definition 3.1. *Let E be a $\mathbb{Z}/2$-equivariant commutative associative ring spectrum. A Real orientation of E is a map*

$$u : B\mathbb{S}^1 \to \Sigma^{1+\alpha} E$$

which makes the following diagram commute

$$\begin{array}{ccc} S^{1+\alpha} & \xrightarrow{\eta} & B\mathbb{S}^1 \\ & \searrow^{1} & \downarrow{\scriptstyle u} \\ & & \Sigma^{1+\alpha} E \end{array}$$

First we describe the $\mathbb{Z}/2$-spectrum $M\mathbb{R}$ that represents Real complex cobordism. Let $MU(n)$ denote the Thom space of the universal bundle γ_n over $BU(n)$. Complex conjugation induces an action of $\mathbb{Z}/2$ on $MU(n)$. The space $M\mathbb{U}(n)$ (with conjugation action) is placed in dimension $n(1+\alpha)$. The canonical Real bundle γ_n of dimension

n over $BU(n)$ gives maps between Thom spaces,

$$\Sigma^{1+\alpha} BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}$$

which gives our required structure maps. Spectrification makes $M\mathbb{R}$ a $\mathbb{Z}/2$-spectrum. Recall that $RO(\mathbb{Z}/2) = \mathbb{Z} \oplus \mathbb{Z}\alpha$ where α represents the sign representation. For V in $RO(\mathbb{Z}/2)$ define $M\mathbb{R}(V)$ as $\operatorname{colim} \Omega^{n(1+\alpha)-V} M\mathbb{U}(n)$. $M\mathbb{R}$ is a multiplicative Real-oriented spectrum in the sense of [9]. We shall write the coefficient ring $M\mathbb{R}_\star$, the $\star$ denoting the $RO(\mathbb{Z}/2)$-grading.

Proposition 3.1. *The spectrum $M\mathbb{R}$ is Real-oriented.*

Proof As in the complex case, $B\mathbb{S}^1$ is the Thom space of the canonical Real bundle on $B\mathbb{S}^1$, and therefore the $(1+\alpha)$-term of the prespectrum defining $M\mathbb{R}$.

Proposition 3.2. *The spectrum KR representing Atiyah's Real K-theory is Real-oriented.*

Proof The tensor product of Real bundles makes KR into a commutative associative ring spectrum. The 0th space of the spectrum KR is $\mathbb{Z} \times B\mathbb{U}$. The Real Bott periodicity gives an equivalence of spectra

$$\Sigma^{1+\alpha} KR \to KR.$$

By a Real CW-complex we will mean a $\mathbb{Z}/2$-equivariant space $K = \bigcup_{n \geq 0} K_n$ where K_0 is a discrete space and K_n is obtained by the pushout diagram

$$\begin{array}{ccc} \coprod \mathbb{S}^{2n-1} & \longrightarrow & K_{n-1} \\ \downarrow & & \downarrow \\ \coprod \mathbb{D}^{2n} & \longrightarrow & K_n \end{array}$$

where $\mathbb{S}^{2n-1} = S(\mathbb{C}^n)$ and $\mathbb{D}^n = D(\mathbb{C}^n)$. The filtration $K_0 \subset K_1 \subset K_2 \subset \ldots \subset K$ is the Real filtration of K. A map of Real spectra is a $\mathbb{Z}/2$-equivariant map that preserves the Real filtration.

Now let E be a $\mathbb{Z}/2$-equivariant spectrum and let K be a Real CW-complex. Then analogous to the non-equivariant case there are cofibrations,

$$K_{n-1} \to K_n \to \bigvee_{\text{Real}\, n-\text{cells}} S^{n(1+\alpha)}.$$

Applying E-cohomology we get an exact couple which gives a conditionally convergent spectral sequence

$$E_1^{p,q} = \oplus_{\text{Real}\, p-\text{cell of } X} E^q \Rightarrow E^{p(1+\alpha)+q}(K)$$

where $q \in RO(\mathbb{Z}/2)$. If E is a ring spectrum then the spectral sequence has a ring structure. We see that $B\mathbb{S}^1$ is a Real CW-spectrum. $E_1 = E^\star[[u]]$ for the spectral sequence given above for $K = B\mathbb{S}^1$, and if E is Real-oriented u is a permanent cycle. Similarly for $K = B\mathbb{S}^1 \times B\mathbb{S}^1$.

Theorem 3.1. *If E is a Real-oriented spectrum, then*

$$E^\star(B\mathbb{S}^1) = E^\star[[u]], \, \dim(u) = -(1+\alpha)$$

$$E^\star(B\mathbb{S}^1 \times B\mathbb{S}^1) = E^\star[[u \otimes 1, 1 \otimes u]].$$

Therefore the map on classifying spaces induced by the product $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ gives a formal group law over the ring $E_\star$.

The grading in $E_\star$ is such that the formal group law $F(x,y)$ is homogeneous of degree $-(1+\alpha)$ if both x and y are of degree $-(1+\alpha)$. Also isomorphisms of formal group laws are formal power series over x of degree $-(1+\alpha)$ where x is of dimension $-(1+\alpha)$.

If E is a Real-oriented spectrum, then multiplication on $E_\star$ is graded commutative. If $x \in E_{k+l\alpha}$ and $y \in E_{m+n\alpha}$, then

$$xy = (-1)^{km+ln} yx \in E_{(k+m)+(l+n)\alpha}.$$

Thus, $E_{*(1+\alpha)} \subset E_\star$ is a commutative ring.

By Theorem 3.1 and the fact that MU_* is the universal formal group law, we get

Proposition 3.3. *The canonical map of rings $M\mathbb{R}_\star \to MU_*$ splits by a map of rings $MU_* \to M\mathbb{R}_\star$ which sends the generator $x_i \in MU_{2i}$ to an element in degree $i(1+\alpha)$.*

Theorem 3.2. *For every Real-oriented spectrum E, every Real bundle is E-oriented. Moreover, $E^\star(M\mathbb{R}) = E^\star[b_1, b_2, \ldots]$, and there is a bijective correspondence between Real orientations of E and maps of ring spectra $M\mathbb{R} \to E$, which again are in bijective correspondence with strict isomorphisms of formal group laws whose source is the formal group law over $E_\star$.*

Following this theorem we obtain a Real-oriented spectrum $BP\mathbb{R}$ using Quillen idempotents. The localized spectrum $M\mathbb{R}_{(p)}$ splits as

$$M\mathbb{R}_{(p)} = \bigvee \Sigma^{m_i(1+\alpha)} BP\mathbb{R}$$

3.1 $E\mathbb{R}(n)$

Here we will use the E_∞-module theory of [5] and construct Real analogues of complex-oriented spectra. We shall construct $E\mathbb{R}(n)$ as a Real-oriented spectrum analogue of the complex oriented $E(n)$. First we need the fact:

Proposition 3.4. *$M\mathbb{R}$ is a an E_∞-ring spectrum.*

Now we have, directly by construction, the underlying non-equivariant spectrum of $M\mathbb{R}$

$$F(\mathbb{Z}/2_+, M\mathbb{R})^{\mathbb{Z}/2} = MU.$$

Let $MU_* = \mathbb{Z}[x_1, \ldots, x_n, \ldots]$. By a derived spectrum of MU we mean an E_∞-MU-module obtained by killing off a regular sequence

$$(z_1, z_2, \ldots) \in MU_*$$

and localizing at elements in MU_*.

By Proposition 3.3 we have a map $\mathbb{Z}[x_1, \ldots, x_n, \ldots] \to M\mathbb{R}_\star$. Therefore for a derived spectrum E of MU we can construct an E_∞-$M\mathbb{R}$-module $\mathbb{E}$ by killing off and localizing at the images in $M\mathbb{R}_\star$ of the relevant elements of MU_*. In more detail, for every element $z \in M\mathbb{R}_\star$ and every E_∞-$M\mathbb{R}$-module $\mathbb{M}$ there is a self-map of $M\mathbb{R}$ modules $z : \mathbb{M} \to \mathbb{M}$. If we denote by the z_i the images in $M\mathbb{R}_\star$ of the elements $z_i \in MU_*$ and let $M_0 = M\mathbb{R}$, we obtain the E_∞-$M\mathbb{R}$-module $\mathbb{M}_i = \mathbb{M}/(z_1, \ldots, z_i)$ using the standard cofibration sequences of $M\mathbb{R}$ modules:

$$\mathbb{M}_{i-1} \xrightarrow{z_i} \mathbb{M}_{i-1} \longrightarrow \mathbb{M}_i$$

The localization $\mathbb{M}[z^{-1}]$ is defined as the colimit in the category of E_∞-$M\mathbb{R}$-modules of the diagram

$$\mathbb{M} \xrightarrow{z} \mathbb{M} \xrightarrow{z} \mathbb{M} \xrightarrow{z} \cdots$$

Observe that the lifts of z_i do not necessarily form a regular sequence in $M\mathbb{R}_\star$. This makes the calculation of the $\mathbb{M}_\star$ non-trivial. However the underlying non-equivariant spectrum of $\mathbb{E}$ is given by $F(\mathbb{Z}/2_+, \mathbb{E})^{\mathbb{Z}/2} = E$. This will enable us in some cases, where there is a strong completion, to compute $\mathbb{E}_\star$ using the Borel spectral sequence.

3.2 The elements $\sigma^{\pm 1}$ and a

There is an obvious homeomorphism of spaces $S^\alpha \wedge \mathbb{Z}/2_+ \simeq S^1 \wedge \mathbb{Z}/2_+$. In the Borel spectral sequence, $E_1^0 = [(E_o\mathbb{Z}/2)_+, E\mathbb{R}(n)]_\star = [\mathbb{Z}/2_+, E\mathbb{R}(n)]_\star$, define $\sigma \in E_1^0$ as

$$\Sigma^{\alpha-1}\mathbb{Z}/2_+ = \Sigma^{-1}(S^\alpha \wedge \mathbb{Z}/2_+) \simeq \Sigma^{-1}(S^1 \wedge \mathbb{Z}/2_+) = \mathbb{Z}/2_+ \to S^0 \to E\mathbb{R}(n)$$

so $\sigma^{-1} \in E_1^0$ is

$$\Sigma^{1-\alpha}\mathbb{Z}/2_+ = \Sigma^{-\alpha}(S^1 \wedge \mathbb{Z}/2_+) \simeq \Sigma^{-\alpha}(S^\alpha \wedge \mathbb{Z}/2_+) = \mathbb{Z}/2_+ \to S^0 \to E\mathbb{R}(n),$$

using the unit map $S^0 \to E\mathbb{R}(n)$.

We write a for the class of the inclusion $a : S^0 \subset S^\alpha$ induced by $\mathbb{R}^0 \subset \mathbb{R}^\alpha$. It acts on everything, i.e on $[X, Y]_\star$, all X, Y, naturally in X and Y. In particular, it acts on the element $1 \in [E\mathbb{Z}/2_+, E\mathbb{R}(n)]_\star$ to produce the element $a = a.1 \in D_1^0$ of degree $-\alpha$, which may be written

$$a.1 : S(\mathbb{R}^{\infty\alpha})_+ \longrightarrow S^0 \xrightarrow{a} S^\alpha \xrightarrow{\Sigma^\alpha \eta} \Sigma^\alpha E\mathbb{R}(n)$$

Since $\mathbb{Z}/2_+ \to S^0 \to S^\alpha$ is obviously zero, $a.1$ maps to $0 \in E_1^0$ and therefore lifts to D_1^1. Further, this lift is canonical: the orthogonal projection $P : \mathbb{R}^{\infty\alpha} \to \mathbb{R}^\alpha$ induces a map $P : (S(\mathbb{R}^{\infty\alpha}), S(\mathbb{R}^\alpha)) \to (D(\mathbb{R}^\alpha), S(\mathbb{R}^\alpha))$ and we may describe the lift as the element

$$S(\mathbb{R}^{\infty\alpha})_+/S(\mathbb{R}^\alpha)_+ \xrightarrow{P_+} D(\mathbb{R}^\alpha)_+/S(\mathbb{R}^\alpha)_+ \xrightarrow{\simeq} S^\alpha \xrightarrow{=} \Sigma^\alpha S^0 \xrightarrow{\Sigma^\alpha \eta} \Sigma^\alpha E\mathbb{R}(n)$$

of $D_1^1(c)$. This maps to an element of $E_1^1(c)$, of degree $-\alpha$, that we also call a.

3.3 The Tate spectral sequence for $E\mathbb{R}(n)$

For the E_1-term we have

$$E_1^{p,q}(t) = [R_p, E\mathbb{R}(n)]^{p+q} = [\mathbb{Z}/2_+, E\mathbb{R}(n)]^q = E(n)^q$$

where $q \in RO(\mathbb{Z}/2)$. We have the element $a \in E_1^1$ and the invertible element $\sigma \in E_1^0$ of degrees $-\alpha$ and $\alpha - 1$, introduced above. Further, one can show that multiplication by a induces an isomorphism $E_1^{p,q} \simeq E_1^{p+1,q-1+\alpha}$ for all $p \in \mathbb{Z}$ and $q \in RO(\mathbb{Z}/2)$. This is all there is:

$$E_1(t) = E(n)_*[a^{\pm 1}, \sigma^{\pm 1}]$$

where $E(n)_* = \mathbb{Z}_{(2)}[v_1, v_2, \ldots, v_n^{\pm 1}]$ and $\deg v_i = (2^i - 1)(1 + \alpha)$. We also write $v_0 = 2$. We have a spectral sequence of $E(n)_*$-modules.

Following [8], the differential d_1 is determined by $d_1(\sigma^{-1}) = v_0 a$, with the result that

$$E_2(t) = \mathbb{F}_2[v_1, v_2, \ldots, v_n^{\pm 1}][\sigma^{\pm 2}, a^{\pm 1}]$$

(the exponent of a simply indicates the filtration). From the E_2-term on we have a spectral sequence of graded commutative rings, and each d_r is a derivation.

We proceed by induction on s, for $1 \leq s < n$, and assume

$$E_{2^s}(t) = \mathbb{F}_2[v_s, v_{s+1}, \ldots, v_n^{\pm 1}][\sigma^{\pm 2^s}, a^{\pm 1}]. \tag{4}$$

By sparseness, $d_r = 0$ for $2^s \leq r \leq 2^{s+1} - 2$. By [9], the next differential is determined by $d_{2^{s+1}-1}(\sigma^{-2^s}) = v_s a^{2^{s+1}-1}$, so that the non-zero differentials are

$$d_{2^{s+1}-1}(v^R \sigma^{2^s(2l-1)} a^k) = v_s v^R \sigma^{2^{s+1}l} a^{k+2^{s+1}-1} \qquad (l \in \mathbb{Z}, k \in \mathbb{Z}, 0 \leq s \leq n)$$

where $v^R = v_s^{r_s} v_{s+1}^{r_{s+1}} \ldots v_n^{r_n}$. The ideal (v_s) is killed and we find $E_{2^{s+1}}(t)$ as (4) above, with s replaced by $s+1$.

Then $E_{2^n}(t) = \mathbb{F}_2[v_n^{\pm 1}][\sigma^{\pm 2^n}, a^{\pm 1}]$. By the same argument, $d_{2^{n+1}-1}$ kills (v_n); but v_n is invertible, so $E_{2^{n+1}}(t) = 0$. Thus the spectral sequence converges strongly to 0. We would like to conclude that $t(E\mathbb{R}(n))$ is trivial. Unfortunately, this is a whole plane spectral sequence with mysterious target, and we have to proceed indirectly.

3.4 The Borel spectral sequence for $E\mathbb{R}(n)$

We have a morphism of spectral sequences $\omega_r : E_r(c) \to E_r(t)$ of $E(n)_*$-modules. The first lemma is easily proved.

Lemma 3.1. *$\omega_r : E_r^p(c) \to E_r^p(t)$ is*

1. *surjective for $p \geq 0$*
2. *bijective for $p \geq r-1$*

Corollary 3.1. *For $x \in E_r^p(c)$, $\omega_r x = 0 \Rightarrow d_r x = 0$.*

Proof $d_r x \in E_r^{p+r}$. $\omega_r d_r x = d_r \omega_r x = 0$, therefore $d_r x = 0$.

Corollary 3.2. *$d_r = 0$ in $E_r(c)$ unless $r+1$ is a power of 2.*

Proof Since true in $E_r(t)$.

Therefore we have a filtration

$$0 \subset \mathrm{Ker}\omega_2 \subset \mathrm{Ker}\omega_4 \subset \ldots \subset \mathrm{Ker}\omega_{2^{n+1}} = E_{2^{n+1}}(c) = E_\infty(c)$$

by $E(n)_*$-modules. $\mathrm{Ker}\omega_2$ is a free $E(n)_*$-module on generators $v_0\sigma^{2s} = 2\sigma^{2s} (s \in \mathbb{Z})$. Generally for $0 \leq s \leq n$, $\mathrm{Ker}\omega_{2^s} \subset \mathrm{Ker}\omega_{2^{s+1}}$ introduces new $E(n)_*$-module generators $v_s\sigma^{2^{s+1}l}a^u$ $(l \in \mathbb{Z}, 0 \leq u \leq 2^{s+1}-2)$. This gives new abelian group generators

$v_s v^R \sigma^{2^{s+1}l} a^u$ from $d_{2^{s+1}-1}(\sigma^{2^s(2l-1)} a^{u-2^{s+1}+1}) = v_s \sigma^{2^{s+1}l} a^u$ in $E_{2^{s+1}-1}(t)$ which yield a basis over $\mathbb{F}_2$ of $\mathrm{Ker}\omega_{2^{s+1}}/\mathrm{Ker}\omega_{2^s}$.

We first describe $E_\infty(c)$ as a filtered module over $E(n)_*[a]$. If $b \in E_1(c)$ is a permanent cycle, we write $[b]$ for its class in $E_\infty(c)$. As a module, it is generated by the elements

$$[v_s \sigma^{2^{s+1}} l] \;\; (0 \le s \le n, l \in \mathbb{Z}) \tag{5}$$

with module relations

$$a^{2^{s+1}-1}[v_s \sigma^{2^{s+1}l}] = 0, \qquad v_i[v_s \sigma^{2^{s+1}l}] = v_s[v_i \sigma^{2^{i+1}(2^{s-i}l)}] \;\; (\text{for } i < s) \tag{6}$$

It is often useful to replace the generator $[v_n \sigma^{2^{n+1}(0)}]$ by $v_n^{-1}[v_n \sigma^{2^{n+1}(0)}]$, the identity element of the ring $E_\infty(c)$. More generally we can replace $[v_n \sigma^{2^{n+1}l}]$ by $v_n^{-1}[v_n \sigma^{2^{n+1}l}] = [\sigma^{2^{n+1}l}]$. The degrees of a, v_k and σ are $-\alpha$,$(2^k-1)(1+\alpha)$ and $(\alpha-1)$ respectively.

Multiplication (which is commutative) is given on the generators by

$$[v_s \sigma^{2^{s+1}l}][v_m \sigma^{2^{m+1}q}] = v_m[v_s \sigma^{2^{s+1}(l+2^{m-s}q)}] \;\; (\text{for } s \le m) \tag{7}$$

This is clearly inherited from $E_1(c)$. In particular, since $v_0 = 2$, $[v_0\sigma^{2l}][v_0\sigma^{2q}] = 2[v_0\sigma^{2(l+q)}]$.

We really want $c(E\mathbb{R}(n))_\star$, by unfiltering $E_\infty(c)$. By sparseness of the spectral sequence, when $n = 2$, each generator $[v_s\sigma^{2^{s+1}}l]$ lifts *uniquely* to an element of $c(E\mathbb{R}(2))_\star$ that we also call $[v_s\sigma^{2^{s+1}}l]$; further, the module relations (6) also lift uniquely to $c(E\mathbb{R}(2))_\star$. However even for $n = 2$, a few of the multiplications (7) do not lift uniquely; there is some indeterminacy, but we do have $[\sigma^{2^{n+1}l}][\sigma^{2^{n+1}q}] = [\sigma^{2^{n+1}}(l+q)]$ in $c(E\mathbb{R}(n))_\star$ for any n.

We follow [11] and define the element $y(n) = v_n^{2^n-1}\sigma^{-2^{n+1}(2^{n-1}-1)}$ in degree $\lambda(n)+\alpha$ where $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$. The element $y(n)$ is clearly invertible in the $RO(\mathbb{Z}/2)$-

graded ring $c(E\mathbb{R}(n))_\star$. We shall use it to reduce elements of $c(E\mathbb{R}(n))_\star$ to elements with integer degree. Define $x(n) = y(n)a$, which has degree $\lambda(n)$.

3.5 Strong completion and cofibrations

Assume given a representation V of G whose unit sphere $S(V)$ is a free G-space. The union $S(\infty V)$ of the spheres $S(qV)$ is a model for EG, the union $D(\infty V)$ of unit discs $D(qV)$ is G-contractible, and the quotient $D(\infty V)/S(\infty V) \simeq S^{\infty V}$ is a model for $\widetilde{EG}$.

In homology and cohomology, the description of $\widetilde{EG}$ as $S^{\infty V}$ implies the following canonical isomorphisms for a G-spectrum k_G, an integer n (or more general element of $RO(G)$, and a G-space X, where X is finite in the case of cohomology:

$$(k_G \wedge \widetilde{EG})_n(X) = \mathrm{colim}_q (k_G \wedge S^{qV})_n(X) = \mathrm{colim}_q k^G_{n-qV}(X)$$

and

$$(k_G \wedge \widetilde{EG})^n(X) = \mathrm{colim}_q (k_G \wedge S^{qV})^n(X) = \mathrm{colim}_q k_G^{n+qV}(X)$$

The colimits can be interpreted algebraically. Let V be any representation of G, let $e : S^0 \to S^V$ be induced by $0 \subset V$, and let $a_V \in k_G^V(S^0)$ be the image of the identity element of $k_G^0(S^0)$ under the map $e^* : k_G^0(S^0) = k_G^V(S^V) \to k_G^V(S^0)$. Therefore given a ring spectrum k_G and any G-representation V, $(k_G \wedge S^{\infty V})_\star(X)$ and, if X is finite, $(k_G \wedge S^{\infty V})^\star(X)$, are the localizations of $k^G_\star(X)$ and $k_G^\star(X)$ away from a_V. Thus multiplication by a_V provides a periodicity isomorphism with period V on $(k_G \wedge S^{\infty V})_\star(X)$ and $(k_G \wedge S^{\infty V})^\star(X)$.

Following Proposition 2.1 this has the following implication.

Proposition 3.5. *Let k_G be a ring G-spectrum. If G acts freely on the unit sphere $S(V)$, then $t(k_G)_\star(X)$ and, if X is finite, $t(k_G)^\star(X)$ are localizations of $c(k_G)_\star(X)$*

and $c(k_G)^\star(X)$ away from a_V. Therefore, multiplication by a_V provides a periodicity isomorphism with period V on $t(k_G)_\star(X)$ and $t(k_G)^\star(X)$.

In the case of $E\mathbb{R}(n)$, $G = \mathbb{Z}/2$, $V = \alpha$, $a_V = a$, and there is a commutative square of $\mathbb{Z}/2$-equivariant ring spectra:

$$\begin{array}{ccc} E\mathbb{R}(n) & \longrightarrow & g(E\mathbb{R}(n)) = \widetilde{E\mathbb{Z}/2} \wedge E\mathbb{R}(n) \\ \downarrow & & \downarrow \\ c(E\mathbb{R}(n)) & \longrightarrow & t(E\mathbb{R}(n)) \end{array}$$

The element $a : S^0 \subset S^\alpha$ acts

1. nilpotently on $c(E\mathbb{R}(n))_\star$.

2. invertibly on $g(E\mathbb{R}(n))_\star$ and $t(E\mathbb{R}(n))_\star$.

Theorem 3.3. *The Tate spectrum $t(E\mathbb{R}(n))$ is trivial.*

Proof It is the localization of $c(E\mathbb{R}(n))$ away from a, so on $t(E\mathbb{R}(n))_\star$, a acts invertibly as well as nilpotently.

Since the left vertical arrow of the Tate diagram is an equivalence, we see that, up to equivalence, $t(k_G)$ can be seen as the cofiber of the composite

$$k_G \wedge EG_+ \to k_G \to F(EG_+, k_G).$$

This gives us the following result.

Proposition 3.6. $f(E\mathbb{R}(n)) \simeq c(E\mathbb{R}(n))$.

We will construct a cofibration of spectra connecting $E(n)$ and $ER(n)$. We have the following equivalence as part of the Tate diagram.

$$\begin{array}{ccccc} f(E\mathbb{R}(n)) & \longrightarrow & E\mathbb{R}(n) & \longrightarrow & g(E\mathbb{R}(n)) \\ & \searrow^{\simeq} & \downarrow & & \\ & & c(E\mathbb{R}(n)) & & \end{array}$$

This implies a splitting of $\mathbb{Z}/2$- ring spectra

$$E\mathbb{R}(n) \simeq c(E\mathbb{R}(n)) \vee g(E\mathbb{R}(n)).$$

There is a fibration $\mathbb{Z}/2_+ \longrightarrow S^0 \xrightarrow{a} S^\alpha$ inducing a fibration

$$\begin{array}{ccccc} F(S^\alpha, g(E\mathbb{R}(n))) & \xrightarrow{a^*} & F(S^0, g(E\mathbb{R}(n))) & \longrightarrow & F(\mathbb{Z}/2_+, g(E\mathbb{R}(n))) \\ \downarrow \simeq & & \downarrow \simeq & & \\ \Sigma^{-\alpha} g(E\mathbb{R}(n)) & \underset{\simeq}{\xrightarrow{a}} & g(E\mathbb{R}(n)) & & \end{array}$$

This equivalence induced by a implies that $F(\mathbb{Z}/2_+, g(E\mathbb{R}(n)))$ is trivial. We have the analogous fibration

$$\begin{array}{ccccc} \Sigma^{-\alpha} c(E\mathbb{R}(n)) & \xrightarrow{a} & c(E\mathbb{R}(n)) & \longrightarrow & F(\mathbb{Z}/2_+, c(E\mathbb{R}(n))) \\ y(n) \uparrow \simeq & \nearrow_{x(n)} & & & \downarrow = \\ \Sigma^{\lambda(n)} c(E\mathbb{R}(n)) & & & & F(\mathbb{Z}/2_+, E\mathbb{R}(n)) \end{array}$$

On fixed points we deduce the fibration of non-equivariant spectra

$$\Sigma^{\lambda(n)} ER(n) \xrightarrow{x(n)} ER(n) \longrightarrow E(n) \tag{8}$$

where $ER(n) := E\mathbb{R}(n)^{h\mathbb{Z}/2}$, the homotopy fixed points of $E\mathbb{R}(n)$, which is just the ordinary fixed points of the Borel spectrum $c(E\mathbb{R}(n))$. The element $x(n)$ has degree $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$.

3.6 The boundary map

We shall now compute the boundary map

$$\partial : \Sigma^{-1}E(n) \to \Sigma^{\lambda(n)}ER(n) \to \Sigma^{\lambda(n)}E(n)$$

Consider the composite $\Sigma^{\alpha}\mathbb{Z}/2_{+} \to S^{\alpha} \to \Sigma\mathbb{Z}/2_{+}$ Notice that multiplication by σ^{-1} induces a non-equivariant equivalence $\Sigma^{-\alpha}E\mathbb{R}(n) \to \Sigma^{-1}E\mathbb{R}(n)$, whereas multiplication by $y(n)$ induces an equivalence of $\mathbb{Z}/2$-spectra $\Sigma^{-\alpha}E\mathbb{R}(n) \simeq \Sigma^{\lambda(n)}E\mathbb{R}(n)$.

This induces the following diagram of naive non-equivariant spectra:

$$\begin{array}{ccccc}
F(\mathbb{Z}/2_{+}, \Sigma^{-1}E(n)) & \xrightarrow{(-)} & F(S, \Sigma^{-1}E(n)) & \xrightarrow{\Delta} & F(\mathbb{Z}/2_{+}, \Sigma^{-1}E(n)) \\
\simeq \uparrow & & \simeq \uparrow \sigma^{-1} & & \simeq \uparrow \sigma^{-1} \\
i^{*}F(\mathbb{Z}/2_{+}, \Sigma^{-1}E\mathbb{R}(n)) & \longrightarrow & i^{*}F(S, \Sigma^{-\alpha}E\mathbb{R}(n)) & \longrightarrow & i^{*}F(\mathbb{Z}/2_{+}, \Sigma^{-\alpha}E\mathbb{R}(n)) \\
(1,c) \uparrow & & \uparrow & & (1,c) \uparrow \\
F(\mathbb{Z}/2_{+}, \Sigma^{-1}E\mathbb{R}(n))^{\mathbb{Z}/2} & \longrightarrow & F(S^{\alpha}, E\mathbb{R}(n))^{\mathbb{Z}/2} & \longrightarrow & F(\mathbb{Z}/2_{+}, \Sigma^{-\alpha}E\mathbb{R}(n))^{\mathbb{Z}/2} \\
\simeq \uparrow & & \simeq \uparrow y(n) & & \simeq \uparrow y(n) \\
\Sigma^{-1}E(n) & \longrightarrow & \Sigma^{\lambda(n)}ER(n) & \longrightarrow & \Sigma^{\lambda(n)}E(n)
\end{array}$$

where $(-)$ is the difference map, Δ is the twisted diagonal map and c is complex conjugation. It follows from above that

Proposition 3.7. $\partial = y(n)^{-1}\sigma(1-c)$

4 Cohomology of projective spaces

Consider the complex projective space $\mathbb{C}P^{\infty}$, with the action of $\mathbb{Z}/2$ given by complex conjugation. Consider the canonical line bundle γ on $\mathbb{C}P^{\infty}$. This is a Real bundle in Atiyah's sense [2]. Recall that our theory is Real-oriented [9], which means there is a Real Chern class $u \in E\mathbb{R}(n)^{1+\alpha}(\mathbb{C}P^{\infty})$ and the standard Atiyah-Hirzebruch spectral sequence may be invoked to show, following Dold,

$$E\mathbb{R}(n)^\star(\mathbb{C}P^\infty) \cong E\mathbb{R}(n)^\star[[u]]$$

and similarly for $c(E\mathbb{R}(n))$.

Consider the Real bundle $\gamma^{\otimes 2}$. The Real Chern class of this is $[2](u)$. Let $\widetilde{RP}^\infty$ denote the unit sphere bundle of $\gamma^{\otimes 2}$. $\widetilde{RP}^\infty$ may be identified with the space of real lines in $\mathbb{C}^\infty$ and it admits an involution given by complex conjugation. The real projective space RP^∞ (with trivial involution) sits inside $\widetilde{RP}^\infty$ as a non-equivariant deformation retract. We therefore have the isomorphism $c(E\mathbb{R}(n))^\star(RP^\infty) \simeq c(E\mathbb{R}(n))^\star(\widetilde{RP}^\infty)$. We can calculate the cohomology of $\widetilde{RP}^\infty$ using the Gysin sequence for the bundle $\gamma^{\otimes 2}$:

$$\ldots \to c(E\mathbb{R}(n))^{(k-1)+(l-1)\alpha}(\mathbb{C}P^\infty) \to c(E\mathbb{R}(n))^{k+l\alpha}(\mathbb{C}P^\infty) \to c(E\mathbb{R}(n))^{k+l\alpha}(\widetilde{RP}^\infty)$$

$$\to c(E\mathbb{R}(n))^{k+(l-1)\alpha}(\mathbb{C}P^\infty) \to \ldots$$

Since multiplication by the Chern class of $\gamma^{\otimes 2}$

$$[2](u) = 2u +_F v_1 u^2 +_F \ldots +_F v_n u^{2^n} = 2u + \ldots + v_n u^{2^n} + \ldots \tag{9}$$

(the dots indicate terms with higher powers of u) is injective, we conclude from the Gysin sequence

$$c(E\mathbb{R}(n))^\star(RP^\infty) \cong c(E\mathbb{R}(n))^\star[[u]]/([2](u)).$$

Let $c(E\mathbb{R}(n))^*(RP^\infty)$ be the subring of elements in integral degree. Then

$$E\mathbb{R}(n)^*(RP^\infty) \cong ER(n)^*(RP^\infty)$$

since RP^∞ has trivial $\mathbb{Z}/2$ action and $ER(n) = c(E\mathbb{R}(n))^{\mathbb{Z}/2}$.

We wish to convert this result to a corresponding statement for $ER(n)$. We recall the invertible element $y(n)$ in degree $\lambda(n)+\alpha$ and use it to convert the Gysin sequence to an exact sequence with integer degrees.

$$\begin{array}{ccccccc}
\cdots \longrightarrow & E\mathbb{R}(n)^{k-1-\alpha}(\mathbb{C}P^\infty) & \xrightarrow{[2](u)} & E\mathbb{R}(n)^{k}(\mathbb{C}P^\infty) & \longrightarrow & E\mathbb{R}(n)^{k}(\widetilde{RP^\infty}) & \longrightarrow \cdots \\
& \simeq \uparrow y(n) & \nearrow & & & = \uparrow & \\
& E\mathbb{R}(n)^{k-1+\lambda(n)}(\mathbb{C}P^\infty) & & & & E\mathbb{R}(n)^{k}(RP^\infty) &
\end{array}$$

We replace u by $u' = uy(n)$, v_i by $v_i' = v_i y(n)^{-2^i+1}$, and in the formal group law $F(p,q) = p + q + \Sigma_{i,j} a_{i,j} p^i q^j$, replace $a_{i,j}$ by $a'_{i,j} = a_{i,j} y(n)^{-i-j+1}$ and $F(p,q)$ by $F'(p,q) = p + q + \Sigma_{i,j} a'_{i,j} p^i q^j$, to obtain the new [2]-series $[2]'(u') = F'(u', u')$.

Now we drop all the $'$. As in [12], we obtain the Gysin sequence of integer-graded groups

$$\cdots \longrightarrow ER(n)^*(\mathbb{C}P^\infty) \xrightarrow{[2](u)} ER(n)^*(\mathbb{C}P^\infty) \longrightarrow ER(n)^*(RP^\infty) \longrightarrow \cdots$$

Also, we work with *cohomology* degrees from now on, by changing all the signs from homology degrees, since we work in cohomology. So u now has degree $-\lambda(n) + 1$. Therefore we have

$$ER(n)^*(RP^\infty) \cong ER(n)^*[[u]]/[2](u)$$

where $u \in ER(n)^{1-\lambda(n)}(RP^\infty)$ and $v_k \in ER(n)^{(\lambda(n)-1)(2^k-1)}$

5 The Bockstein spectral sequence

We have the stable cofibration

$$\Sigma^{\lambda(n)} ER(n) \xrightarrow{x} ER(n) \longrightarrow E(n)$$

where $x \in ER(n)^{-\lambda(n)}$ and $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$. The fibration gives us a long exact sequence

$$\begin{array}{ccc} ER(n)^*(X) & \xrightarrow{\quad x \quad} & ER(n)^*(X) \\ & {}_{\partial}\nwarrow \qquad \swarrow_{\rho} & \\ & E(n)^*(X) & \end{array} \tag{10}$$

where x lowers the degree by $\lambda(n)$ and ∂ raises the degree by $\lambda(n)+1$. This leads to the Bockstein spectral sequence, which will completely determine $M = ER(n)^*(X)/(x)$ as a subring of $E(n)^*(X)$. We know that $x^{2^{n+1}-1} = 0$ so there can be only $2^{n+1} - 1$ differentials.

We filter M,

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_{2^{n+1}-1} = M$$

by submodules

$$M_r = \mathrm{Ker}\left[x^r : \frac{ER(n)^*(X)}{x} \to \frac{x^r ER(n)^*(X)}{x^{r+1}}\right]$$

so that M_r/M_{r-1} gives the x^r-torsion elements of $ER(n)^*(X)$ that are non-zero in M.

We collect the basic facts about the spectral sequence in the following theorem in [12]. $E(n)$ is a complex oriented spectrum with a complex conjugation action. Denote this action by c.

Theorem 5.1. *In the Bockstein spectral sequence for $ER(n)^*(X)$*

1. *The exact couple (10) gives rise to a spectral sequence, E^r, of $ER(n)^*$-modules, starting with*

$$E^1 \simeq E(n)^*(X).$$

2. $E^{2^{n+1}} = 0$

3. $Im\, d^r \simeq M_r/M_{r-1}$.

4. *The degree of* d^r *is* $r\lambda(n)+1$.

5. $d^r(ab) = d^r(a)b + c(a)d^r(b)$

6. $d^1(z) = v_n^{-(2^n-1)}(1-c)(z)$ *where* $c(v_i) = -v_i$.

7. *If* $c(z) = z$ *in* E^1, *then* $d^1(z) = 0$. *If* $c(z) = z$ *in* E^r *then* $d^r(z^2) = 0$.

8. *The following are all vector spaces over* $\mathbb{Z}/2$:

$$M_j/M_i,\ (j \geq i > 0)\ \textit{and}\ E^r, (r \geq 2).$$

Proof Most of the results follow from the basic properties of an exact couple and the fact that $x^{2^{n+1}-1} = 0$. We have the complex conjugation for our involution on $E(n)^*(X)$ and trivial involution on $ER(n)^*(X)$. Assuming the formula for d^1 we obtain the product formula for d^1:

$$d^1(ab) = v_n^{-(2^n-1)}(1-c)(ab) = v_n^{-(2^n-1)}(ab - c(ab)) = v_n^{-(2^n-1)}(ab - c(a)c(b))$$

$$= v_n^{-(2^n-1)}((a-c(a))b + c(a)(b-c(b))) = d^1(a)b + c(a)d^1(b).$$

The product rule for d^r follows from that of d^1. If $c(z) = z$:

$$d^1(z) = v_n^{-(2^n-1)}(1-c)(z) = v_n^{-(2^n-1)}(z-z) = 0.$$

For the second case $d^1(z^2) = 0$ because z^2 is invariant under c. For $r > 1$,

$$d^r(z^2) = zd^r(z) + c(z)d^r(z) = 2zd^r(z)$$

which is zero since we are working modulo 2 for $r > 1$.

Let us continue to assume the formula for d^1 and show the $\mathbb{Z}/2$ vector spaces. It is enough to show this for E^2. Start with an arbitrary d^1-cycle $y \in E^1$ such that

$2y \neq 0$. Observe that

$$d^1(v_n^{2^n-1}) = v_n^{-(2^n-1)}(1-c)(v_n^{2^n-1}) = v_n^{-(2^n-1)}(v_n^{2^n-1} + v_n^{2^n-1}) = 2.$$

This shows that $2x = 0$. Consider the element $v_n^{2^n-1}y$.

$$d^1(v_n^{2^n-1}y) = yd^1(v_n^{2^n+1}) + c(v_n^{2^n-1})d^1(y) = 2y + 0.$$

Therefore no multiplication by 2 survives to E^2.

This is all we need to show that the M_j/M_i for $j \geq i > 0$ are $\mathbb{Z}/2$ vector spaces.

Finally the formula for the differential d^1 follows from Propsition 3.7.

Note that the image of $ER(n)^*(X) \to E(n)^*(X)$ consists of targets of the differentials and therefore always have the differentials trivial on them. Also anything in the image is trivial under the action of c.

Since $ER(n)^*(-)$ is $2^{n+2}(2^n-1)$-periodic we will consider it as graded over $\mathbb{Z}/(2^{n+2}(2^n-1))$. We have to do the same then for $E(n)^*(-)$. We can do this by setting the unit $v_n^{2^{n+1}} = 1$ in the homotopy of $E(n)$. (This does not lose any information since we can always recover the original by inserting powers of $v_n^{2^{n+1}}$ to make the degrees match.)

5.1 The spectral sequence for $ER(2)^*$

For the Bockstein spectral sequence of Theorem 5.1 to be useful, we need to know the ring $ER(n)^*$. From now on we concentrate on the case $n = 2$. This spectral sequence begins with $E^1 = E(2)^*$ which is just a free $\mathbb{Z}_{(2)}[v_1]$-module on a basis given by v_2^i for $0 \leq i < 8$. We are grading mod 48. Since all elements of E^1 have even degree and degree $\deg d^r = 17r + 1$, $d^r = 0$ for n even. As $E^8 = 0$, we only have d^1, d^3, d^5, d^7 to consider.

We have the differential d^1 acting as follows:

$$d^1(v_2^{2s+1}) = v_2^{-3}(1-c)v_2^{2s+1} = v_2^{-3}2v_2^{2s+1} = 2v_2^{2s-2}.$$

Similarly $d^1(v_2^{2s}) = 0$.

However multiplication by v_1 doesn't behave well with respect to this differential. The problem is that $v_1 \in E(2)^*$ does not lift to $ER(2)^*$. We will need a substitute for v_1. We shall use the element $\alpha = v_1^{ER(2)} = y(2)^{-1}v_1 \in ER(2)^{\lambda(2)-1}$. The image of α is $v_1 v_2^5 \in E(2)^{-32}$. Because v_2 is a unit this a good substitute for ordinary v_1. Furthermore this is invariant under c because it is in the image of the map from $ER(2)^*$ and is a permanent cycle. Or we could just see ths by observing that there are an even number of v's. We rewrite the homotopy of $E(2)$ as $\mathbb{Z}_{(2)}[\alpha, v_2^{\pm 1}]$ but again set $v_2^8 = 1$.

Now back to the computation of d^1 on v_2^{2s+1} where the E_1 term is a free $\mathbb{Z}_{(2)}[\alpha]$-module on generators v_2^i, $0 \leq i < 8$.

$$d^1(\alpha^k v_2^{2s+1}) = d^1(\alpha^k)v_2^{2s+1} + c(\alpha^k)d^1(v_2^{2s+1}) = 0 + \alpha^k 2v_2^{2s-2}.$$

But this really follows from the fact that we have a spectral sequence of $ER(2)^*$-modules. Thus the d^1-cycles form a free $\mathbb{Z}_{(2)}[\alpha]$-module generated by $\{1, v_2^2, v_2^4, v_2^6\}$ and the d^1-boundaries form the free submodule with basis $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_i = 2v_2^{2i}$. In particular, $\alpha_0 = 2$. Thus $E^2 = E^3$ is the free $\mathbb{F}_2[\alpha]$-module with the basis (the images of) $\{1, v_2^2, v_2^4, v_2^6\}$.

By [12], $d^3(v_2^2) = \alpha v_2^4$. Since d^3 is a derivation, $d^3(v_2^6) = \alpha$, and the only elements of E^4 are 1 and v_2^4. Since $\deg d^5 = 38$, $d^5 = 0$. We must have $d^7(v_2^4) = 1$ to make $E^8 = 0$.

We can read off M as a $\mathbb{Z}_{(2)}[\alpha]$-submodule of E^1. $M_1 = \mathrm{Im} d^1$ is the free module on basis $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$. M_3 is generated as a module by adding the elements α and

$w = \alpha v_2^4$, which make M_3/M_1 the free $\mathbb{F}_2[\alpha]$-module with basis $\{\alpha, w\}$. Finally, the only new element of M_7 is 1. The rest of the module structure is given by

$$2\alpha = \alpha\alpha_0, 2w = \alpha\alpha_2, 2.1 = \alpha_0, \alpha.1 = \alpha \tag{11}$$

Further, M is a subring of E^1, generated by $\alpha_1, \alpha_2, \alpha_3, \alpha$ and w. The products not already given are

$$\alpha_s\alpha_t = 2\alpha_{s+t} \text{ (taking } s+t \bmod 4) \tag{12}$$

$$w\alpha_s = \alpha\alpha_{s+2} \text{ (taking } s+2 \bmod 4) \tag{13}$$

$$w^2 = \alpha^2 \tag{14}$$

To obtain $ER(2)^*$, we must unfilter $M = ER(2)^*/(x)$ and add the generator x. We lift each generator α_s and w to $ER(2)^*$, keeping the same names, with the relations

$$\alpha_s x = 0, \alpha x^3 = 0, w x^3 = 0, x^7 = 0. \tag{15}$$

By sparseness, each α_s and w lifts *uniquely* to $ER(2)^*$; further, the module actions (11) and multiplications (12), (13), (14) also lift uniquely and hold in $ER(2)^*$, not merely mod(x).

Proposition 5.1. *$ER(2)^*$ is graded over $\mathbb{Z}/48$. It is generated as a ring by elements,*

$$x, w, \alpha, \alpha_1, \alpha_2, \alpha_3$$

of degrees -17,-8,-32,-12,-24 and -36 respectively, with relations and products as listed above.

5.2 The Bockstein Spectral Sequence for $ER(2)^*(RP^\infty)$

As always, we have the split short exact sequence

$$0 \to ER(2)^*(X,*) \to ER(2)^*(X) \to ER(2)^*(*) \to 0$$

for any space X with baspoint $*$. Now that we know $ER(2)^*(*) = ER(2)^*$, we will concentrate on $ER(2)^*(X,*)$. Nevertheless, we need to use the action of $ER(2)^*$ on $ER(2)^*(X)$ and hence $ER(2)^*(X,*)$; furthermore, this action extends to actions of the Bockstein spectral sequences.

The $E(2)^*$-cohomology of RP^∞ can be computed from the Gysin sequence

$$\dots E(2)^{k-2}(\mathbb{C}P^\infty) \xrightarrow{[2](x_2)} E(2)^k(\mathbb{C}P^\infty) \longrightarrow E(2)^k(\widetilde{RP^\infty}) \longrightarrow E(2)^{k-1}(\mathbb{C}P^\infty) \dots$$

where $E(2)^*(\mathbb{C}P^\infty) \simeq E(2)^*[[x_2]]$, $x_2 \in E(2)^2(\mathbb{C}P^\infty)$. From above $E(2)^*(RP^\infty) \simeq E(2)^*[[x_2]]/([2](x_2))$. Recall that $ER(2)^*(RP^\infty) = ER(2)^*[[u]]/([2](u))$ where we have $u \in ER(2)^{1-\lambda(2)}(RP^\infty)$. We will replace the x_2 by the image of $u \in ER(2)^{-16}(RP^\infty)$ which we also call $u \in E(2)^{-16}(RP^\infty)$, which is really $v_2^3 x_2$. Likewise we replace the usual $v_1 \in E(2)^{-2}$ with $v_2^5 v_1 = \alpha \in E(2)^{-32}$ which comes from $\alpha \in ER(2)^{-32}$. The element $w \in ER(2)^{-8}$ maps to $\alpha v_2^4 = v_2 v_1 \in E(2)^{-8}$. These changes are necessary because x_2 and v_1 are not in the image of $ER(2)$-cohomology.

We will describe our groups in terms of a *2-adic basis* in the sense of [12], i.e, a set of elements such that any element in our group can be written as a unique sum of these elements with coefficients 0 or 1 (where the sum is allowed to be a formal power series in u). In the ring $E(2)^*(RP^\infty)$ we have $2u = \alpha u^2 + \dots$, therefore the 2-adic basis is given by $v_2^i \alpha^k u^j$, $(0 \le i < 8, 0 \le k, 1 \le j)$.

The original relation $[2](u) = 2u +_F v_1 u^2 +_F v_2 u^4$ for $E\mathbb{R}(2)$ converts to the relation

$$[2](u) = 2u +_F \alpha u^2 +_F u^4 \tag{16}$$

since v_1 is replaced by $v_1(v_2^3)^{-1} = v_1 v_2^5 = \alpha$ and v_2 is replaced by $v_2(v_2^3)^{-3} = v_2^{-8} = 1$. Because $2x = 0$, x times the relation (16) gives us $0 = x(\alpha u^2 +_F u^4)$. Therefore from the point of view of x^1-torsion αu^2 can be replaced with $u^4 + \ldots$. Similarly, if we multiply by x^3 and use the relation $x^3\alpha = 0$ we end up with $x^3 u^4 = 0$.

Since $u \in E(2)^*(RP^\infty)$ is in the image from $ER(2)^*(RP^\infty)$ our differentials commute with multiplication by u and also commute with multiplication by α. The d^1 differential creates a relation coming from our relation $0 = 2u +_F \alpha u^2 +_F u^4$ when $2u$ is set to zero. So in E^2, we have $\alpha u^2 \equiv u^4 + \ldots$. The Bockstein spectral sequence goes like this:

Theorem 5.2. *$E^1 = E(2)^*(RP^\infty, *)$ is represented by*

$$v_2^i \alpha^k u^j \quad (0 \leq i \leq 7, \quad 0 \leq k, \quad 1 \leq j).$$

$$d^1(v_2^{2s-5}\alpha^k u^j) = 2v_2^{2s}\alpha^k u^j = v_2^{2s}\alpha^{k+1}u^{j+1} + \ldots$$

$E^2 = E^3$ is given by:

$$v_2^{2s}\alpha^k u, \quad v_2^{2s}u^j \quad (2 \leq j, \quad 0 \leq s \leq 4, \quad 0 \leq k)$$

$$d^3(v_2^{4s-2}\alpha^k u) = v_2^{4s}\alpha^{k+1}u, \;\; d^3(v_2^{4s-2}u^j) = v_2^{4s}\alpha u^j = v_2^{4s}u^{j+2} + \ldots$$

$E^4 = E^5 = E^6 = E^7$ is given by:

$$v_2^4 u^{\{1-3\}}, \quad u^{\{1-3\}}$$

$$d^7(v_2^4 u^{\{1-3\}}) = u^{\{1-3\}}.$$

The x^1-torsion generators are given by:

$$\alpha_i \alpha^k u^j \quad (0 \le i, \quad 0 \le k, \quad 1 \le j)$$

where $\alpha_0 = 2$.

The x^3-torsion generators are given by:

$$\alpha^{k+1}u,\ \alpha^k wu\ (k \ge 0),\ u^j\ (j \ge 4),\ wu^j\ (j \ge 2).$$

The only x^7-torsion generators are

$$u^{\{1-3\}}.$$

In degrees that are multiples of 8 (denoted $8*$), the description of $ER(2)^*(RP^\infty, *)$ simplifies enormously. As x is the only generator whose degree is not a multiple of 4, and x^4 kills everything except powers of u, multiplication by powers of x produces no new elements in degree $8*$.

Corollary 5.1. *The homomorphism*

$$ER(2)^{8*}(RP^\infty, *) \to E(2)^{8*}(RP^\infty, *)$$

is injective, and is almost surjective – the only elements not hit are $v_2^4 u^j$ for $1 \le j \le 3$.

We shall compute the Bockstein spectral sequence associated to $ER(2)$ for the odd-dimensional projective space RP^{16K+9}. This will give us new non-immersion results from [13]. We will have to introduce some new elements for the computations in the next section. The Atiyah-Hirzebruch spectral sequence for $ER(2)^*(RP^2)$ gives elements x_1 and x_2 in filtration degrees 1 and 2. As a $ER(2)^*$ module $ER(2)^*(RP^2)$

is generated by elements we will call z_{-16} represented by xx_1 and z_2 represented by x_2. $z_{-16} = u \in ER(2)^*(RP^2)$.

For the cofibration $S^1 \to RP^2 \to S^2$, the long exact sequence

$$\begin{array}{ccc} ER(2)^*(S^1) & \xleftarrow{\quad i^* \quad} & ER(2)^*(RP^2) \\ & {\scriptstyle \partial} \searrow \qquad \nearrow {\scriptstyle \rho^*} & \\ & ER(2)^*(S^2,*) & \end{array}$$

is given by $\partial(i_1) = 2i_2, \rho^*(i_2) = z_2$, and $i^*(u) = xi_1$.

We know that $\Sigma^{2n-2}ER(2)^*(RP^2,*) \simeq ER(2)^*(RP^{2n}/RP^{2n-2},*)$. We have elements $z_{2n-18}, z_{2n} \in ER(2)^*(RP^{2n}/RP^{2n-2},*)$. These elements map to $v_2^{5n+3}u^n$ and $v_2^{5n}u^n$ respectively in $E(2)^*(RP^{2n}/RP^{2n-2},*)$ by [12].

6 $ER(2)^*(RP^{16K+9},*)$

In [12], Kitchloo and Wilson considered only even dimensional real projective spaces. Our object is to extend the results to certain odd-dimensional cases.

Proposition 6.1. *The element* $u^{8K+5} \in ER(2)^*(RP^{16K+10})$ *maps to a non-zero element of* $ER(2)^*(RP^{16K+9})$.

Note that this cannot happen for a complex oriented cohomology theory.

Proof We use the exact sequence

$$ER(2)^*(S^{16K+10},*) \xrightarrow{q^*} ER(2)^*(RP^{16K+10}) \longrightarrow ER(2)^*(RP^{16K+9})$$

We only have to show that u^{8K+5} is not in the image of q^*. Now $ER(2)^*(S^{16K+10},*)$ is the free $ER(2)^*$-module generated by the element i_{16K+10}, and q^* is known.

The structure of $ER(2)^*(RP^{16K+10})$ is given by Theorems 13.2 and 13.3 of [12].

We give a complete description of M, where

$$ER(2)^*(RP^{16K+10},*)/xER(2)^*(RP^{16K+10},*) \simeq M \subset E(2)^*(RP^{16K+10},*)$$

as a submodule of $E(2)^*(RP^{16K+10}) = \mathbb{Z}_{(2)}[\alpha, v_2^{\pm 1}][u]/(u^{8K+6}, [2](u))$. We describe M by specifying a 2-adic basis. As $d^r = 0$ for r even, M is filtered by $0 = M_0 \subset M_1 = M_2 \subset M_3 = M_4 \subset M_5 = M_6 \subset M_7 = M$, where $M_r/M_{r-1} \simeq \mathrm{Im} d^r$ and d^r is the differential in the Bockstein spectral sequence. Elements of M_r not in M_{r-1} lift to x^r-torsion elements of $ER(2)^*(RP^{16K+10},*)$.

As both α and u come from $ER(2)$-cohomology, we may describe M from section 5 as a filtered $\mathbb{Z}_2[\alpha, u]$-module. We write z_t for various elements of $ER(2)^*(RP^{16K+10},*)$ and $\bar{z}_t$ for it's image in M, where t denotes the degree.

$\alpha^k u^j(u\alpha_s) \in M_1$ for $k \geq 0$, $0 \leq j \leq 8K+4$, $0 \leq s \leq 3$, where $u\alpha_s = 2v_2^{2s}u = d^1(v_2^{2s+3}u)$. Note that $\alpha_0 = 2$.

$\alpha^k(u\alpha) \in M_3$ for $k \geq 0$, where $u\alpha = d^3(v_2^{-2}u)$.

$\alpha^k(uw) \in M_3$ for $k \geq 0$, where $uw = d^3(v_2^2 u) = uv_2^4\alpha$.

$\alpha^k u^j \beta_0 \in M_3$ for $k \geq 0$, $0 \leq j \leq 8K+1$, where $\beta_0 = d^3(v_2^{-2}u^2) = \alpha u^2 \equiv u^4 + \dots$ mod 2$= \alpha_0$.

$\alpha^k u^j \beta_1 \in M_3$ for $k \geq 0$, $0 \leq j \leq 8K+1$ where $\beta_1 = d^3(v_2^2 u^2) = wu^2$.

$\alpha^k \gamma_0 \in M_3$ for $k \geq 0$, where $\gamma_0 = v_2 \alpha u^{8K+5} = d^3(v_2^{-1}u^{8K+5})$.

$\alpha^k \gamma_1 \in M_3$ for $k \geq 0$, where $\gamma_1 = v_2^5 \alpha u^{8K+5} = d^3(v_2^3 u^{8K+5})$.

$\bar{z}_{16K+10} \in M_5$, where $z_{16K+10} = q^* i_{16K+10}$ is induced from S^{16K+10}. Then $\bar{z}_{16K+10} = v_2 u^{8K+5} = d^5(v_2^2 u^{8K+4})$. Note that $\alpha \bar{z}_{16K+10} = \gamma_0$ and $w\bar{z}_{16K+10} = \gamma_1$.

$\bar{z}_{16K-14} \in M_5$, where $\bar{z}_{16K-14} = v_2^5 u^{8K+5} = d^5(v_2^6 u^{8k+4})$. Note that $\alpha \bar{z}_{16K-14} = \gamma_1$ and $w\bar{z}_{16K-14} = \gamma_0$.

$\bar{z}_{16K+4} \in M_7$, where $\bar{z}_{16K+4} = v_2^2 u^{8K+5} = d^7(v_2^6 u^{8K+5})$.

$u^j \in M_7$, for $1 \leq j \leq 3$, since $d^7(v_2^4 u^j) = u^j$.

The action of α on M is clear, except for $\alpha\bar{z}_{16K+4} = -u^{8K+4}\alpha_1$. The relations involving u can also be determined, but are not useful. We really need the corresponding lifted relations in $ER(2)$-cohomology, where they hold only mod(x). Again, we quote

$$u^{8K+6} = x^2 z_{16K-14},\ u^{8K+7} = x^4 z_{16K+4},\ u^{8K+8} = 0. \tag{17}$$

Now in $ER(2)$-cohomology, $q^* i_{16K+10} = v_2 u^{8K+5}$, from (13.1) [12]. Since the degrees of α_2, α, w and x are $-12s, 16, 40$ and -17 respectively, mod 48, the only elements in $ER(2)^* i_{16K+10}$ of degree $-16(8K+5)$ have the form $\alpha^k w x^2 i_{16K+10}$. Then we have $q^*(\alpha^q w x^2 i_{16K+10}) = v_2 \alpha^q w x^2 u^{8K+5}$, which lies in the ideal (x) and is not the same as u^{8K+5}.

6.1 The Bockstein spectral sequence for $ER(2)^*(RP^{16K+9})$

We compute this cohomology by sandwiching it between the $ER(2)$-cohomology of RP^{16K+10} and RP^{16K+8}, which we know, in the commutative diagram of exact sequences

$$\begin{array}{ccccc}
ER(2)^*(S^{16K+10},*) & \xrightarrow{=} & ER(2)^*(S^{16K+10},*) & & \\
\downarrow \rho^* & & \downarrow & & \\
ER(2)^*(\frac{RP^{16K+10}}{RP^{16K+8}},*) & \longrightarrow & ER(2)^*(RP^{16K+10},*) & \longrightarrow & ER(2)^*(RP^{16K+8},*) \\
\downarrow i^* & & \downarrow & & \downarrow = \\
ER(2)^*(S^{16K+9},*) & \longrightarrow & ER(2)^*(RP^{16K+9},*) & \longrightarrow & ER(2)^*(RP^{16K+8},*) \\
\downarrow 2 & & & & \\
ER(2)^*(S^{16K+9},*) & & & &
\end{array} \tag{18}$$

The E^1-term for the Bockstein spectral sequence is just $E(2)^*(RP^{16K+9},*)$, which decomposes [4] as

$$E(2)^*(RP^{16K+8},*) \oplus E(2)^*(S^{16K+9},*).$$

via the maps

$$RP^{16K+8} \to RP^{16K+9} \to S^{16K+9}$$

(In general, $E(2)^*(RP^{2n}) = E(2)^*[u]/(u^{n+1},[2](u))$, as $E(2)$ is complex oriented and $E(2)^*$ has no 2-torsion.) The even part of the E^1-term has the 2-adic basis

$$v_2^i \alpha^k u^j \quad (0 \le i \le 7, \quad 0 \le k, \quad 0 \le j \le 8K+4)$$

The odd part is a free $\mathbb{Z}_{(2)}$-module with basis

$$v_2^i \alpha^k i_{16K+9} \quad (0 \le i \le 7, \quad 0 \le q, \quad 0 \le k)$$

Since d^1 is of even degree we can just read off our d^1 from Theorem 13.2 [12] for RP^{16K+8} and section 5 of [12] for the S^{16K+9} part.

$$d^1(v_2^{2s-5}\alpha^k u^j) = 2v_2^{2s}\alpha^k u^j = v_2^{2s}\alpha^{k+1}u^{j+1} + \ldots \quad (j \le 8K+4)$$

$$d^1(v_2^{2s+1}\alpha^k i_{16K+9}) = 2v_2^{2s-2}\alpha^k i_{16K+9}$$

Thus E^2 is given by

$$v_2^{2s}\alpha^k u \quad (k \ge 0), \quad v_2^{2s}u^j \quad (1 \le j \le 8K+4), \quad v_2^{2s+1}\alpha^k u^{8K+4} \quad (0 \le k),$$

$$v_2^{2s}\alpha^k i_{16K+9}$$

d^2 has odd degree 35. Since we have both odd and even degree elements in the E^2-term, d^2 might very well be non-trivial. If it is, then by naturality, it must have its source in the RP^{16K+8} part and target in the S^{16K+9} part. Also, the source cannot be anything from the E^2-term for the BSS for RP^∞, for we know that d^2 is trivial there. Therefore the only possible sources are $v_2^{2s+1}\alpha^k u^{8K+4}$ with possible targets $v_2^{2s}\alpha^k i_{16K+9}$.

Now, since v_2^2 is a unit, if there is a d^2, it must be non-zero on $v_2^{-1}u^{8K+4}$ which has degree $+6-16(8K+4)$ which is $-10+16K$ mod 48. The degree of the target must be this plus 35. The possible targets have degrees $-12s-32k+16K+9$. The only solutions are αi_{16K+9}, $\alpha^4 i_{16K+9}$ etc.

If d^2 is non-zero our guess would be $d^2(v_2^{-1}u^{8K+4}) = \alpha i_{16K+9}$. Thus we need to show that $x^2\alpha i_{16K+9} = 0$ in order for our guess to be correct.

The left column of (18) shows that the $ER(2)^*$-module $ER(2)^*(RP^{16K+10}/RP^{16K+8},*)$ is generated by two elements $z_{16K+10} = \rho^* i_{16K+10}$ and z_{16K-8}, where $i^* z_{16K-8} = xi^{16K+9}$.

We want to show that $x^2\alpha i_{16K+9} = 0$ in $ER(2)^*(RP^{16K+9},*)$. This is the image of the same named element in $ER(2)^*(S^{16K+9},*)$ which lifts to $x\alpha z_{16K-8} \in ER(2)^*(\frac{RP^{16K+10}}{RP^{16K+8}})$. This maps to $x\alpha v_2^4 u^{8K+5}$ in $ER(2)^*(RP^{16K+10},*)$ from (13.1) of [12]. Since $2x = 0$, we can use the relation $[2](u) = 2u +_F \alpha u^2 +_F u^4 = 0$ on αv_2^2, and we get

$$x\alpha v_2^4 u^{8K+5} = xv_2^4 u^{8K+7} + \dots$$

The least power of u which is zero in $ER(2)^*(RP^{16K+10})$ is $8K+8$. We have noted that $u^{8K+7} = x^4 z_{16K+4}$, so that we have $x^5 v_2^4 z_{16K+4}$, which is non-zero and does not lift to S^{16K+10}. It follows that $x^2\alpha i_{16K+9} \neq 0$.

However, if we multiply the whole calculation by α^3, we get $\alpha^3 x^5 v_2^4 z_{16K+4} = 0$, as $x^3\alpha = 0$. So $x^2\alpha^4 i_{16K+9} = 0$, and we conclude that

$$d^2(v_2^{2s+1}\alpha^k u^{8K+4}) = v_2^{2s+2}\alpha^{k+4} i_{16K+9}$$

Then, E^3 is given by

$$v_2^{2s}\alpha^k u \quad (0 \le k),\ \ v_2^{2s}u^j \quad (1 \le j \le 8K+4),\ \ v_2^{2s}\alpha^{\{0-3\}}i_{16K+9}.$$

d^3 is even degree so the even and odd parts don't mix under the differential. On both parts we already know the d^3 differential:

$$d^3(v_2^{\{6,2\}}\alpha^k u) = v_2^{\{0,4\}}\alpha^{k+1}u$$

$$d^3(v_2^{\{6,2\}}u^j) = v_2^{\{0,4\}}\alpha u^j = v_2^{\{0,4\}}u^{j+2} \quad (1 \le j \le 8K+2)$$

$$d^3(v_2^{4s-2}\alpha^{\{0-2\}}i_{16K+9}) = v_2^{4s}\alpha^{\{1-3\}}i_{16K+9}$$

$$d^3(v_2^{4s}\alpha^{\{0-3\}}i_{16K+9}) = 0$$

Thus E^4 is given by

$$v_2^{\{0,4\}}u^{\{1-3\}}, \quad v_2^{\{6,2\}}u^{8K+3,8K+4}, \quad v_2^{4s}i_{16K+9}.$$

d^4 has degree 21, which is odd. So it must go from the RP^{16K+8} part to the S^{16K+9} part. d^4 must be zero on anything in the image from RP^∞. So our non-zero differentials have possible sources $v_2^{\{6,2\}}u^{\{8K+3,8K+4\}}$, and possible targets $v_2^{4s}i_{16K+9}$ and $v_2^{\{6,2\}}\alpha^3 i_{16K+9}$. Let's compute the degrees (mod 48).

$$v_2^6 u^{8K+3} : -36 - 16(8K+3) \equiv -36 + 16K \longrightarrow -15 + 16K$$

$$v_2^2 u^{8K+3} : -12 - 16(8K+3) \equiv -12 + 16K \longrightarrow 9 + 16K$$

$$v_2^6 u^{8K+4} : -36 - 16(8K+4) \equiv -4 + 16K \longrightarrow 17 + 16K$$

$$v_2^2 u^{8K+4} : -12 - 16(8K+4) \equiv 20 + 16K \longrightarrow 41 + 16K$$

Comparing with degrees of the possible targets we see that the only possible differentials are:

$$d^4(v_2^{\{6,2\}}u^{8K+3}) = v_2^{\{0,4\}}i_{16K+9}$$

This must be true if i_{16K+9} is x^4-torsion. We invoke the commutative diagram (18) used before. Consider x^3z_{16K-8} in $ER(2)^*(\frac{RP^{16K+10}}{RP^{16K+8}})$. The following diagram shows its images in the lower left hand square of the commutative diagram.

$$\begin{array}{ccc} x^3z_{16K-8} & \longmapsto & x^3v_2^4u^{8K+5} \\ \downarrow & & \downarrow \\ x^4i_{16K+9} & \longmapsto & x^4i_{16K+9} \end{array}$$

The element in the upper right-hand corner is zero since $x^3u^4 = 0$. This shows i_{16K+9} is indeed x^4-torsion. We obtain our E^5-term

$$v_2^{\{0,4\}}u^{\{1-3\}}, \quad v_2^{\{6,2\}}u^{8K+4}.$$

d^5 has even degree. For dimensional reasons the differentials must be zero in the odd part, and Theorem 13.2 [12] determines that it is zero for the even part.

d^6 has degree 7. Again, it must go from even part to odd part by naturality. By sparseness, $d^6 = 0$.

Since d^7 has even degree, the differential does not mix odd and even degrees. First of all in the even part we have

$$d^7(v_2^4u^{\{1-3\}}) = u^{\{1-3\}}$$

Also by mapping to RP^{16K+8}(page 23 [12]) we get that

$$d^7(v_2^6u^{8K+4}) = v_2^2u^{8K+4}$$

We collect our results in the following theorem.

Theorem 6.1. *The Bockstein spectral sequence for $ER^*(RP^{16K+9}, *)$ is as follows:*

E^1

$$v_2^i \alpha^k u^j \quad (0 \leq i \leq 7, \quad 0 \leq k, \quad 0 \leq j \leq 8K+4)$$

$$v_2^i \alpha^k i_{16K+9} \quad (0 \leq i \leq 7, \quad 0 \leq k)$$

$$d^1(v_2^{2s-5} \alpha^k u^j) = 2v_2^{2s} \alpha^k u^j = v_2^{2s} \alpha^{k+1} u^{j+1} + \ldots \quad (j \leq 8K+3)$$

$$d^1(v_2^{2s+1} \alpha^k i_{16K+9}) = 2v_2^{2s-2} \alpha^k i_{16K+9}$$

where $v_2^i \alpha^k i_{16K+9}$ generates a free $\mathbb{Z}_{(2)}$-module in E^1.

E^2

$$v_2^{2s} \alpha^k u \quad (k \geq 0), \quad v_2^{2s} u^j \quad (1 \leq j \leq 8K+4), \quad v_2^{2s+1} \alpha^k u^{8K+4} \quad (0 \leq k),$$

$$v_2^{2s} \alpha^k i_{16K+9}$$

$$d^2(v_2^{2s+1} \alpha^k u^{8K+4}) = v_2^{2s+2} \alpha^{k+4} i_{16K+9}$$

E^3

$$v_2^{2s} \alpha^k u \quad (0 \leq k), \;\; v_2^{2s} u^j \quad (1 \leq j \leq 8K+4), \;\; v_2^{2s} \alpha^{0-3} i_{16K+9}$$

$$d^3(v_2^{\{6,2\}} \alpha^k u) = v_2^{\{0,4\}} \alpha^{k+1} u$$

$$d^3(v_2^{\{6,2\}} u^j) = v_2^{\{0,4\}} \alpha u^j = v_2^{\{0,4\}} u^{j+2} \quad (2 \leq j \leq 8K+2)$$

$$d^3(v_2^{4s-2} \alpha^{\{0-2\}} i_{16K+9}) = v_2^{4s} \alpha^{\{1-3\}} i_{16K+9}$$

$$d^3(v_2^{4s} \alpha^{\{0-3\}} i_{16K+9}) = 0$$

E^4

$$v_2^{\{0,4\}}u^{\{1-3\}}, \quad v_2^{\{6,2\}}u^{\{8K+3,8K+4\}}, \quad v_2^{4s}i_{16K+9}$$

$$d^4(v_2^{\{6,2\}}u^{8K+3}) = v_2^{\{0,4\}}i_{16K+9}$$

$E^5 = E^6 = E^7$

$$v_2^{\{0,4\}}u^{\{1-3\}}, \ v_2^{\{6,2\}}u^{8K+4}, \ v_2^{\{6,2\}}\alpha^3 i_{16K+9}$$

$$d^7(v_2^4 u^{\{1-3\}}) = u^{\{1-3\}}, \quad d^7(v_2^6 u^{8K+4}) = v_2^2 u^{8K+4}$$

Next we identify all the elements in degree 8*.

Theorem 6.2. *A 2-adic basis of $ER(2)^{8*}(RP^{16K+9}, *)$ is given by the elements*

$$\alpha^k u^j, \ (k \geq 0, 1 \leq j \leq 8K+4)$$

$$v_2^4 \alpha^k u^j, \ (k \geq 1, 1 \leq j \leq 8K+4)$$

$$v_2^4 u^j, \ (4 \leq j \leq 8K+4)$$

$$x\alpha^k i_{16K+9}, \ xv_2^4\alpha^k i_{16K+9}, \ (k \geq 0)$$

Proof The first classes of elements represent the images of differentials in the spectral sequence that do not involve i_{16K+9}. As in [13], multiplication by powers of x leads to no new elements in degree $8*$. Those images involving i_{16K+9} provide x^2,x^3, or x^3-torsion, which may be multiplied by x.

Corollary 6.1. *There is an algebraic map*

$$ER(2)^{8*}(RP^{16K+9}) \to E(2)^{8*}(RP^{16K+10})$$

which only misses the elements $v_2^4 u^{\{1-3\}}$.

7 Non-Immersions

If RP^b immerses in $\mathbb{R}^c$, James showed [10] that there is an axial map

$$m : RP^b \times RP^{2^L-c-2} \to RP^{2^L-b-2}$$

for large L (meaning a map that is non-trivial on both axes). Specifically, to show that RP^{2n} does not immerse in $\mathbb{R}^{2k+1}$ we need to prove there is no axial map

$$m : RP^{2n} \times RP^{2^L-2k-3} \to RP^{2^L-2n-2} \tag{19}$$

Our strategy is to consider the class $u \in ER(2)^*(RP^{2^L-2n-2})$, which satisfies $u^{2^{L-1}-n} = 0$ when $n \equiv 0$ or 7 mod 8 (Theorem 1.6.[12]). We shall see that m^*u is known, in principle. If we can show that $(m^*u)^{2^{L-1}-n} \neq 0$, we have a contradiction.

Davis [4] used this approach, by using the complex-oriented cohomology theory $E(2)$ to deduce that RP^{2n} does not immerse in $\mathbb{R}^{2k}$ by showing there is no axial map

$$m : RP^{2n} \times RP^{2^L-2k-2} \to RP^{2^L-2n-2}$$

when $n = m + \alpha(m) - 1$ and $k = 2m - \alpha(m)$ for some m, where $\alpha(m)$ denotes the number of 1's in the binary expansion of m. We wish to improve this result to show that for certain n and k, (19) does not exist.

There is an axial map $m : RP^\infty \times RP^\infty \to RP^\infty$, which is the restriction of the map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ induced by the tensor product of the canonical Real line bundles. Therefore $m^*u = u_1 +_F u_2$, where u_1, u_2 and u are the Chern classes for the three copies of RP^∞.

If $m : RP^b \times RP^c \to RP^d$ is an axial map, the diagram

$$\begin{array}{ccc} RP^b \times RP^c & \xrightarrow{m} & RP^d \\ \downarrow \subset & & \downarrow \subset \\ RP^\infty \times RP^\infty & \xrightarrow{m} & RP^\infty \end{array}$$

commutes, as all axial maps $RP^b \times RP^c \to RP^\infty$ are homotopic. It follows that the same formula $m^*u = u_1 +_F u_2$ holds for this m. As the formal group law F is a formal power series in u_1 and u_2 over $\mathbb{Z}_{(2)}[\alpha]$ and $\deg u = -16$ and $\deg \alpha = 16$, we are interested only in degrees that are multiples of 16. This simplifies our work, as $ER(2)^{16*}(RP^\infty) \to E(2)^{16*}(RP^\infty)$ is a an isomorphism by [12].

We assume $k = 2 \bmod 8$, so that $2^L - 2k - 2 = 16K + 10$ and we can use Theorem 6.2. Consider the diagram

$$\begin{array}{ccc} ER(2)^{16*}(RP^\infty \times RP^\infty) & \longrightarrow & E(2)^{16*}(RP^\infty \times RP^\infty) \\ \downarrow & & \downarrow \\ ER(2)^{16*}(RP^{2n} \times RP^{16K+10}) & \longrightarrow & E(2)^{16*}(RP^{2n} \times RP^{16K+10}) \\ \downarrow & & \downarrow \\ ER(2)^{16*}(RP^{2n} \times RP^{16K+9}) & \longrightarrow & E(2)^{16*}(RP^{2n} \times RP^{16K+9}) \end{array}$$

From Don Davis's work, the image of $(u_1 +_F u_2)^{2^{L-1}-n} \in ER(2)^{16*}(RP^\infty \times RP^\infty)$ in $E(2)^{16*}(RP^{2n} \times RP^{16K+10})$ is non-zero. We need to show that the image in $ER(2)^{16*}(RP^{2n} \times RP^{16K+9})$ is non-zero, (Note that we cannot use $E(2)$-cohomology for this purpose, as it is complex-oriented, which implies that $u^{8K+5} \in E(2)^*(RP^{16K+10})$ maps to zero in $E(2)^*(RP^{16K+9})$.

The two end terms in $(u_1 +_F u_2)^{2^{L-1}-n}$ are $u_1^{2^{L-1}-n}$ and $u_2^{2^{L-1}-n}$, which are plainly zero; all the other terms have the form $\lambda \alpha^k u_1^i u_2^j$, where $\lambda \in \mathbb{Z}_{(2)}, k \geq 0, i \geq 1, j \geq 1,$

and $i+j \geq 2^{L-1}-n$. Following [12], we may use the formulae

$$2u_1 = -\alpha u_1^2 + \dots,\ \ \alpha u_1 u_2^2 = \alpha u_1^2 u_2 + \dots$$

and induction to reduce $(u_1 +_F u_2)^{2^{L-1}-n}$ to a sum of distinct terms of the forms $\alpha^k u_1^i u_2$ and $u_1^i u_2^j$, with no numerical cofficient. (The first formula comes from $[2](u_1) = 0$; the second from $u_1[2](u_2) - u_2[2](u_1) = 0$.) Further, again by [12], $u_1^{n+1} = 0$ since $n \equiv 0$ or 7 mod 8. Then $\alpha^k u_1^i u_2 = 0$, since we still have $i+1 \geq 2^{L-1}-n$. We do not know (or need to know) exactly which terms are present; all we have to do is show that the monomials $u_1^i u_2^j$ (for $1 \leq i \leq n$ and $1 \leq j \leq 8K+5$) remain linearly independent in $ER(2)^*(RP^{2n} \times RP^{16K+9})$, which we defer to the next section.

Meanwhile, let us review the various numerical conditions. We need $n = m + \alpha(m) - 1$, $k = 2m - \alpha(m)$,$k \equiv 2$ mod 8 and $n \equiv 0$ or 7 mod 8. So $2m - \alpha(m) \equiv 2$ and $m + \alpha(m) \equiv 0$ or 1. Solving these, we get $(m, \alpha(m)) \equiv$ (6,2) or (1,0).

Theorem 7.1. *If $(m, \alpha(m)) \equiv$ (6,2) or (1,0) mod 8,*

$RP^{2(m+\alpha(m)-1)}$ does not immerse in $\mathbb{R}^{2(2m-\alpha(m))+1}$.

7.1 Products with an odd space

We shall look into the Bockstein spectral sequence for

$$ER(2)^*(RP^{2n} \wedge RP^{16K+9}, *)$$

where $2n < 16K+9$.

The E^1-term is the usual

$$E(2)^*(RP^{2n} \wedge RP^{16K+9}, *)$$

$$\simeq E(2)^*(RP^{2n},*)\otimes E(2)^*(RP^{16K+9},*)\oplus \Sigma^{-16(8K+4)-1}E(2)^*(RP^{2n},*)$$

(from [7]) Also, we know that

$$E(2)^*(RP^{16K+9},*)\cong E(2)^*(RP^{16K+8},*)\oplus E(2)^*(S^{16K+9},*).$$

Since $E(2)^*(S^{16K+9},*)$ is free it does not affect the Tor term, only the tensor product. So our E^1-term is:

$$E(2)^*(RP^{2n},*)\otimes E(2)^*(RP^{16K+8},*)\oplus E(2)^*(RP^{2n},*)\otimes E(2)^*(S^{16K+9},*)$$

$$\oplus\Sigma^{16K-17}E(2)^*(RP^{2n},*)$$

A 2-adic basis for this is given by

$$v_2^s\alpha^k u_1^i u_2 \quad (0\le k, \quad 0<i\le n, \quad s<8)$$

$$v_2^s u_1^i u_2^j \quad (0<i\le n, \quad 1<j\le 8K+4, \quad s<8)$$

by the same reduction as before, and

$$v_2^s\alpha^k i_{16K+9} \quad (0\le k, \quad 0<i\le n, \quad s<8)$$

$$v_2^s\alpha^k u_1^i z_{16K-33} \quad (0\le k, \quad 0\le i<n, \quad s<8).$$

We know that xi_{16K+9} represents $v_2^4u_2^{8K+4}$. So $v_2^4xu_1^ni_{16K+9}$ represents $u_1^nu_2^{8K+5}$. There is no differential on $u_1^ni_{16K+9}$. Also there is no differential on $v_2^4u_1^ni_{16K+9}$. All we have to do is show that $u_1^ni_{16K+9}$ is not in the image of d^1. Since d^1 has even degree we only have to worry about the odd degree elements since $u_1^ni_{16K+9}$ is odd degree.

d^1 has degree 18 so if it is to hit $u_1^ni_{16K+9}$ it must start at some $\alpha^ku_1^iz_{16K-33}$

because they are the only elements in the correct degree modulo 16. Then we would have d^1 non-trivial on z_{16K-33}.

In the Bockstein spectral sequence for $ER(2)^*(RP^{16M+16} \wedge RP^{16K+10}, *)$, $8M+8 < 8K+5$, we have from Theorem 19.2 [12], $d^1(z_{16K-1}) = 0$. From Theorem 1.2 of [7] we have that z_{16K-1} maps to $u_1 z_{16K-33}$ in the spectral sequence for $ER(2)^*(RP^{16K+16} \wedge RP^{16K+10}, *)$. Since this passes through the spectral sequence for $ER(2)^*(RP^{16M+16} \wedge RP^{16K+9}, *)$, z_{16K-1} maps to $u_1 z_{16K-33}$ here as well. So $d^1(u_1 z_{16K-33}) = u_1 d^1(z_{16K-33}) = 0$.

$$\begin{array}{ccc} ER(2)^*(RP^{16K+16} \wedge RP^{16K+10}) & \longrightarrow & ER(2)^*(RP^{16M+16} \wedge RP^{16K+8}) \\ \downarrow & \nearrow & \\ ER(2)^*(RP^{16M+16} \wedge RP^{16K+9}) & & \end{array}$$

All elements killed by multiplication by u_1 go to zero under the map to $RP^{16K} \wedge RP^{16K+9}$, so our $d^1(z_{16K-33})$ is zero.

Theorem 7.2. *When $n \leq 8M < 8M + 8 < 8K + 5$, in*

$$ER(2)^{16*}(RP^{2n} \wedge RP^{16K+9})$$

the element $u_1^n u_2^{8K+5}$ is non-zero.

References

[1] S.Araki, M.Murayama, *τ-cohomology theories*, Japan J. Math (N.S.), 4(1978) 363-416.

[2] M.F.Atiyah, *K-theory and Reality*, Quar. J. Math., 17(1966), 367-386.

[3] D.Davis, *Table of immersions and embeddings of real projective spaces*, http://www.math.lehigh.edu/dmd1/immtable

[4] D.Davis, *A strong immersion theorem for real projective spaces*, Ann. of Math., 120:510-520, 1984.

[5] A.D.Elmendorf, I.Kriz, M.Mandell, J.P.May, *Rings, Modules, and Algebras in Stable Homotopy Theory*, Mathematical Surveys and Monographs, 47, Amer. Math. Soc.

[6] J.P.C.Greenlees, J.P.May, *Generalized Tate Cohomology*, Mem. Amer. Math. Soc., 113.

[7] Jesus González, W.S.Wilson, *The BP-theory of two-fold products of projective spaces.*

[8] P.Hu, *The Ext^0-term of the Real-oriented Adams-Novikov spectral sequence*, Homotopy Methods in algebraic topology, 271, Contemporary Mathematics, 141-153, AMS.

[9] P.Hu, I.Kriz, *Real-oriented Homotopy Theory and an analogue of the Adams-Novikov Spectral Sequence*, Topology, 40(2):317-399, 2001.

[10] I.M.James, *On the immersion problem of real projective spaces*, Bull. Amer. Math., Soc. 69:231-238, 1963.

[11] N.Kitchloo, W.S.Wilson, *On fibrations related to real spectra.* Proceedings of the Nishida fest (Kinosaki 2003), volume 10, Geometry and Topology Monographs, pages 237-244, 2007.

[12] N.Kitchloo, W.S.Wilson, *The second real Johnson-Wilson theory and non-immersions of RP^n*, Homology, Homotopy Appl., 10(3), 2008, 223-268.

[13] N.Kitchloo, W.S.Wilson *The second real Johnson-Wilson theory and non-immersions of RP^n, Part 2*, Homology, Homotopy Appl., 10(3), 2008, 269-290.

[14] L.G.Lewis, Jr., J.P.May, M.Steinberger *Equivariant Stable Homotopy Theory*, Lecture Notes in Math., 1213, Springer Verlag, New York, 1985.

[15] P.Landweber. *Conjugations of complex manifolds and equivariant homotopy of MU*, Bull. Amer. Math. Soc., 74, 1968.

Vita

Romie Banerjee was born in August 1982 and was raised in Haldia, West Bengal, India. In 2004, he received his Bachelor of Science degree in mathematics from Chennai Mathematical Institue, Chennai, India. In 2005, he received his Master of Arts degree in mathematics from Johns Hopkins University. He defended his thesis on March 23, 2010.