Balanced Metrics in Kähler Geometry

by

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Updated

Abstract

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Acknowledgments

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Chapter 1

Introduction

A central notion in geometric invariant theory (GIT) is the concept of stability. Stability plays a significant role in forming quotient spaces of projective varieties for which geometric invariant theory was invented. One can define Mumford-Takemoto slope stability for holomorphic vector bundles, and also there is a notion of Gieseker stability which is more in the realm of geometric invariant theory. It is well-known that over algebraic curves these different notions coincide.

On the other hand, in differential geometry, metrics with certain curvature property have been interesting to mathematicians for years. One of the earliest examples of such metrics are Einstein metrics. Einstein metrics are metrics which are proportional to their Ricci curvature. Einstein introduced the concept of Einstein metrics in order to formulate relativity theory. Later Yang and Mills introduce the Yang-Mills equations which are the generalization of Maxwells equations. Solutions to the Yang-Mills equations are connections over vector bundles which satisfy certain curvature property. In the context of holomorphic vector bundles over Kähler manifolds, the Yang-Mills equation corresponds to the Hermitian-Einstein equation which is analogue of Einstein metrics in the setting of holomorphic vector bundles.

There is a close relationship between the concept of stability coming from the algebraic geometric side and the existence of Hermitian-Einstein metrics. In ?, Seshadri and Narasimhan prove that a holomorphic vector bundle over a compact Riemann surface is poly-stable if and only if it admits a Hermitian-Einstein metric. The picture becomes complete after the work of Donaldson, Uhlenbeck and Yau. They prove the following which is known as the Hitchin-Kobayashi correspondence.

Theorem 1.0.1. *([D1],[D2],[UY]) Let (X, ω) be a compact Kähler manifold and $E \to X$ be a holomorphic vector bundle. Then E is Mumford poly-stable if and only if E admits a Hermitian-Einstein metric.*

We recall the definition of Mumford slope stability. Let (X, ω) be a compact Kähler manifold of dimension n and $E \to X$ be a holomorphic vector bundle of rank r. We can define the slope of the bundle E by $\mu(E) = deg(E)/r$, where $deg(E) = \int_X c_1(E) \wedge \omega^{n-1}$. Notice that for $n \geq 2$, the slope depends on the cohomology class of ω as well as $c_1(E)$. A holomorphic vector bundle E is called Mumford (semi)stable if for any coherent subsheaf F of E with lower rank,($\mu(F) \leq \mu(E)$) $\mu(F) < \mu(E)$.

Beside the notion of Mumford slope stability, there is another notion of stability introduced by Gieseker which is more in the realm of GIT. Let (X, L) be a polarized algebraic manifold and E be a holomorphic vector bundle over X. The bundle E is called Gieseker stable if for any proper coherent subsheaf F of E

$$\frac{h^0(X, F \otimes L^k)}{rk(F)} < \frac{h^0(X, E \otimes L^k)}{rk(E)},$$

for $k \gg 0$. The recent work of X. Wang ([W1], [W2]) gives a geometric interpretation of Gieseker stability. Wang proves that there is a relation between Gieseker poly-stability and existence of so-called balanced metrics. This relation was conjectured first by Donaldson in ([D5]).

The situation is more complicated in the case of polarized varieties. Canonical metrics on polarized varieties have been studied for years. Some of the earliest work was done by Calabi who introduced the notion of extremal metrics. He also proved some uniqueness results and conjectured the existence of Kähler-Einstein metrics on certain types of complex manifolds. The celebrated work of Yau solves the problem of the existence of Kähler-Einstein metrics on compact complex manifolds with trivial canonical class([Y1], [Y2]). Also, Aubin and Yau independently proved the existence of Kähler-Einstein metrics on compact complex manifolds with negative first chern class ([A],[Y1], [Y2]). The case of positive chern class corresponds to manifolds with negative canonical class which are called Fano manifolds. It is known that there are obstructions to the existence of Kähler-Einstein metrics in this case. The first known obstruction is due to Matsushima. He shows that if such a metric exists, then the Lie algebra of holomorphic vector fields must be reductive. Another obstruction to the existence of Kähler-Einstein metrics is the Futaki invariant which is coming from holomorphic vector fields on the manifold. Tian proves that vanishing of Futaki invariants on smooth Fano surfaces implies the existence of Kähler-Einstein metrics([T1]). Later in ([T2]), Tian constructs a three dimensional Fano manifold which has no nontrivial holomorphic vector fields (hence vanishing Futaki invariant) and yet does not admit any Kähler-Einstein metric. He shows that this example does not satisfy the so-called weak K-stability condition which is introduced by Tian in the same paper. Then he conjectures that for Fano manifolds the weak K-stability is a necessary and sufficient condition for the existence of Kähler-Einstein metrics.

Inspired by the Hitchin-Kobayashi correspondence, Yau conjectures that there should be a similar correspondence in the case of polarized varieties. More precisely, Yau conjectures that for any smooth Fano variety X, there is a relationship between the existence of Kähler-Einstein metrics and the stability of polarized variety (X, K_X^{-1})

in some GIT sense. Yau's conjecture was generalized by Tian and Donaldson. They develop the notion of K-stability for polarized varieties and conjecture that for a polarized variety (X, L), the existence of constant scalar curvature Kähler (cscK) metrics in the class of $2\pi c_1(L)$ is equivalent to the K-polystability of (X, L). In ([D3]), Donaldson proves the following

Theorem 1.0.2. *Let (X, L) be a polarized variety. Assume that $Aut(X, L)/\mathbb{C}^*$ is discrete. If there exists a constant scalar curvature Kähler metric ω_∞ in the class of $2\pi c_1(L)$, then (X, L^k) admits a balanced metric for $k \gg 0$ and the sequence of rescaled balanced metrics ω_k converges to ω_∞ in the C^∞-norm.*

By the earlier result of Zhang ([Zh]), we know that the Chow stability of (X, L) is equivalent to the existence of balanced metrics on L. Therefore, Donaldson's theorem implies that asymptotically Chow stable is a necessary condition for the existence of cscK metrics. In some sense, this basically proves one direction of Donaldson-Tian-Yau's conjecture.

1.1 Space of Fubini-Study metrics

Let X be a compact complex manifold of dimension n and L be a positive line bundle over X. By Kodaira embedding theorem, for $k \gg 0$ we get a sequence of embeddings

$$\iota_k : X \to \mathbb{P}(H^0(X, L^k)^*),$$

such that $\iota_k^* \mathcal{O}(1) = L^k$. Any hermitian inner product on $H^0(X, L^k)$ induces a Fubini-Study metric on the line bundle $\mathcal{O}(1)$ and therefore on the line bundle L. We denote the space of all such metrics on L by $\mathcal{K}_k$.

Tian proved that any positive metric g on L can be approximated by a sequence of metrics g_k, where $g_k \in \mathcal{K}_k$. More precisely, he proved the following

Theorem 1.1.1. *Let $\underline{s}^{(k)} = (s_0^{(k)}, \ldots, s_{N_k}^{(k)})$ be an orthonormal basis for $H^0(X, L^k)$ with respect to the following hermitian inner product.*

$$\langle s, t \rangle = \int_X \langle s(x), t(x) \rangle_{g^{\otimes k}} \frac{\omega_g(x)^n}{n!}.$$

Let $g_k = \iota^*_{\underline{s}^{(k)}} h_{FS}$, where $\iota_{\underline{s}^{(k)}} : X \to \mathbb{P}^{N_k}$ is the Kodaira embedding using the basis $\underline{s}^{(k)}$. Then

$$\big(g_k\big)^{\frac{1}{k}} \to g \quad as \;\; k \to \infty,$$

in C^2- topology.

Later Ruan proved the convergence in C^∞-topology. A major development regarding the behavior of the sequence g_k in the statement of the above theorem was made by fundamental result of Catlin and Zelditch. They show the existence of a complete asymptotic expansion for the sequence.

Theorem 1.1.2. *With the notation of the above theorem, define*

$$\rho_k(g)(x) = \sum_{i=0}^{N_k} |s_i^{(k)}(x)|^2_{g^{\otimes k}}.$$

Then there exist functions $a_0(x), a_1(x,)\ldots$ which such that the following asymptotic expansion holds in C^∞.

$$\rho_k(g)(x) \sim a_0 k^n + a_1 k^{n-1} + \ldots.$$

Moreover, $a_0 = 1$ and $a_1 = \frac{S(\omega_g)}{2}$, where $S(\omega_g)$ is the scalar curvature of ω_g.

The same result holds if we twist L^k with a holomorphic vector bundle.

Mabuchi introduce a functional on the space of positive metrics on a an ample line bundle L. This functional has the feature that if we restrict it to the space of Fubini-Study metrics on L^k, its critical points (if there is any) are exactly balanced metrics. As it mentioned before, by a result of Zhang existence of balanced metric on L is equivalent to the Chow stability of the polarized manifold (X, L). In the case of holomorphic bundle E, there is a similar functional introduced by Donaldson. If we restrict this functional to the space of Fubini-Study metrics on E, its critical points are exactly balanced metrics. Again the existence of balanced metric on E is equivalent to the stability of Gieseker point of E. In this thesis, we introduce a functional F on the space of positive metrics on E. We define that a metric h on E is strongly balance if it is a critical point of the restriction of F to the space of Fubini-Study metrics on E. It is trivial that a strongly balanced metric on E is balanced in the sense of Wang. Therefore the existence of strongly balanced metric on E implies the stability of Gieseker point of E. We also find a GIT interpretation for the restriction of F to the space of Fubini-Study metrics on E (cf. Proposition 3.3.4.).

1.2 Numerical algorithm to find balanced metrics

As it mentioned before, we can approximate the space of positive metrics on a line bundle L by the space of Fubini-Study metrics on L coming from high power of L. Every such a Fubini-Study metric corresponds to a hermitian inner product on the space of global sections of some power of L. Fix a large integer k. We can define a map FS from the space of hermitian inner products on $H^0(X, L^k)$ to the space of positive metrics on L^k using Kodaira embedding. On the other, any positive metric on L^k induces an L^2- inner product on the space of sections which we call it Hilb. Therefore, we obtain a map $T = \mathrm{Hilb} \circ \mathrm{FS}$ from the space of hermitian inner product on $H^0(X, L^k)$ to itself. It is easy to see that balanced metrics are correspond to fixed points of T. Starting with a hermitian inner product H on $H^0(X, L^k)$, if the sequence $\{T^l(H)\}$ converges, then the limit must be a fixed point of T and therefore a balnced metric. Donaldson mentions that if there exists a unique balanced metric on L^k to a constant, then the sequence $\{T^l(H)\}$ converges ([D6]). In [Sa], Sano proves the following.

Theorem 1.2.1. *Suppose that $Aut(X, L)/\mathbb{C}^*$ is discrete. If L admits a balanced metric, then the sequence $\{T^l(H)\}$ converges as $l \to \infty$.*

Notice that the discreteness of $Aut(X, L)/\mathbb{C}^*$ implies that the balance metric is unique up to a constant if it exists.

The same picture holds for holomorphic vector bundles. The following is conjectured by Douglas, et. al. in [DKLR].

Theorem 1.2.2. *Suppose that E is simple and admits a balanced metric. Then for any $H_0 \in M_E$, the sequence $T^r(H_0)$ converges to H_∞, where H_∞ is a balanced metric on E.*

1.3 Chow stability of ruled manifolds

Prior to the developments discussed in the begining, in ([M]), Morrison proved that for a rank two vector bundle over a compact Riemann surface, slope stability of the bundle is equivalent to the Chow stability of the corresponding ruled surface with respect to certain polarizations.

One of the earliest results in this spirit is the work of Burns and De Bartolomeis in [BD]. They construct a ruled surface which does not admit any extremal metric in a certain cohomology class. In [H1], Hong proves that there are constant scalar curvature Kähler metrics on the projectivization of stable bundles over curves. In [H2] and [H3], he generalizes this result to higher dimensions with some extra assumptions. Combining Hong's results with Donaldson's, one can see that $(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*_m}(n))$ is Chow stable for $m, n \gg 0$ when the bundle E is poly-stable and the base manifolds admits a constant scalar curvature Kähler metric. Note that it concerns with Chow stability of $(\mathbb{P}E^*\mathcal{O}_{\mathbb{P}E^*_m}(n))$ for n big enough.

In [RT], Ross and Thomas develop the notion of slope stability for polarized algebraic manifolds. As one of the applications of their theory, they prove that if $(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L^k)$ is slope semi-stable for $k \gg 0$, then E is a slope semi-stable bundle and (X, L) is a slope semi-stable manifold. For the case of one dimensional base of genus $g \geq 1$, they show stronger results. In this case they prove that if $(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L^k)$ is slope (semi, poly) stable for some k, then E is a slope (semi, poly) stable bundle.

In this thesis, We generalize one direction of Morrison's result to higher rank vector bundles over compact algebraic manifolds. Let (X, ω) be a compact Kähler manifold of dimension n and $L \to X$ be a polarization for X such that $\omega \in 2\pi c_1(L)$. Let $E \to X$ be a holomorphic vector bundle over X and $\pi : \mathbb{P}E^* \to X$ be the projection map. We have proven the following in Section 5.6.

Theorem 1.3.1. *([S2]) Suppose that $Aut(X)$ is discrete. If E is Mumford slope stable and X admits a constant scalar curvature Kähler metric in the class of $2\pi c_1(L)$, then*

$$(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L^k)$$

is Chow stable for $k \gg 0$.

Since Chow stability is equivalent to the existence of balanced metrics, in order to prove Theorem 1.3.1, it suffices to show that $(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L^k)$ admits balanced metrics for $k \gg 0$. The strategy of the proof is as follows:

First we show that there exists an asymptotic expansion for the Bergman kernel of $(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L^k)$(Theorem 1.3.2). Let σ be a positive hermitian metric on L such that $Ric(\sigma) = \omega$. For any hermitian metric g on $\mathcal{O}_{\mathbb{P}E^*}(1)$, we define the volume form $d\mu_{g,k}$ as follows

$$d\mu_{g,k} = k^{-n}\frac{(\omega_g + k\pi^*\omega)^{n+r-1}}{(n+r-1)!} = \sum_{j=0}^{n} k^{j-n}\frac{\omega_g^{n+r-1-j}}{(n+r-j)!}\wedge\frac{\pi^*\omega^j}{j!},$$

where $\omega_g = Ric(g)$. We prove the following in Section 5.4.

Theorem 1.3.2. *([S2]) Let h be a hermitian metric on E and g be the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}E^*}(1)$ induced by the hermitian metric h. Then there exist smooth endomorphisms $A_i \in \Gamma(X, E)$ such that*

$$\rho_k(g,\omega)([v]) \sim C_r Tr\big(\lambda(v,h)(k^n + A_1k^{n-1} + ...)\big),$$

where $\rho_k(g,\omega)$ is the Bergman kernel of $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1)\otimes\pi^*L^k)$ with respect to the L^2-inner product $L^2(g\otimes\sigma^{\otimes k}, d\mu_{k,g})$, C_r is a positive constant depends only on the rank of the vector bundle E and $\lambda(v,h) = \frac{1}{||v||_h^2}v\otimes v^{*_h}$ is an endomorphism of E. Moreover

$$A_1 = \frac{i}{2\pi}\Lambda F_{(E,h)} - \frac{i}{2\pi r}tr(\Lambda F_{(E,h)})I_E + \frac{(r+1)}{2r}S(\omega)I_E,$$

where Λ is the trace operator acting on $(1,1)$-forms with respect to the Kähler form ω.

Finding balanced metrics on $\mathcal{O}_{\mathbb{P}E^*}(1)\otimes\pi^*L^k$ is basically the same as finding solutions to the equations $\rho_k(g,\omega) = \text{Constant}$. Therefore in order to prove Theorem 1.3.1, we need to solve the equations $\rho_k(g,\omega) = \text{Constant}$ for $k \gg 0$. Now if ω has constant scalar curvature and h satisfies the Hermitian-Einstein equation $\Lambda_\omega F_{(E,h)} = \mu I_E$, then $A_1(h,\omega)$ is constant. Notice that in order to make A_1 constant, existence of Hermitian-Einstein is not enough. We need the existence of constant scalar curvature Kähler metric as well. The crucial fact is that the linearization of A_1 at (h,ω) is surjective. This enables us to construct formal solutions as power series in k^{-1} for the equation $\rho_k(g,\omega) = \text{Constant}$. Therefore, for any positive integer q, we can construct a sequence of metrics g_k on $\mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L^k$ and bases $\underline{s}^k = (s_1^{((k))}, ..., s_N^{(k)})$ for $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1))$ such that

$$\sum |s_i^{(k)}|^2_{g_k} = \text{Constant},$$

$$\int_{\mathbb{P}E^*} \langle s_i^{(k)}, s_j^{(k)} \rangle_{g_k} dvol_{g_k} = D_k I + M_k,$$

where $D_k \to 1$ as $k \to \infty$, and M_k is a trace-free hermitian matrix such that $||M_k||_{op} = o(k^{-q-1})$ as $k \to \infty$ for big enough positive constant q.

Now the next step is to perturb these almost balanced metrics to get balanced metrics. As pointed out by Donaldson, the problem of finding balanced metric can be viewed also as a finite dimensional moment map problem solving the equation $M_k = 0$. Indeed, Donaldson shows that M_k is the value of a moment map μ_D on the space of ordered bases with the obvious action of $SU(N)$. Now, the problem is to show that if for some ordered basis $\underline{s}$, the value of moment map is very small, then we can find a basis at which moment map is zero. The standard technique is flowing down $\underline{s}$ under the gradient flow of $|\mu_D|^2$ to reach a zero of μ_D. We need a lower bound for $d\mu_D$ to guarantee that the flow converges to a zero of the moment map. We do this by adapting Phong-Sturm proof to our situation ([PS2]).

1.4 Outline

In the second chapter, we give some background in Kähler geometry and geometric invariant theory. Also, we define and state basic facts about balanced metrics. In the third chapter, we construct a functional on the space of Fubini-Study metrics on a very ample holomorphic vector bundle. We show that this functional is convex along geodesics. We give GIT interpretation of this functional. In Chapter 4, we define a discrete dynamical system in the space of Fubini-Study metrics on a very ample holomorphic vector bundle and prove the convergence of the dynamical system under the assumption of stability of the Gieseker point of the bundle and simplicity of the bundle. In Chapter 5, we study projectivization of vector bundle. The main result of this chapter is Theorem 1.3.1 which gives a sufficient condition for stability of projectivization with respect to certain polarizations. There are two main steps for the proof of Theorem 1.3.1. The first step is constructing almost balanced metrics on the projectivization. In order to construct such metrics, we show an asymptotic expansions for the Bergman kernel of metrics on the projectivization which come from the bundle. It is done in Section 5.4. The second step is to perturb these almost balanced metrics to get balanced metrics. In order to do this, we need some eigenvalue estimates which is done in Section 5.2. In the last chapter, we give a simple construction of almost balanced metric in the case of one dimensional base manifold.

Chapter 2

Background

2.1 Stability of vector bundles

Let X be a compact complex manifold of complex dimension m. A positive-definite $(1,1)$-form ω on X is called Kähler if $d\omega = 0$. Let E be a holomorphic vector bundle on X of rank r. We define the ω-degree of E by

$$\deg(E) = \int_X c_1(E) \wedge \omega^{n-1},$$

and ω-slope of E by

$$\mu(E) = \text{degree}(E)/rk(E).$$

Notice that if the complex dimension of X is one, then the degree(E) and $\mu(E)$ do not depend on the choice of Kähler metric ω.

Definition 2.1.1. *A vector bundle E is called Mumford-Takemoto stable (semistable respectively) if for any coherent subsheaf $\mathcal{F}$ of E satisfying* $0 < rk(F) < rk(E)$*, we have $\mu(\mathcal{F}) < \mu(E)$ ($\mu(\mathcal{F}) \leq \mu(E)$ respectively). E is called polystable if E is the direct sum of stable vector bundles with the same slope.*

There is a differential geometric interpretation for stability of vector bundles known as the Hitchin-Kobayashi correspondence. We start with the definition of Hermitian-Einstein metric.

Definition 2.1.2. *A hermitian metric h on E is called Hermitian-Einstein if*

$$\Lambda_\omega F_{(E,h)} = \mu I_E,$$

where Λ_ω is the contraction of $(1,1)$-form with respect to the Kähler form ω and $F_{(E,h)}$ is the curvature of Chern connection on (E,h).

The following is called the Hitchin-Kobayashi correspondence.

Theorem 2.1.3. *Let (X,ω) be a compact Kähler manifold and $E \to X$ be a holomorphic vector bundle. Then E is Mumford poly-stable if and only if E admits a Hermitian-Einstein metric.*

Another notion of stability for vector bundles is due to Gieseker.

Definition 2.1.4. *Let (X,L) be a polarized manifold. A coherent sheaf $\mathcal{E}$ on X is called Gieseker stable(resp. semistable) if for any proper coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ and $k \gg 0$, we have*

$$\frac{h^0(\mathcal{F}\otimes L^k)}{rank(\mathcal{F})} < \frac{h^0(\mathcal{E}\otimes L^k)}{rank(\mathcal{E})} \;\; (\leq \textit{respectively }).$$

2.2 Geometric invariant theory

This section gives some background on GIT. The goal of GIT is constructing quotient spaces X/G when an algebraic group G acts on projective variety X. In order to obtain a "nice" quotient space, one needs to through out "bad locus" of X. More precisely, we need to take the quotient on the semi stable locus of X denoted by X^{ss}. We will give the definition of stability in the case $X = \mathbb{P}V$, where V is a complex vector space. Let an algebraic group G acts on $\mathbb{P}V$ via a linear representation $\rho : G \to GL(V)$. Therefore, we can lift the action of G to V.

Definition 2.2.1. *Let $x \in \mathbb{P}V$ and $\widehat{x} \in V$ be a nonzero lift of x.*

- *x is called stable if the orbit $G.\widehat{x}$ is closed in V and the stabilizer of x is finite.*
- *x is called poly-stable if the orbit $G.\widehat{x}$ is closed in V*
- *x is called semi stable if $\{0\} \nsubseteq \overline{G.\widehat{x}}$.*

In order to check that whether an element $x \in \mathbb{P}V$ is stable, one needs to study the whole orbit $G.\widehat{x}$ which can be quite complicated. There is a numerical criteria known as Hilbert-Mumford criterion to check the stability condition. First, we need to introduce the concept of one parameter subgroup and corresponded weight to it.

Definition 2.2.2. *A one parameter subgroup of G is a nontrivial algebraic homomorphism $\lambda : \mathbb{C}^* \to G$. Let $x \in \mathbb{P}V$. Therefore $x_0 = \lim_{t \to 0} \lambda(t)x$ exists and is a fixed point for the action of $\lambda(t)$. Let $\widehat{x_0}$ be a nonzero lift of x_0 to V. Then there exists a real number $w(x, \lambda)$ so that*

$$\lambda(t)\widehat{x_0} = t^{-w(x,\lambda)}\widehat{x_0}.$$

We have the following

Theorem 2.2.3. *Let $x \in \mathbb{P}V$.*

- *x is stable iff $w(x, \lambda) > 0$ for any one parameter subgroup λ of G.*
- *x is semistable iff $w(x, \lambda) \geq 0$ for any one parameter subgroup λ of G.*
- *x is polystable iff $w(x, \lambda) \geq 0$ for any one parameter subgroup λ of G and equality holds only if λ fixes x.*

One of the main applications of GIT is to form moduli spaces of varieties and vector bundles.

2.2.1 Gieseker point

Let E be a very ample vector bundle over a polarised algebraic manifold $(X, \mathcal{O}_X(1))$. We have a natural map

$$T : \bigwedge^r H^0(X, E) \to H^0(X, \det(E)),$$

which for any $s_1, ..., s_r$ in $H^0(X, E)$ is defined by

$$T(s_1 \wedge ... \wedge s_r)(x) = s_1(x) \wedge ... \wedge s_r(x).$$

Since E is globally generated T is surjective. The image of T in $\mathbb{P}(\hom(\bigwedge^r H^0(X, E), H^0(X, \det(E)))$ is called the Gieseker point of E.

The following is proven by Gieseker:

Theorem 2.2.4. *The bundle E is Gieseker stable (semistable respectively) iff the Giseker point of $E(k)$ is stable (semistable respectively) with respect to the action of $SL(H^0(X, E(k)))$ for $k \gg 0$.*

2.2.2 Chow point

Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension m and degree d. Define

$$Z = \{P \in Gr(N - m - 1, \mathbb{P}^N) \mid P \bigcap X \neq \emptyset\}.$$

One can see that Z is a hypersurface in the Grassmannian $Gr(N - m - 1, \mathbb{P}^N)$ of degree d. Therefore, there exists $R_X \in H^0(Gr(N - m - 1, \mathbb{P}^N), \mathcal{O}(d))$ such that $Z = \{R_x = 0\}$. The section R_X is called the Chow form of X and X is called Chow stable (semistable) if $[R_X] \in \mathbb{P}H^0(Gr(N - m - 1, \mathbb{P}^N), \mathcal{O}(d))$ is stable (semistable) under the action of $SL(N+1, \mathbb{C})$. Let $(X, \mathcal{O}_X(1))$ be a polarized variety. Let m be a positive integer such that $\mathcal{O}_X(m))$ is very ample. Then $(X, \mathcal{O}_X(m))$ is called Chow stable if the image of X under the Kodaira embedding

$$\iota : X \to \mathbb{P}(H^0(X, \mathcal{O}_X(m))^*)$$

is Chow stable.

2.3 Balanced metrics on vector bundles

As above, let (X, ω) be a Kähler manifold and E a very ample holomorphic vector bundle on X. Using global sections of E, we can map X into the Grassmannian $Gr(r, H^0(X, E)^*)$. Indeed, for any $x \in X$, we have the evaluation map $H^0(X, E) \to E_x$, which sends s to $s(x)$. Since E is globally generated, this map is a surjection. So its dual is an inclusion of $E_x^* \hookrightarrow H^0(X, E)^*$, which determines a r-dimensional subspace of $H^0(X, E)^*$. Therefore we get an embedding $i : X \hookrightarrow Gr(r, H^0(X, E)^*)$. Clearly we have $i^* U_r = E^*$, where U_r is the tautological vector bundle on $G(r, H^0(X, E)^*)$, i.e. at any r-plane in $G(r, H^0(X, E)^*)$, the fibre of U_r is exactly that r-plane. A choice of basis for $H^0(X, E)$ gives an isomorphism between $Gr(r, H^0(X, E)^*)$ and the standard $Gr(r, N)$, where $N = \dim H^0(X, E)$. We have the standard Fubini-Study hermitian metric on U_r, so we can pull it back to E and get a hermitian metric on E. Using $i^* h_{FS}$ and ω, we get an L^2 inner product on $H^0(X, E)$.

Definition 2.3.1. *The embedding is called balanced if $\int_X \langle s_i, s_j \rangle \frac{\omega^n}{n!} = C\delta_{ij}$, for some constant C which is determined by X and E.*

One can view the balanced condition as a fixed point of some map on the space of Fubini-Study metrics. Let K and M be the space of Hermitian metrics on E and Hermitian inner products on $H^0(X, E)$, respectively. Following Donaldson ([D2]), one defines the following maps

-
$$\mathrm{Hilb} : K \to M, \qquad h \mapsto \mathrm{Hilb}(h)$$
$$\langle s, t \rangle_{\mathrm{Hilb}(h)} = \frac{N}{Vr} \int \langle s(x), t(x) \rangle_h \frac{\omega^n}{n!},$$

where $N = \dim(H^0(X,E))$ and $V = \mathrm{Vol}(X,\omega)$. Note that Hilb only depends on the volume form $\omega^n/n!$.

- For the metric $H \in M$, $FS(H)$ is the unique metric on E such that $\sum s_i \otimes s_i^{*_{FS(H)}} = I$, where $s_1, ..., s_N$ is an orthonormal basis for $H^0(X,E)$ with respect to H. This gives the map $FS : M \to K$.

- Define
$$T : M \to M$$
$T(H) = \mathrm{Hilb} \circ FS(H)$. This map T is called the generalized T-operator in [DKLR].

It is easy to see that a metric h is balanced if and only if $\mathrm{Hilb}(h)$ is a fixed point of the map T.

The following describes the balanced condition in terms of Gieseker stability.

Theorem 2.3.2. *(?) Let E be a holomorphic vector bundle over a polarized manifold $(X, \mathcal{O}_X(1))$. Then E is Gieseker polystable if and only if there is a positive integer m_0 such that for any integer $m \geq m_0$, $E \otimes \mathcal{O}_X(m)$ admits a balanced metric.*

Fixing a nonzero element $\Theta \in \bigwedge^N H^0(X,E)$, we can define the determinant of any element in M. Thus we can define a map

$$\log\det : M \to \mathbb{R}.$$

A different choice of Θ only changes this map by an additive constant.

Also, we define a functional $I : K \to \mathbb{R}$ again unique up to an additive constant. Fix a background metric h_0. For a path $h_t = e^{\phi_t} h_0$ in K,

$$\frac{dI}{dt} = \int_X tr(\dot{\phi})\ \mathrm{dVol}_\omega \tag{2.1}$$

This functional is a part of Donaldson's functional independent the path. We define:

$$Z = -I \circ FS : M \to \mathbb{R} \tag{2.2}$$

We have the following scaling identities:

$$\mathrm{Hilb}(e^{\alpha}h) = e^{\alpha}\mathrm{Hilb}(h),$$

$$FS(e^{\alpha}h) = e^{\alpha}FS(h),$$

$$I(e^{\alpha}h) = I(h) + \alpha r V,$$

where α is a real number.

Following Donaldson, define:

$$\widetilde{Z} = Z + \frac{rV}{N}\log\det. \tag{2.3}$$

So $\widetilde{Z}$ is invariant under constant scaling of the metric.

This functional Z is studied by Wang in [W1] and Phong and Sturm in [PS]. They consider this as a functional on $SL(N)/SU(N)$. In order to see this, we observe that there is a correspondence between M and $GL(N)/U(N)$. Fix an element $H_0 \in M$ and an orthonormal basis $s_1, ..., s_N$ for $H^0(X, E)$ with respect to H_0. Now for any $H \in M$ we assign $[H(s_i, s_j)] \in GL(N)$. Notice that a change of the orthonormal basis only changes this matrix by multiplication by elements of $U(N)$. So we get a well-defined element of $GL(N)/U(N)$. The subset

$$M_0 = \{H \in M | \det[H(s_i, s_j)] = 1\}$$

corresponds to $SL(N)/SU(N)$.

Recall the definition of the Gieseker point of the bundle E.

$$T(E) : \bigwedge^{r} H^0(X, E) \to H^0(X, \det(E)).$$

Notice that fixing a basis for $H^0(X,E)$ gives an isomorphism between $\bigwedge^r H^0(X,E)$ and $\bigwedge^r \mathbb{C}^N$. Hence, there is a natural action of $GL(N)$ on $\mathrm{Hom}(\bigwedge^r H^0(X,E), H^0(X, \det(E)))$. Phong-Sturm ([PS]) and Wang ([W1]) prove that Z is convex along geodesics of $SL(N)/SU(N)$ and its critical points are corresponding to balanced metrics on E. Phong and Sturm prove the following

Theorem 2.3.3. *([PS, Theorem 2]) There exists a $SU(N)$- invariant norm $||.||$ on Hom$(\bigwedge^r H^0(X,E), H^0(X,\det(E)))$ such that for any $\sigma \in SL(N)$*

$$Z(\sigma) = \log \frac{||\sigma.T(E)||^2}{||T(E)||^2}$$

Theorem 2.3.4. *([W1, Lemma 3.5], [PS, Lemma 2.2]) The functional Z is convex along geodesics of M.*

The Kempf-Ness theorem ([KN]) shows that Z is proper and bounded from below if $T(E)$ is stable under the action of $SL(N)$.

The following is an immediate consequence of the above theorem and the fact that balanced metrics are critical points of Z. Also notice that $\widetilde{Z}$ is invariant under the scaling of a metric by a positive real number.

Theorem 2.3.5. *Assume that H_0 is a balanced metric on E. Then $\widetilde{Z}|_{M_0}$ is proper and bounded from below. Moreover $\widetilde{Z}(H) \geq \widetilde{Z}(H_0)$ for any $H \in M$.*

Lemma 2.3.6. *For any $H \in M$, we have*

$$Tr(T(H)H^{-1}) = N$$

Proof. Let $h = FS(H)$ and let $s_1, ..., s_N$ be an H-orthonormal basis. We have,

$$\sum s_i \otimes s_i^{*_h} = I$$

Therefore,

$$r = Tr\big(\sum s_i \otimes s_i^{*_h}\big) = \sum |s_i|_h^2.$$

Integrating the above equation implies the result.

$\square$

Lemma 2.3.7. *For any $H \in M$,*

- $Z(H) \geq Z(T(H))$.
- $\log\det(H) \geq \log\det(T(H))$.
- $\widetilde{Z}(H) \geq \widetilde{Z}(T(H))$.

Proof. Put $h = FS(H)$, $H' = \mathrm{Hilb} \circ FS(H)$ and $h' = FS(H') = e^{\varphi}h$. Let $s_1, ..., s_N$ be an H'-orthonormal basis. We have,

$$\sum s_i \otimes s_i^{*_h} = e^{-\varphi}.$$

Hence,

$$\int_X tr(-\varphi) = \int_X \log\det(e^{-\varphi}) \leq \int_X \log\Big(\frac{tr(e^{-\varphi})}{r}\Big)^r$$

$$= r\int_X \log(tr(e^{-\varphi})) - rV\log r \leq rV\log\Big(\frac{1}{V}\int_X tr(e^{-\varphi})\Big) - rV\log r$$

$$= rV\log\Big(\frac{1}{V}\int_X \sum |s_i|_h^2\Big) - rV\log r = 0$$

This shows the first inequality. For the second one, Lemma 2.3.6 implies that $tr(H'H^{-1}) = N$. Using the arithmetic -geometric mean inequality, we get

$$\det(H'H^{-1})^{\frac{1}{N}} \leq \frac{tr(H'H^{-1})}{N} = 1.$$

This implies that $\log\det(H'H^{-1}) \leq 0$. The third inequality is obtained by summing up the first two.

$\square$

A holomorphic vector bundle E is called simple if $Aut(E) \simeq \mathbb{C}^*$. We will need the following

Lemma 2.3.8. *Suppose that E is simple and admits a balanced metric. Then the balanced metric is unique up to a positive constant.*

Proof. Since $\det(H)^{-1/N}H \in M_0$ for any $H \in M$, it suffices to prove that a balanced metric in M_0 is unique. Let $H_\infty \in M_0$ be a balanced metric on E and $s_1, ..., s_N$ be an orthonormal basis of $H^0(X, E)$ with respect to H_∞. This basis gives an embedding $\iota : X \to Gr(r, N)$ such that $\iota^* U_r = E$, where $U_r \to Gr(r, N)$ is the universal bundle over the Grassmannian. Assume that H is another element of M_0. Therefore, there exists an element $a \in su(N)$ such that $e^{ia}.H_\infty = H$. Then $\{e^{ita}\}$ gives a one parameter family of automorphism of $(Gr(r, N), U_r)$ and therefore a one parameter family in $\mathrm{Aut}(X, E)$. From lemma 3.5 in [?], we have

$$\frac{d^2}{dt^2} Z(e^{ita}) = \int_{\iota(X)} ||\tilde{a}||^2 dvol_X, \tag{2.4}$$

where $\tilde{a}$ is the vector field on $Gr(r, N)$ generated by the infinitesimal action of a and $||\tilde{a}||$ is the Fubini-Study norm of $\widetilde{a}$. Suppose that H is a balanced metric. Therefore it is a minimum for the functional Z. This implies that

$$\frac{d^2}{dt^2} Z(e^{ita}) = 0,$$

and hence by (5.12) that $\tilde{a}\big|_{\iota(X)} \equiv 0$. This implies that the one parameter family $\{e^{ita}\}$ fixes $\iota(X)$ pointwise and therefore it induces a one parameter family of endomorphisms of E. By the simplicity of E, this induced map must be a constant scalar of identity. Since it also has determinant 1, this concludes the proof.

□

2.4 Balanced metrics on manifolds

Let X be a compact Kähler manifold of dimension n and $\mathcal{O}_X(1) \to Y$ be a very ample line bundle on X. Since $\mathcal{O}(1)$ is very ample, using global sections of $\mathcal{O}_X(1)$, we can embed X into $\mathbb{P}(H^0(X, \mathcal{O}_X(1))^*)$. A choice of ordered basis $\underline{s} = (s_1, ..., s_N)$ of $H^0(X, \mathcal{O}_X(1))$ gives an isomorphism between $\mathbb{P}(H^0(X, \mathcal{O}_X(1))^*)$ and $\mathbb{P}^{N-1}$. Hence for any such $\underline{s}$, we have an embedding $\iota_{\underline{s}} : X \hookrightarrow \mathbb{P}^{N-1}$ such that $\iota_{\underline{s}}^* \mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}_X(1)$. Using $\iota_{\underline{s}}$, we can pull back the Fubini-Study metric and Kähler form of the projective space to $\mathcal{O}(1)$ and X respectively.

Definition 2.4.1. *An embedding $\iota_{\underline{s}}$ is called balanced if*

$$\int_X \langle s_i, s_j \rangle_{\iota_{\underline{s}}^* h_{FS}} \frac{\iota_{\underline{s}} \omega_{FS}}{n!} = \frac{V}{N} \delta_{ij},$$

where $V = \int_Y \frac{\omega_0^n}{n!}$. A hermitian metric(respectively a Kähler form) is called balanced if it is the pull back $\iota_{\underline{s}}^* h_{FS}$ (respectively $\iota_{\underline{s}}^* \omega_{FS}$) where $\iota_{\underline{s}}$ is a balanced embedding.

2.5 Some basics of ruled manifolds

Let V be a hermitian vector space of dimension r.

Definition 2.5.1. *There is a natural isomorphism $\hat{}: V \to H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(1))$, which sends $v \in V$ to $\hat{v} \in H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(1))$ so that for any $f \in V^*, \hat{v}(f) = f(v)$. Therefore, any hermitian product h on V defines a metric $\widehat{h}$ on $H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(1))$. Indeed, for any $f \in V^*$ and $v, w \in V$, we define*

$$\widehat{h}(\hat{v}([f]), \hat{w}([f])) = \frac{f(v)\overline{f(w)}}{h(f, f)}. \tag{2.5}$$

Lemma 2.5.2. *For any $v, w \in V$, we have*

$$\langle v, w \rangle = C_n \int_{\mathbb{P}V^*} < \hat{v}, \hat{w} > \frac{\omega_{FS}^{r-1}}{(r-1)!} \tag{2.6}$$

where C_r is a constant defined by

$$C_r = \int_{\mathbb{C}^{r-1}} \frac{d\xi \wedge d\bar{\xi}}{(1+\sum_{j=1}^{r-1}|\xi_j|^2)^{r+1}}. \tag{2.7}$$

Here $d\xi \wedge d\bar{\xi} = (\sqrt{-1}d\xi_1 \wedge d\bar{\xi}_1) \wedge \cdots \wedge (\sqrt{-1}d\xi_{r-1} \wedge d\bar{\xi}_{r-1})$.

Proof. Let $e_0, ..., e_r$ be an orthogonal basis for V. So for any e_i, we get a section $\hat{e_i} \in H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(1))$. For $f \in V^*$, we can write $f = \sum w_j e_j^*$. By definition, we have $|\hat{e_i}|^2_{[f]} = \frac{|f(e_i)|^2}{|f|^2}$. Then,

$$\int_{\mathbb{P}V^*} |\hat{e_i}|^2_{[f]} \, \mathrm{d}[f] = \int \frac{|f(e_i)|^2}{\sum |w_j|^2} \, \mathrm{d}[f] = \int_{\mathbb{P}V^*} \frac{|w_i|^2}{\sum |w_j|^2} \, \mathrm{dVol} = c_n.$$

This number is independent of i and only depends on $n = \dim \mathbb{P}V^*$. Also one can check that for $i \neq j$, we have

$$\int_{\mathbb{P}V^*} \langle \hat{e_i}, \hat{e_j} \rangle_{[f]} = \int_{\mathbb{P}V^*} \frac{w_i \bar{w}_j}{\sum |w_j|^2} \, \mathrm{dVol} = 0.$$

□

Similar to the case of vector spaces, we have the natural isomorphism $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1)) = H^0(X, E)$.

Also, for any Hermitian metric h on E, we have a Hermitian metric $\hat{h}$ on $\mathcal{O}_{\mathbb{P}E^*}(1)$. For any metric H in $\mathcal{M}_E$, we can naturally define a metric $j(H)$ on $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1))$. Indeed, for any $s, t \in H^0(X, E)$, we define

$$H(s,t) = j(H)(\hat{s}, \hat{t}).$$

Theorem 2.5.3. *For any H in $\mathcal{M}_E$, We have*

$$\widehat{FS(H)} = FS(j(H)).$$

Proof. Let $\hat{H} := j(H)$ and $h := FS(H)$. Also we will use $||.||$ to denote $i(h)$. Let $s_1, ..., s_N$ be an orthonormal basis for $H^0(X, E)$ with respect to H, and let $\hat{s}_1, ..., \hat{s}_N$

be the corresponding basis for $H^0(\mathbf{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1))$. By definition of $\hat{H}$, $\hat{s}_1, ..., \hat{s}_N$ is an orthonormal basis for $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1))$. Thus, we have

$$\sum |\hat{s}_i|_{FS(\hat{H})} = 1$$

and

$$\sum s_i \otimes s_i^{*FS(H)} = I.$$

Let $e_1, ..., e_r$ be a local orthonormal frame for E with respect to h and $e_1^*, ..., e_r^*$ be its dual basis. Also let $\hat{e}_1, ..., \hat{e}_r$ be the corresponding local sections for $\mathcal{O}_{\mathbf{P}E^*}(1)$. For $e \in E$ we have

$$||\hat{e}||^2_{[e_i^*]} = |\langle e_i^*, e > |^2 = |e_i^*(e)|^2.$$

Therefore, for any $v^* \in E^*$, where $v^* = \sum \lambda_i e_i^*$, we have

$$||\hat{e}||^2_{[\sum \lambda_i e_i^*]} = \frac{\sum |\lambda_i|^2 |\langle e_i, e > |^2}{\sum |\lambda_i|^2}.$$

Writing $s_1, ..., s_N$ in terms of the local frame, we have $s_i = \sum a_{ij} e_j$. We denote the matrix $[a_{ij}]$ by A. Notice

$$\sum s_i \otimes s_i^{*FS(H)} = I \text{ if and only if } A^* A = I.$$

We also have

$$||\hat{s}_i||^2_{e_k^*} = |\langle s_i, e_k > |^2 = |\langle \sum a_{ij} e_j, e_k \rangle|^2 = |a_{ik}|^2.$$

Summing them, we get

$$\sum ||\hat{s}_i||^2_{e_k^*} = \sum |a_{ik}|^2 = \sum a_{ik} \bar{a_{ik}} = (A^* A)_{k,k} = 1.$$

As above, let $v^* = \sum \lambda_i e_i^*$. Without loss of generality we can assume $\|v^*\| = 1$. Thus we have

$$\|\hat{s}_i\|^2_{v^*} = \sum |\lambda_k|^2 |a_{ik}|^2,$$

then we get

$$\sum \|\hat{s}_i\|_{v^*}^2 = \sum \sum |\lambda_k|^2 |a_{ik}|^2 = \sum |\lambda_k|^2 \sum |a_{ik}|^2 = \sum |\lambda_k|^2 = \|v^*\|^2 = 1.$$

Since the identity $\sum |\hat{s}_i|_{FS(\hat{H})} = 1$ determines $FS(\hat{H})$ uniquely, we conclude that $FS(\hat{H}) = i(FS(H))$.

$\square$

Chapter 3

A functional on the space of Fubini-Study metrics

3.1 Definitions

Let (X, ω) be a projective Kähler manifold and E be a holomorphic vector bundle on X. We also assume that E is very ample, so in particular any fibre is generated by global sections of E. Since E is globally generated, using global sections of E, we can embed X into $G(r, H^0(X, E)^*)$. Indeed, for any $x \in X$, we have the evaluation map $H^0(X, E) \to E_x$, which sends s to $s(x)$. Since E is globally generated, this map is a surjection. So its dual is an inclusion of $E_x^* \hookrightarrow H^0(X, E)^*$, which determines a r-dimensional subspace of $H^0(X, E)^*$. Therefore we get an embedding $i : X \hookrightarrow G(r, H^0(X, E)^*)$. Clearly we have $i^*U_r = E^*$, where U_r is the tautological vector bundle on $G(r, H^0(X, E)^*)$, i.e. at any r-plane in $G(r, H^0(X, E)^*)$, the fibre of U_r is exactly that r-plane. A choice of basis for $H^0(X, E)$ gives an isomorphism between $G(r, H^0(X, E)^*)$ and the standard $G(r, N)$, where $N = dimH^0(X, E)$. We have the standard Fubini-Study hermitian metric on U_r, so we can pull it back to E and get a hermitian metric on E. Also, since X is a smooth subvariety of $G(r,N)$, the restriction of the Fubini-Study Kähler form of $G(r,N)$ to X is a Kähler form. Using i^*h_{FS} and $\omega_{FS}|_X$, we get an L^2 inner product on $H^0(X,E)$. The embedding is called balanced if $\int_X \langle s_i, s_j\rangle\, \omega_{FS}^n = C\delta_{ij}$.

We can also define another kind of balanced embedding by fixing some Kähler form ω on X. More precisely, we call the embedding ω-balanced if $\int_X \langle s_i, s_j\rangle\, \omega^n = C\delta_{ij}$. Note that in the definition of strongly balanced embedding we do not need to fix Kähler form on X, but being ω- balanced depends on the choice of Kähler form, or more precisely on the volume form of the Kähler form. We are going to phrase the above discussion in slightly different langauge.

Let h be a hermitian metric on E. We define a $(1,1)$-form ω_h on X by $\omega_h = \partial\bar{\partial}\log\det(h)$. For any bundle endomorphism Φ, we have

$$\omega_{e^{\Phi}h} = \omega_h + \partial\bar{\partial}tr(\Phi). \tag{3.1}$$

We let $\mathcal{K}_E$ be the space of all hermitian metrics h on E, with the property that ω_h is positive and we let $\mathcal{M}_E$ be the space of hermitian inner products on $H^0(X,E)$. We will construct the following

- Given h in $\mathcal{K}$, we define a hermitian inner product $Hilb(h)$ on $H^0(X,E)$ by
$$\langle s,t\rangle_{Hilb(h)} = \frac{N}{Vr}\int \langle s(x),t(x)\rangle_h dVol_h,$$
where $N = dim(H^0(X,E))$, $dVol_h = \frac{\omega_h^n}{n!}$ and $V =$ Vol(X,h). In this way we get a map $Hilb : \mathcal{K} \to \mathcal{M}$.

Note that if E is a line bundle, then the map $Hilb$ becomes the usual map defined by Donaldson. We have the following definition.

Definition 3.1.1. *A metric h on E is called strongly balanced if $FS \circ Hilb(h) = h$.*

Recall the definition of T_E the Gieseker point of E. For simplicity through this chapter we denote it by T. After possibly tensoring by a high power of an ample line bundle, we may assume T is surjective. This implies that

$$T^* : H^0(X, \det(E))^* \to \bigwedge^r H^0(X, E)^*$$

is injective. Let m be a positive number. One can construct in a similar way

$$T^{(m)} : S^m \bigwedge^r H^0(X, E) \to H^0(X, (\det(E))^{\otimes m}). \tag{3.2}$$

This gives the inclusion

$$(T^{(m)})^* : H^0(X, (\det(E))^{\otimes m})^* \hookrightarrow S^m \bigwedge^r H^0(X, E)^*.$$

Definition 3.1.2. *The map $(T^{(m)})^*$ and an inner product H on $H^0(X, E)$ induce a hermitian inner product $T^{(m)}(H)$ on $H^0(X, (\det(E))^{\otimes m})$.*

Since E is very ample, we have the following embeddings:

- Using global sections of E, we can embed X into $G(r, H^0(X, E)^*)$. Indeed, we get an embedding $f : X \hookrightarrow G(r, H^0(X, E)^*)$, where $f(x)$ is the r-dimensional subspace of $H^0(X, E)^*$ defined by

  $$H^0(X, E) \to E_x \to 0.$$

- Using global sections of $(\det(E))^m$, we can embed X into $\mathbb{P}(H^0(X, (\det(E))^m)^*)$. This embedding,

  $$j_m : X \hookrightarrow \mathbb{P}(H^0(X, (\det(E))^m)^*),$$

  is defined by

  $$H^0(X, (\det(E))^m)^* \to ((\det(E))^m)_x \to 0.$$

- We have the embedding $X \hookrightarrow \mathbb{P}(\bigwedge^r H^0(X,E)^*)$, $x \mapsto i(x)$ defined by

$$i(x)(s_1 \wedge ... \wedge s_r) = [s_1(x) \wedge ... \wedge s_r(x)]$$

for any $s_1, ..., s_r \in H^0(X,E)$.

- Using global sections of $\mathcal{O}(m)$ on $\mathbb{P}(\bigwedge^r H^0(X,E)^*)$, we have the embedding

$$\mathbb{P}(\bigwedge^r H^0(X,E)^*) \hookrightarrow \mathbb{P}(S^m \bigwedge^r H^0(X,E)^*).$$

Composing this with i, we get the embedding

$$i_m : X \hookrightarrow \mathbb{P}(S^m \bigwedge^r H^0(X,E)^*).$$

We have the following lemma:

Lemma 3.1.3.
- $(T^{(m)})^* \circ j_m = i_m$
- *$pl \circ f = i$, where $pl : G(r, H^0(X,E)^*) \hookrightarrow \mathbb{P}(\bigwedge^r H^0(X,E)^*)$ is the Plucker embedding.*

Proof. Let $x \in X$, $s_1^{(i)}, ..., s_r^{(i)} \in H^0(X,E)$ and $X_i = s_1^{(i)} \wedge ... \wedge s_r^{(i)}$. Then we have

$$(T^{(m)})^*(j_m(x))(X_1...X_m) = j_m(x)(T^{(m)}(X_1...X_m))$$

$$= j_m(x)(T(X_1)....T(X_m))) = j_m(x)((s_1^{(1)} \wedge .... \wedge s_r^{(1)})...(s_1^{(m)} \wedge .... \wedge s_r^{(mi)}))$$

$$= [(s_1^{(1)}(x) \wedge ... \wedge s_r^{(1)}(x)) \otimes ...(s_1^{(m)}(x) \wedge ... \wedge s_r^{(m)}(x))] = i_m(x)(X_1...X_r)$$

The second part is obvious from the definitions.

□

We also have the following:

Proposition 3.1.4. *For any metric H on $H^0(X,E)$, we have*

$$\det(FS(H))^{\otimes m} = FS(T^m(H)).$$

Proof. The metric H on $H^0(X,E)$ induces a Fubini-Study metric h_{FS}^{Gr} on $U_r \to G(r, H^0(X,E)^*)$ and a Fubini-Study metric h_{FS} on $\mathcal{O}(1) \to \mathbb{P}(\bigwedge^r H^0(X,E)^*)$. The latter metric induces a metric $h_{FS,m}$ on $\mathcal{O}(1) \to \mathbb{P}(S^m \bigwedge^r H^0(X,E)^*)$. Therefore,

$$pl^* h_{FS} = det(h_{FS}^{Gr}), \;\; v_m^* h_{FS,m} = h_{FS}^{\otimes m},$$

Where

$$v_m : \mathbb{P}(\bigwedge^r H^0(X,E)^*) \to \mathbb{P}(S^m \bigwedge^r H^0(X,E)^*)$$

is the Veronese embedding. By definition, we have $FS(H) = f^* h_{FS}^{Gr}$. Therefore we have

$$(\det(FS(H)))^{\otimes m} = (\det(f^* h_{FS}^{Gr}))^{\otimes m} = (f^* \det(h_{FS}^{Gr}))^{\otimes m}$$

$$= f^* pl^* h_{FS}^{\otimes m} = f^* pl^* v_m^* h_{FS,m} = (v_m \circ pl \circ f)^* h_{FS,m}$$

$$= (v_m \circ i)^* h_{FS,m} = i_m^* h_{FS,m} = j_m^*((T^m)^*))^* h_{FS}.$$

On the other hand $T^m(H)$ induces a FS metric $h'_{FS} = ((T^m)^*)^* h_{FS,m}$ on $\mathbb{P}(H^0(X, \det(E)^{\otimes m})^*)$. By definition, $FS(T^m(H)) = j_m^* h'_{FS}$.

$\square$

We fix a metric H_0 on $H^0(X,E)$, an orthogonal basis $s_1, ..., s_N$ for $H^0(X,E)$ with respect to H_0 and an orthonormal basis $t_1, ..., t_k$ for $H^0(X, \det(E))$ with respect to $T(H_0)$. We can write T as

$$T = \sum T_\alpha \otimes t_\alpha,$$

where

$$T_\alpha = T^*(t_\alpha^*) = \sum_{i_1 < ... < i_r} a^\alpha_{i_1,...,i_r} s^*_{i_1} \wedge ... \wedge s^*_{i_r}.$$

Hence, we have

$$\bigwedge^k T : \bigwedge^k (\bigwedge^r H^0(X,E)) \to \bigwedge^k H^0(X, \det(E)) \simeq \mathbb{C}. \tag{3.3}$$

So $\bigwedge^k T$ can be viewed as an element of $\bigwedge^k(\bigwedge^r H^0(X,E)^*)$. We note that any inner product on $H^0(X,E)$ induces an inner product on $\bigwedge^k(\bigwedge^r H^0(X,E)^*)$.

Proposition 3.1.5. *For any inner product H on $H^0(X,E)$, we have*

$$\det(T(H)) = ||\bigwedge^k T||_H^2,$$

where $\det(T(H)) := \det(\langle t_i, t_j\rangle_{T(H)})$.

Proof. We have

$$\langle t_\alpha^*, t_\beta^*\rangle_{T(H)^*} = \langle T^*(t_\alpha^*), T^*(t_\beta^*)\rangle_{H_\alpha} = \langle T_\alpha, T_\beta\rangle_{H^*}.$$

So we have

$$\log\det(T(H)^*) = \log\det\langle T_\alpha, T_\beta\rangle_H,$$

which gives the following

$$\log\det(T(H)) = -\log\det\langle T_\alpha, T_\beta\rangle_H.$$

□

The conclusion follows from the following general linear algebra fact:

Lemma 3.1.6. *Let V and W be vector spaces of dimension N and k, respectively. Let A be a linear map from V to W. Fixing a basis $w_1, ..., w_k$, we can write $A = \sum A_i \otimes w_i$, where $A_i \in V^*$. Then for any inner product $\langle , \rangle$ on V, we have*

$$\det(\langle A_i, A_j\rangle) = ||\bigwedge^k A||^2.$$

Similarly, if we fix an orthonormal basis $t_1, ..., t_{p(m)}$ for $H^0(X, \det(E)^{\otimes m})$, we can prove the following:

Proposition 3.1.7. *For any inner product H on $H^0(X,E)$, we have*

$$\det(T^{(m)}(H)) = || \bigwedge^{p(m)} T^m ||_H^2,$$

where $\det(T^m(H)) := \det(\langle t_i, t_j \rangle_{T^m(H)})$.

3.2 Existence of strongly balanced metrics

The goal of this section is proving the following theorem:

Theorem 3.2.1. *Let X be a compact Riemann surface. If $(X, (\det E)^{\otimes m})$ is Chow semistable for some sufficiently large m and E is stable, then E admits a strongly balanced metric.*

In order to prove the above theorem, first we prove theorem 3.2.6. A natural question arising from theorem 3.2.6 regards sufficient conditions for the stability of the point $\bigwedge^{p(m)} T^{(m)}$. Gieseker and Morrison proved that for rank two vector bundles on a smooth curve, the stability of E implies the stability of the point $\bigwedge^{p(m)} T^{(m)}$ for $m \gg 0$. Later Schmitt generalized their theorem to higher rank vector bundles over smooth curves. Theorem 3.2.6 combined with Gieseker, Morrison and Schmitt results imply theorem 3.2.1. Thus, in the remainder of this section we proceed by proving theorem 3.2.6.

Let L be a holomorphic line bundle over X. We introduce there is a unique functional $I : \mathcal{K}_L \to \mathbb{R}$ defined using the variational formula,

$$\frac{d}{dt} I(g(t)) = \frac{1}{V} \int_X \dot{\varphi}(t) \, dVol_{g_t}, \tag{3.4}$$

where $g_t = e^{\varphi_t} g_0$ is a smooth path in $\mathcal{K}_L$. This functioanl is defined up to a constant which can be fixed by fixing the metric g_0 in $\mathcal{K}_L$. This functional is analogue of functionals defined by Donaldson and Mabuchi.

Let E be a holomorphic vector bundle of rank r on X. For any hermitian metric h on E, we have a hermitian metric $\det(h)$ on $\det(E)$. Obviously, for any $h \in \mathcal{K}_E$, we have $\det(h) \in \mathcal{K}_{\det(E)}$. Now we can define a functional $F : \mathcal{K}_E \to \mathbb{R}$, unique up to a constant, which is defined using the following variational formula:

$$\frac{d}{dt}F(h_t) = \frac{1}{V}\int_X Tr(\dot{\phi}(t))\, dVol_{h_t}, \tag{3.5}$$

where $h_t = e^{\phi(t)}h_0$ is a smooth path in $\mathcal{K}_E$. Therefore, since the functional I is well-defined, then the functional F is well defined on $\mathcal{K}_E$. We now define the function $L : \mathcal{K}_E \to \mathbb{R}$ by

$$L(H) = -F \circ FS(H).$$

Lemma 3.2.2. *1)Let $H_t = e^{t\delta H}H$ be a path in $\mathcal{M}$. where δH is a hermitian matrix. We have:*

$$\frac{d}{dt}\Big|_{t=0} L(H_t) = \int_X Tr(\delta H[\langle s_i, s_j\rangle_{FS(H)}])d\mu_{FS(H)},$$

where $s_1, ..., s_N$ is an orthonormal basis for $H^0(X, E)$ with respect to H

2)The functional L is convex along geodesics in M_E.

Proof. Let $s_1(t), ..., s_N(t)$ be an orthonormal basis of $H^0(X, E)$ with respect to H_t. Thus, we have

$$\delta_{ij} = \langle s_i(t), s_j(t)\rangle_{FS(H_t)} = \langle e^{t\delta H}s_i(t), s_j(t)\rangle_{FS(H)}.$$

Differentiating it with resect to t at $t = 0$, we get

$$\langle \delta H s_i, s_j\rangle_{FS(H)} + \langle s_i'(o), s_j\rangle_{FS(H)} + \langle s_i, s_j'(0)\rangle_{FS(H)} = 0,$$

which implies $\delta H = D + D^*$, where D is defined by $s_i'(0) = \sum d_{ij}s_j$. On the other hand, by the definition of $FS(H_t)$, we have

$$\sum s_i(t) \otimes s_i(t)^{FS(H_t)} = Id.$$

It implies for any v in E, we have

$$v = \sum \langle v, s_i(t) \rangle_{FS(H_t)} s_i(t) = \sum \langle e^{\phi_t} v, s_i(t) \rangle_{FS(H)} s_i(t),$$

where ϕ_t is defined by $FS(H_t) = e^{\phi_t} FS(H)$.

Again differentiating the above equation with resect to t at $t = 0$, we obtain

$$\sum \langle \dot{\phi} v, s_i \rangle_{FS(H)} s_i + \sum \langle v, s_i'(0) \rangle_{FS(H)} s_i + \sum \langle v, s_i \rangle_{FS(H)} s_i'(0) = 0,$$

which gives

$$\begin{aligned} \dot{\phi} v &= -\sum \langle v, s_i'(0) \rangle_{FS(H)} s_i - \sum \langle v, s_i \rangle_{FS(H)} s_i'(0) \\ &= \sum \overline{d_{ij}} \langle v, s_j \rangle_{FS(H)} s_i - \sum d_{ij} \langle v, s_i \rangle_{FS(H)} s_j \\ &= -\sum (\delta H)_{ij} \langle v, s_i \rangle_{FS(H)} s_j. \end{aligned}$$

Let $e_1, .., e_r$ be an orthonormal frame for E with respect to $h_0 = FS(H)$. The matrix AA^* is $\langle s_i, s_j \rangle_{FS(H)}$ we can write

$$s_i = \sum a_{ij} e_j.$$

Therefore, we get

$$\dot{\phi} e_p = -\sum (\delta H)_{ij} \overline{a_{ip}} a_{jq} e_q$$

So the matrix of $\dot{\phi}$ in the local frame $e_1, .., e_r$ is given by$A^* \delta H A$. So, we have

$$tr(\dot{\phi}) = tr(A^* \delta H A) = tr(\delta H A A^*).$$

This proves the first part.

□

Corollary 3.2.3. *H is a critical point of L if and only if it is strongly balanced.*

As above we have fixed the metric H_0 on $H^0(X, E)$, so we have the metric $T(H_0)$ on $H^0(X, \det(E))$ and $\det(FS(H_0))$ on $\det(E)$. Let $t_1, ..., t_k$ be an orthonormal basis for $H^0(X, \det(E))$ with respect to $T(H_0)$. Using this, we can define $\log\det : \mathcal{M} \to \mathbb{R}$ and $I : \mathcal{K}_{\det(E)} \to \mathbb{R}$. We recall that the functional I is defined using the following variational property:

$$\frac{dI}{dt} = \int_X \dot{\varphi}\, \omega_\varphi^n,$$

where $g_t = e^{\varphi_t} \det(h_0)$ is a path in $K_{\det(E)}$. Now we define the functional Z on $\mathcal{M}_{\det(E)}$ by

$$Z = -I \circ FS, \quad \tilde{Z} = Z + \frac{V}{k} \log\det \tag{3.6}$$

In the rest of this section, we relate the functional $\tilde{Z}$ to the functional L from the previous section.

Proposition 3.2.4.

$$\tilde{Z}(T(H)) = L(H) - \frac{V}{k} \log || \bigwedge^k T||_H^2.$$

Proof.

$$\begin{aligned}
\tilde{Z}(T(H)) &= Z(T(H)) + \frac{V}{k} \log\det(T(H)) \\
&= -I(FS(T(H))) + \frac{V}{k} \log\det(T(H)) \\
&= -I(\det FS(H)) + \frac{V}{k} \log\det(T(H)) \\
&= L(H) - \frac{V}{k} \log || \bigwedge^k T||_H^2.
\end{aligned}$$

□

Similarly we can prove the following:

Proposition 3.2.5.

$$\tilde{Z}_m(T^{(m)}(H)) = L(H) - \frac{V}{p(m)} \log || \bigwedge^{p(m)} T^{(m)}||_H^2,$$

where $\tilde{Z}_m$ is the related functional on $M_{(\det E)^{\otimes m}}$ and $p(m) = h^0((\det E)^{\otimes m})$.

Now we can prove the following

Theorem 3.2.6. *If $(X, (\det E)^{\otimes m})$ is Chow semistable and the point $\bigwedge^{p(m)} T^{(m)}$ is stable under the action of $SL(H^o(X, E))$, then E admits a strongly balanced metric.*

Proof. Since $(X, (\det E)^{\otimes m})$ is Chow semi-stable, we have that $\tilde{Z}_m$ is bounded from below. Using the Kempf-Ness theorem,[KN], the stability of the point $\bigwedge^{p(m)} T^{(m)}$ implies that the function $V : \frac{SL(N)}{SU(N)} \to \mathbb{R}$, defined by $V(\sigma) = \log || \bigwedge^{p(m)} \sigma T^{(m)} ||^2_{H_0}$, is proper and bounded from below. On the other hand, we can see that

$$|| \bigwedge^{p(m)} \sigma T^{(m)} ||^2_{H_0} = || \bigwedge^{p(m)} T^{(m)} ||^2_{\sigma H_0},$$

which implies $\log || \bigwedge^{p(m)} T^{(m)} ||^2_H$ is proper and bounded from below. So the formula

$$\tilde{Z}_m(T^{(m)}(H)) = L(H) - \frac{V}{p(m)} \log || \bigwedge^{p(m)} T^{(m)} ||^2_H$$

implies that L is proper and bounded from below. Therefore, L has a critical point which is a strongly balanced metric.

□

3.3 Convexity of the functional L

In this section we follow Donaldson [D4]. Our proof is essentially Donaldson's proof with some very minor modifications. Let L be a line bundle and X be a Kähler manifold. Assume E is very ample. Let $A = \{s_0, ..., s_N\}$ such that it contains a basis for $H^0(X, L)$. Using A, we can embed X into $\mathbb{CP}^N$. Let h be the FS metric on L and let ω be the pull back of the FS Kähler form on X. We have

$$\sum |s_i|^2_h = 1.$$

Following Donaldson, we denote the following pairings by $(.,.)$

$$T^*X \times (T^*X \otimes L) \to T^*X$$

$$L \times (T^*X \otimes L) \to T^*X$$

$$T^*X \times T^*X \to \mathbb{R}$$

$$L \times L \to \mathbb{R},$$

which are obtained by using h and ω.

Lemma 3.3.1. *For any function f on X we have*

$$|\nabla f|^2 = 2 \sum |(\nabla f, \nabla s_i)|_h^2.$$

Proof. Assume at the point x, we have $s_0 \neq 0$ and $\nabla s_0 = 0$. We can find local holomorphic functions near x so that $s_i = f_i s_0$. We have

$$|s_0|^2 = (1 + \sum |f_i|^2)^{-1},$$

and

$$\omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(1 + \sum |f_i|^2)^{-1}.$$

Thus, at the point x, we have

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum \partial f_i \wedge \overline{\partial f_i}.$$

Now for the function f, we can write

$$\partial f = \sum g_i \partial f_i.$$

So,

$$(\nabla f, \nabla s_i) = tr_\omega(\partial f \wedge \partial f_i) s_0.$$

At the point x, we have

$$tr_\omega(\partial f \wedge \partial f_i) = tr_\omega(\sum \partial f_\alpha g_\alpha \wedge \partial f_i) = g_\alpha.$$

Hence, at the point x, we have

$$\sum |(\nabla f, \nabla s_i)|_h^2 = \sum |g_i|^2 |s_0|_h^2 = \frac{|\nabla f|^2}{(1 + \sum |f_i|^2)} |s_0|_h^2 = |\nabla f|^2.$$

$\square$

We are ready to prove the convexity of functional L.

Let H_t be a geodesic in M_E. One can find an orthonormal basis $s_1, ..., s_N$ for $H^0(X, E)$ with respect to H_0 so that

$$H_t(s_\alpha, s_\beta) = \delta_{\alpha\beta} e^{\lambda_\alpha t}.$$

So $e^{\frac{-\lambda_1}{2}t} s_1, ..., e^{\frac{-\lambda_N}{2}t} s_N$ is an orthonormal basis with respect to H_t. We let $h_t = FS(H_t) = e^{\phi_t} h_0$.

Thus, we have

$$\sum e^{-\lambda_i} s_i \otimes s_i^{*h_t} = Id.$$

Since $h_t = e^{\phi_t} h_0$, we get

$$e^{-\phi_t} = \sum e^{-\lambda_i t} s_i \otimes s_i^{*h_0}.$$

Taking the determinant and log gives

$$tr(-\phi_t) = \log\det\left(\sum e^{-\lambda_i t} s_i \otimes s_i^{*h_0}\right).$$

Differentiating with respect to t, we get

$$tr(\dot{\phi_t}) = tr\left(\left(\sum e^{-\lambda_i t} s_i \otimes s_i^{*h_0}\right)^{-1}\left(\sum \lambda_i e^{-\lambda_i t} s_i \otimes s_i^{*h_0}\right)\right), \tag{3.7}$$

So

$$tr(\dot{\phi_t})\Big|_{t=0} = tr\left(\sum \lambda_i s_i \otimes s_i^{*h_0}\right), \tag{3.8}$$

$$tr(\ddot{\phi_t})\Big|_{t=0} = tr\left(\left(\sum \lambda_i s_i \otimes s_i^{*h_0}\right)^2 - \left(\sum \lambda_i^2 s_i \otimes s_i^{*h_0}\right)\right). \tag{3.9}$$

We denote $\dot{\phi_t}\Big|_{t=0}$ simply by $\dot{\phi}$ and $\ddot{\phi_t}\Big|_{t=0}$ by $\ddot{\phi}$.

Lemma 3.3.2.

$$Tr(s_i \otimes s_i^*) = |s_i|^2,$$

$$Tr((s_i \otimes s_i^*) \circ (s_j \otimes s_j^*)) = |\langle s_i, s_j\rangle|^2.$$

Using the above lemma, we have

$$tr(\dot{\phi}) = \sum \lambda_i |s_i|^2, \tag{3.10}$$

$$tr(\dot{\phi}^2) = \sum \lambda_i \lambda_j |\langle s_j, s_i \rangle|^2, \tag{3.11}$$

$$tr(\ddot{\phi}) = -\lambda_i^2 |s_i|^2 + \sum \lambda_i \lambda_j |\langle s_j, s_i \rangle|^2. \tag{3.12}$$

These imply

$$\ddot{L} = \int_X -\frac{1}{2} |\nabla tr(\dot{\phi})|^2 + \sum \lambda_i^2 |s_i|^2 - \sum \lambda_i \lambda_j |\langle s_j, s_i \rangle|^2.$$

Thus, we can embed X into some big projective space using $(s_{i_1} \wedge ... \wedge s_{i_r})_{i_1 < ... < ir}$ as a set in $H^0(X, \det E)$. The pull back of the FS Kähler form is $Ric(h_0)$. We have

$$\sum |s_{i_1} \wedge ... \wedge s_{i_r}|^2_{\det(h_0)} = 1.$$

Using the first lemma, we have

$$|\nabla tr(\dot{\phi})|^2 = \sum |(\nabla tr(\dot{\phi}), \nabla s_{i_1} \wedge ... \wedge s_{i_r})|^2. \tag{3.13}$$

Let $\varphi = tr(\phi)$. We use the inner products H_t to embed X into Grassmaninan and projective space. Let $s_1, ..., s_N$ be an orthonormal basis for $H^0(X, E)$ with respect to H_t. So we get the FS metric h_t on E. As before we have

$$\sum s_i(t) \otimes s_i(t)^{*h_t} = Id$$

and

$$\sum |s_{i_1}(t) \wedge ... \wedge s_{i_r}(t)|^2_{\det(h_t)} = 1$$

We have $e^{\frac{-\lambda_1}{2}t} s_1, ..., e^{\frac{-\lambda_N}{2}t} s_N$ is an orthonormal basis with respect to H_t. So we can put $s_i(t) = e^{\frac{-\lambda_i}{2}t} s_i$. We have

$$1 = \sum |s_{i_1}(t) \wedge ... \wedge s_{i_r}(t)|^2_{\det(h_t)} = \sum e^{tr(\phi_t)} |s_{i_1}(t) \wedge ... \wedge s_{i_r}(t)|^2_{\det(h_0)},$$

which implies

$$\sum e^{-t\lambda_{i_1 \dots i_r}} |s_{i_1} \wedge \dots \wedge s_{i_r}|^2_{\det(h_0)} = e^{-tr(\phi_t)}. \tag{3.14}$$

Differentiating with respect to t, we get

$$\sum \lambda_{i_1 \dots i_r} e^{-t\lambda_{i_1 \dots i_r}} |s_{i_1} \wedge \dots \wedge s_{i_r}|^2_{\det(h_0)} = tr(\dot{\phi}_t) e^{-tr(\phi_t)}. \tag{3.15}$$

$$-\sum \lambda^2_{i_1 \dots i_r} e^{-t\lambda_{i_1 \dots i_r}} |s_{i_1} \wedge \dots \wedge s_{i_r}|^2_{\det(h_0)} = -tr(\dot{\phi}_t)^2 e^{-tr(\phi_t)} + tr(\ddot{\phi}_t) e^{-tr(\phi_t)}. \tag{3.16}$$

Evaluating at $t = 0$, we obtain

$$\sum \lambda_{i_1 \dots i_r} |s_{i_1} \wedge \dots \wedge s_{i_r}|^2_{\det(h_0)} = tr(\dot{\phi}) \tag{3.17}$$

$$-\sum \lambda^2_{i_1 \dots i_r} |s_{i_1} \wedge \dots \wedge s_{i_r}|^2_{\det(h_0)} = -tr(\dot{\phi})^2 + tr(\ddot{\phi}). \tag{3.18}$$

Using (3.10), (3.11) and (3.12), we get

$$\sum \lambda_i |s_i|^2 = \sum \lambda_{i_1 \dots i_r} |s_{i_1} \wedge \dots \wedge s_{i_r}|^2_{\det(h_0)} \tag{3.19}$$

$$(\sum \lambda_{i_1 \dots i_r} |s_{i_1} \wedge \dots \wedge s_{i_r}|^2_{\det(h_0)})^2 - \sum \lambda^2_{i_1 \dots i_r} |s_{i_1} \wedge \dots \wedge s_{i_r}|^2_{\det(h_0)}$$
$$= -\sum \lambda_i^2 |s_i|^2 + \sum \lambda_i \lambda_j |\langle s_j, s_i \rangle|^2.$$

Lemma 3.3.3. *Consider the positive function F*

$$F = \sum |\nabla \dot{\varphi}, \nabla(s_{i_1} \wedge \dots \wedge s_{i_r}) - (\lambda_{i_1 \dots i_r} - \dot{\varphi})(s_{i_1} \wedge \dots \wedge s_{i_r})|^2$$

on X, where $\lambda_{i_1 \dots i_r} = \sum \lambda_{i_\alpha}$. Then

$$\ddot{L} = \int_X F \omega^n_{h_0}.$$

Proof. Let I denote multi index $(i_1, \dots, i_r)$ and $S_I = s_{i_1} \wedge \dots \wedge s_{i_r}$. Using Donaldson's pairing for the line bundle, $\det(E)$, we have

$$F = \sum |(\nabla \dot{\varphi}, \nabla S_I) - (\lambda_I - \dot{\varphi}) S_I|^2,$$

where $\varphi = tr(\phi)$. Thus,

$$F = \frac{1}{2}|\nabla\dot{\varphi}|^2 + \sum|(\lambda_I - \dot{\varphi})|^2|S_I|^2 - 2\sum((\nabla\dot{\varphi}, \nabla S_I), S_I)(2\lambda_I - \dot{\varphi}).$$

Again, according to Donaldson, we have

$$((\nabla\dot{\varphi}, \nabla S_I), S_I) = (\nabla\dot{\varphi}, (\nabla S_I, S_I)),$$

which implies

$$2((\nabla\dot{\varphi}, \nabla S_I), S_I)(\lambda_I - \dot{\varphi}) = (\nabla\dot{\varphi}, \nabla|S_I|^2)(\lambda_I - \dot{\varphi}).$$

Since we have

$$\nabla|S_I|^2 = 2(S_I, \nabla S_I),$$

and $\sum|S_I|^2 = 1$, we obtain

$$\sum((\nabla\dot{\varphi}, \nabla S_I), S_I)\dot{\varphi} = (\nabla\dot{\varphi}, \nabla\sum|S_I|^2)\dot{\varphi} = 0.$$

So,

$$F = \frac{1}{2}|\nabla\dot{\varphi}|^2 + \sum|(\lambda_I - \dot{\varphi})|^2|S_I|^2 - \nabla|\dot{\varphi}|^2 = -\frac{1}{2}|\nabla\dot{\varphi}|^2 + \sum|(\lambda_I - \dot{\varphi})|^2|S_I|^2,$$

Since $\sum\lambda_I|S_I|^2 = tr(\dot{\phi}) = \dot{\varphi}$. Hence,

$$\begin{aligned} F &= -\frac{1}{2}|\nabla\dot{\varphi}|^2 + \sum(\lambda_I^2 + \dot{\varphi}^2 - 2\lambda_I\dot{\varphi})|S_I|^2 \\ &= -\frac{1}{2}|\nabla\dot{\varphi}|^2 + \sum\lambda_I^2|S_I|^2 + \dot{\varphi}^2\sum|S_I|^2 - 2\dot{\varphi}\sum\lambda_I|S_I|^2 \\ &= -\frac{1}{2}|\nabla\dot{\varphi}|^2 + \sum\lambda_I^2|S_I|^2 - \dot{\varphi}^2 = -\frac{1}{2}|\nabla\dot{\varphi}|^2 - tr(\ddot{\phi}). \end{aligned}$$

□

From the above computation, we can derive the following amusing linear algebra identities:

Corollary 3.3.4. *Let A be a $N \times r$ $(r < N)$ matrix and $\lambda_1, ..., \lambda_N$ real numbers .If $A^*A = Id$, then*

$$\sum_{i,j} \lambda_i |a_{ij}|^2 = \sum_{i_1 < ... < ir} \lambda_{i_1 \dots i_r} |\det(A_{i_1 \dots i_r})|^2$$

$$\sum_{i,j} \lambda_i^2 |a_{ij}|^2 + 2 \sum_{k<l} \lambda_i \lambda_j |a_{ik}|^2 |a_{il}|^2 = \sum_{i_1 < ... < ir} \lambda^2_{i_1 \dots i_r} |\det(A_{i_1 \dots i_r})|^2,$$

where $A_{i_1 \dots i_r}$ is the $r \times r$ matrix whose rows are the respective $i_1^{th}, ..., i_r^{th}$ rows of A and

$$\lambda_{i_1 \dots i_r} = \sum_{j=1}^{r} \lambda_{i_j}.$$

Chapter 4

Donaldson's dynamical system

In [D3], Donaldson defines a dynamical system on the space of Fubini-Study metrics on a polarized compact Kähler manifold. Sano proved that if there exists a balanced metric for the polarization, then this dynamical system always converges to the balanced metric ([Sa]). In [DKLR], Douglas, et. al., conjecture that the same holds in the case of vector bundles. In this paper, we give an affirmative answer to their conjecture.

Let (X, ω) be a Kähler manifold of dimension n and E be a very ample holomorphic vector bundle on X. Let h be a Hermitian metric on E. We can define a L^2-inner product on $H^0(X, E)$ by

$$\langle s, t \rangle = \int_X h(s,t) \frac{\omega^n}{n!}.$$

Let $s_1, ..., s_N$ be an orthonormal basis for $H^0(X, E)$ with respect to this L^2-inner product. The Bergman kernel of h is defined by

$$B(h) = \sum s_i \otimes s_i^{*_h}.$$

Note that $B(h)$ does not depend on the choice of the orthonormal basis $s_1, ..., s_N$.

A metric h is called balanced if $B(h)$ is a constant multiple of the identity. By the theorem of Wang, we know that the existence of balanced metrics is closely related to the stability of the vector bundle E. Indeed E admits a unique (up to a positive constant) balanced metric if and only if the Gieseker point of E is stable (([W1, Theorem 1.1]), [PS]). On the other hand, a balanced metric is unique (up to a constant) provided the bundle is simple.

The main theorem of this chapter is the following

Theorem 4.0.5. *Suppose that E is simple and admits a balanced metric. Then for any $H_0 \in M_E$, the sequence $T^r(H_0)$ converges to H_∞, where H_∞ is a balanced metric on E.*

Our proof follows Sano's argument in [Sa] with the necessary modifications for the bundle case.

In order to prove the theorem, we consider the functional Z that is used by Wang ([W1]) and Phong, Sturm ([PS]) in order to study the existence and uniqueness of balanced metrics on holomorphic vector bundles. The key property of this functional is that its critical points are balanced metrics. In the second section we recall some properties of the functionals Z and $\widetilde{Z}$. In the second section, we give an appropriate notion of boundedness for subsets of M, defined in [**?**]a. With this definition, any bounded sequence has a convergent subsequence after a suitable rescaling of the sequence. Therefore in order to prove that the sequence $H_n = T^n(H)$ converges , we need to show that H_n is bounded. On the other hand, existence of a balanced metric implies that $\widetilde{Z}$ is bounded from below and proper in a suitable sense. Hence it shows that $\widetilde{Z}(H_n)$ is bounded. Now properness of $\widetilde{Z}$ implies that H_n is bounded.

4.1 Proof

In this section, we closely follow Sano's argument in ([Sa, Section 3]). Let $s_1, ..., s_N$ be a basis for $H^0(E)$. With this basis, we can view elements of M as $N \times N$ matrices. Now using this identification, we state the following definition introduced in Sano ([Sa]).

Definition 4.1.1. *A subset $U \subseteq M$ is called bounded if there exists a number $R > 1$, satisfying the following: For any $H \in U$, there exists a positive number γ_H so that*

$$\frac{\gamma_H}{R} \leq \min \frac{|H(\xi)|}{|\xi|} \leq \max \frac{|H(\xi)|}{|\xi|} \leq \gamma_H R \tag{4.1}$$

Note that boundedness does not depend on the choice of the basis. Also notice that $\min |H(\xi)|/|\xi|$ is the smallest eigenvalue of the matrix $[H(s_i, s_j)]$ and $\max |H(\xi)|/|\xi|$ is the largest eigenvalue of the matrix $[H(s_i, s_j)]$.

From the definition, one can see that U is bounded if and only if there exists $R > 1$ satisfying the following: For any $H \in U$, there exists a positive number γ_H so that

$$||[H(s_i, s_j)]||_{op} \leq \gamma_H R,$$

$$||[H(s_i, s_j)]^{-1}||_{op} \leq \gamma_H^{-1} R.$$

Proposition 4.1.2. *Any bounded sequence H_i has a subsequence H_{n_i} such that $\gamma_{n_i}^{-1} H_{n_i}$ converges to some point in M. Here $\gamma_i = \gamma_{H_i}$ in Definition 4.1.1.*

Proof. The sequence $\gamma_{n_i}^{-1} H_{n_i}$ is a bounded sequence in the space of $N \times N$ matrices with respect to the standard topology. Hence the proposition follows from the fact that the closure of bounded sets are compact.

□

Notice that the standard topology on the space of $N \times N$ matrices is induced by the standard Euclidean norm. Since all norms on a finite dimensional vector space are equivalent, we can use the operator norm. Therefore a sequence $\{H_\alpha\}$ in M converges to $H \in M$ if and only if

$$\big|[H_\alpha(s_i,s_j)] - [H(s_i,s_j)]\big|_{op} \to 0 \quad \text{as} \quad \alpha \to 0.$$

Lemma 4.1.3. *The set $U \subseteq M$ is bounded if and only if there exists a number $R > 1$ so that for any $H \in U$, we have*

$$\frac{1}{R} \le \min \frac{|\widetilde{H}(\xi)|}{|\xi|} \le \max \frac{|\widetilde{H}(\xi)|}{|\xi|} \le R,$$

where $\widetilde{H} = (\det(H))^{-\frac{1}{N}} H$.

Proof. Assume that U is bounded. So by definition there exists a number $R > 1$, satisfying the following:

For any $H \in U$, there exists a positive number γ_H so that

$$\frac{\gamma_H}{R} \le \min \frac{|H(\xi)|}{|\xi|} \le \max \frac{|H(\xi)|}{|\xi|} \le \gamma_H R$$

Let H be an element of U. Without loss of generality we can assume that $H(s_i,s_j) = e^{\lambda_i}\delta_{ij}$ and $\lambda_1 \le ... \le \lambda_N$. For any i, we have

$$\frac{\gamma_H}{R} \le e^{\lambda_i} \le \gamma_H R.$$

This implies that $\gamma_H \le Re^{\lambda_i}$ and $\gamma_H \ge R^{-1}e^{\lambda_i}$. Therefore

$$e^{\lambda_N} \le \gamma_H R \le R^2 e^{\lambda_i},$$

and

$$e^{\lambda_1} \ge \gamma_H R^{-1} \ge R^{-2} e^{\lambda_i},$$

for any $1 \le i \le N$. Hence

$$(\det(H))^{\frac{-1}{N}} e^{\lambda_N} = e^{\lambda_N - \frac{\sum \lambda_i}{N}} = \left(\prod e^{\lambda_N - \lambda_i}\right)^{\frac{1}{N}} \le R^2.$$

and

$$(\det(H))^{\frac{-1}{N}} e^{\lambda_1} = e^{\lambda_1 - \frac{\sum \lambda_i}{N}} = \left(\prod e^{\lambda_1 - \lambda_i}\right)^{\frac{1}{N}} \geq R^{-2}.$$

□

Let H_0 be an element in M. We define the sequence $\{H_n\}$ by $H_n = \text{Hilb} \circ FS(H_{n-1})$.

Lemma 4.1.4. *If $\{H_n\}$ is a bounded sequence in M, then $\det(H_n)$ is bounded and*

$$\det(H_{n+1} H_n^{-1}) \to 1 \text{ as } n \to \infty.$$

Proof. $\widetilde{Z}(H_n)$ is bounded since the sequence $\{H_n\}$ is bounded. On the other hand, lemma 2.3.7 implies that the sequences $Z(H_n)$ and $\log\det(H_n)$ are decreasing. So, $\log\det(H_n)$ is bounded and decreasing. Hence, it converges to some real number. This implies that $\det(H_{n+1} H_n^{-1}) \to 1$ as $n \to \infty$.

□

Lemma 4.1.5. *Assume $\{H_n\}$ is a bounded sequence in M. Let H be a fixed element of M and $s_1^{(l)}, ..s_N^{(l)}$ be an orthonormal basis with respect to H_l so that the matrix $[H(s_i^{(l)}, s_j^{(l)})]$ is diagonal. Then*

$$\frac{N}{Vr} \int_X \left|s_i^l\right|^2_{h_l} dvol_X \to 1 \quad as \quad l \to \infty,$$

where $h_n = FS(H_n)$.

Proof. Let $\hat{s}_1^{(l)}, ..\hat{s}_N^{(l)}$ be an orthonormal basis with respect to H_l so that $H_{l+1}(\hat{s}_i^{(l)}, \hat{s}_j^{(l)})$ is diagonal. Hence

$$\det\left[H_{l+1}(\hat{s}_i^{(l)}, \hat{s}_j^{(l)})\right] = \prod_{i=1}^{N} H_{l+1}(\hat{s}_i^{(l)}, \hat{s}_1^{(l)}).$$

Lemma 4.1.4 implies that

$$\det\left[H_{l+1}(\hat{s}_i^{(l)}, \hat{s}_j^{(l)})\right] \to 1.$$

On the other hand lemma 2.3.7 implies that

$$tr\big[H_{l+1}(\hat{s}_i^{(l)}, \hat{s}_j^{(l)})\big] = N.$$

We define $A_l(i) = H_{l+1}(\hat{s}_i^{(l)}, \hat{s}_i^{(l)})$. Therefore, we have

$$\prod_{i=1}^{N} A_l(i) \to 1 \quad \text{as} \quad l \to \infty, \tag{4.2}$$

$$\sum_{i=1}^{N} A_l(i) = N, \quad \text{for any } \ 1 \leq l \leq N. \tag{4.3}$$

We claim that for any i,

$$A_l(i) \to 1 \quad \text{as} \quad l \to \infty. \tag{4.4}$$

Suppose that for some $1 \leq \alpha \leq N$, $\{A_l(\alpha)\}$ does not converge to 1 as $l \to \infty$. This means that there exists a positive number $\epsilon > 0$ and a subsequence $\{A_{l_q}(\alpha)\}$ such that

$$\big|A_{l_q}(\alpha) - 1\big| \geq \epsilon. \tag{4.5}$$

On the other hand, (5.5) implies that $A_l(i) \leq N$ since $A_l(i) \geq 0$ and therefore the sequences $\{A_{l_q}(i)\}$ are bounded for any $1 \leq i \leq N$. Hence there exist nonnegative numbers $A(1), ...A(N)$ and a subsequence $\{l_{q_j}\}$ so that

$$A_{l_{q_j}}(i) \to A(i) \quad \text{as} \quad j \to \infty. \tag{4.6}$$

Therefore, (5.4), (5.5) and (5.8) imply that

$$\prod_{i=1}^{N} A(i) = 1 \quad \text{and} \quad \sum_{i=1}^{N} A(i) = N.$$

By arithmetic-geometric mean inequality, we always have

$$\big(\prod_{i=1}^{N} A(i)\big)^{\frac{1}{N}} \leq \frac{1}{N}\sum_{i=1}^{N} A(i)$$

and equality holds if and only if all A_i's are equal. Since equality holds in this case, we conclude that $A(1) = ... = A(N) = 1$. In particular

$$A_{l_{q_j}}(\alpha) \to 1 \quad \text{as} \quad j \to \infty,$$

which contradicts (5.7). This implies that $H_{l+1}(\hat{s}_i^{(l)}, \hat{s}_i^{(l)}) \to 1$ for all i.

On the other hand, there exists $[a_{ij}^l] \in U(N)$ such that $s_i^{(l)} = \sum_{j=1}^N a_{ij}\hat{s}_j^l$. Since $U(N)$ is compact, we can find a subsequence of $[a_{ij}^l]$ which converges to an element of $U(N)$. Without loss of generality, we can assume that there exists $[a_{ij}] \in U(N)$ such that $a_{ij}^l \to a_{ij}$ as $l \to \infty$. We have,

$$H_{l+1}(s_i^{(l)}, s_i^{(l)}) = \sum a_{ij}^l \overline{a_{ik}^l} H_{l+1}(\hat{s}_j^{(l)}, \hat{s}_k^{(l)}) \to \sum_{j=1}^N |a_{ij}|^2 = 1$$

□

Proposition 4.1.6. *(cf. [Sa, Proposition]) If $\{H_n\}$ is a bounded sequence in M, then for any $H \in M$ and any $\epsilon > 0$,*

$$\widetilde{Z}(H) > \widetilde{Z}(H_n) - \epsilon, \tag{4.7}$$

for sufficiently large n.

Proof. Let $s_1^{(l)}, ..., s_N^{(l)}$ be an orthonormal basis with respect to H_l such that $H(s_i^{(l)}, s_j^{(l)}) = \delta_{ij}e^{\lambda_i^{(l)}}$. We fix a positive integer l. Define $H_t(s_i^{(l)}, s_i^{(l)}) = \delta_{ij}e^{t\lambda_i^{(l)}}$. We have $H_0 = H_l$ and $H_1 = H$. Let $f_l(t) = f(t) = \widetilde{Z}(H_t)$. We have

$$f(1) - f(0) = \int_0^1 f'(t)\, dt = \int_0^1 \Big(f'(0) + \int_0^t f''(s)\, ds\Big)\, dt$$
$$= f'(0) + \int_0^1 \int_0^1 f''(s)\, ds\, dt \geq f'(0),$$

since $\widetilde{Z}$ is convex along geodesics. On the other hand, we have

$$f'(t) = \frac{d}{dt}\Big(- I(FS(H_t)) + \frac{Vr}{N} \log\det(H_t)\Big)$$

$$= -\int_X \frac{d}{dt}\big(FS(H_t)\big)dvol_X + \frac{Vr}{N}\sum \lambda_i^{(l)}$$

Therefore,

$$f_l'(0) = -\int_X \big(\sum \lambda_i^{(l)} |s_i^{(l)}|^2_{h_l}\big)dvol_X + \frac{Vr}{N}\sum \lambda_i^{(l)}, \tag{4.8}$$

where $h_l = FS(H_l)$.

We have that $e^{\frac{-\lambda_1^{(l)}}{2}} s_1^{(l)}, ..., e^{\frac{-\lambda_N^{(l)}}{2}} s_N^{(l)}$ is an orthonormal basis with respect to H for any l. Hence lemma 4.1.3 implies that there exists $R > 1$ so that

$$\frac{(\det(H_l))^{\frac{1}{N}}}{R} < H_l(e^{\frac{-\lambda_i^{(l)}}{2}} s_i^{(l)}, e^{\frac{-\lambda_i^{(l)}}{2}} s_1^{(i)}) < (\det(H_l))^{\frac{1}{N}} R,$$

for any i and l. Therefore

$$\frac{1}{N}\log(\det(H_l)) - \log R < -\lambda_i^{(l)} < \frac{1}{N}\log(\det(H_l)) + \log R.$$

This implies that $\{\lambda_i^{(l)}\}$ is bounded since $\{\det(H_l)\}$ is bounded by Lemma 4.1.4. Hence (4.8) implies that $f_l'(0) \longrightarrow 0$, as $l \longrightarrow \infty$.

$\square$

Corollary 4.1.7. *If $\{H_n\}$ is a bounded sequence in M, then*

$$\widetilde{Z}(H_n) \longrightarrow \inf\{\widetilde{Z}(H) \mid H \in M\}.$$

Proof of Theorem 4.0.5. As before, fix $H_0 \in M$ and an orthonormal basis $s_1, ..., s_N$ for $H^0(X, E)$ with respect to the metric H_0. As in Section 2, let

$$M_0 = \big\{H \in M | \det[H(s_i, s_j)] = 1\big\}.$$

Assume that there exists a balanced metric on E. Since the balanced metric is unique up to a positive constant, there exists a unique balanced metric $H_\infty \in M_0$. As before, for any $H \in M$, we define

$$\widetilde{H} = (\det H)^{-\frac{1}{N}} H.$$

Clearly $\widetilde{H} \in M_0$ and

$$\widetilde{Z}(\widetilde{H}) = \widetilde{Z}(H) = Z(\widetilde{H}).$$

Since there exists a balanced metric on E, theorem 2.3.5 implies that the functional $Z_{|M_0}$ is proper and bounded from below. Hence the sequence $Z(\widetilde{H_n})$ is a bounded sequence in $\mathbb{R}$ since the sequence $\widetilde{Z}(H_n) = Z(\widetilde{H_n})$ is decreasing. Therefore the sequence $\{\widetilde{H_n}\}$ is bounded in M_0 since $Z_{|M_0}$ is proper. We claim that

$$\widetilde{H_n} \longrightarrow H_\infty \quad \text{as} \quad n \to \infty.$$

Suppose that the sequence $\{\widetilde{H_n}\}$ does not converge to H_∞. Then there exists $\epsilon > 0$ and a subsequence $\{H_{n_j}\}$ such that

$$||\widetilde{H_{n_j}} - H_\infty||_{op} \geq \epsilon. \tag{4.9}$$

On the other hand, we know that the sequence $\{\widetilde{H_{n_j}}\}$ is bounded. Therefore there exist a subsequence $\{\widetilde{H_{n_{j_q}}}\}$ and an element $\widehat{H} \in M$ such that

$$\widetilde{H_{n_{j_q}}} \to \widehat{H} \quad \text{as} \quad q \to \infty.$$

Therefore,

$$1 = \det[\widetilde{H_{n_{j_q}}}(s_\alpha, s_\beta)] \to \det[\widehat{H}(s_\alpha, s_\beta)] \quad \text{as} \quad q \to \infty,$$

which implies that $\widehat{H} \in M_0$. Now, corollary 4.1.7 implies that

$$\widetilde{Z}(\widetilde{H_{n_{j_q}}}) = \widetilde{Z}(H_{n_{j_q}}) \longrightarrow \inf\{\widetilde{Z}(H) \mid H \in M\}.$$

Hence,

$$\widetilde{Z}(\widehat{H}) = \inf\{\widetilde{Z}(H) \mid H \in M\}.$$

This implies that $\widehat{H}$ is a balanced metric and therefore $H_\infty = \widehat{H}$ by lemma 2.3.8. This contradicts (5.9). Thus $\widetilde{H_n} \longrightarrow H_\infty \quad \text{as} \quad q \to \infty$.

Now lemma 4.1.4 implies that $\log\det(H_n)$ is bounded. The sequence $\{\log\det(H_n)\}$ is bounded and decreasing. Therefore there exists $b \in \mathbb{R}$ such that

$$\log\det(H_n) \to b \quad \text{as} \quad n \to \infty.$$

Hence $\det(H_n)$ converges to the positive real number e^b. Thus

$$H_n \longrightarrow e^{\frac{-b}{N}} H_\infty \quad \text{as} \quad n \to \infty.$$

$\square$

Chapter 5

Main Theorem

5.1 Moment map setup

In this section, we review Donaldson's moment map setup. We follow the notation of [PS2].

Let (Y, ω_0) be a compact Kähler manifold of dimension n and $\mathcal{O}(1) \to Y$ be a very ample line bundle on Y equipped with a Hermitian metric g_0 such that $Ric(g_0) = \omega_0$. Since $\mathcal{O}(1)$ is very ample, using global sections of $\mathcal{O}(1)$, we can embed Y into $\mathbb{P}(H^0(Y, \mathcal{O}(1))^*)$. A choice of ordered basis $\underline{s} = (s_1, ..., s_N)$ of $H^0(Y, \mathcal{O}(1))$ gives an isomorphism between $\mathbb{P}(H^0(Y, \mathcal{O}(1))^*)$ and $\mathbb{P}^{N-1}$. Hence for any such $\underline{s}$, we have an embedding $\iota_{\underline{s}} : Y \hookrightarrow \mathbb{P}^{N-1}$ such that $\iota_{\underline{s}}^* \mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}(1)$. Using $\iota_{\underline{s}}$, we can pull back the Fubini-Study metric and Kähler form of the projective space to $\mathcal{O}(1)$ and Y respectively.

Definition 5.1.1. *An embedding $\iota_{\underline{s}}$ is called balanced if*

$$\int_Y \langle s_i, s_j \rangle_{\iota_{\underline{s}}^* h_{FS}} \frac{\iota_{\underline{s}}^* \omega_{FS}}{n!} = \frac{V}{N} \delta_{ij},$$

where $V = \int_Y \frac{\omega_0^n}{n!}$. A hermitian metric(respectively a Kähler form) is called balanced if it is the pull back $\iota_{\underline{s}}^* h_{FS}$ (respectively $\iota_{\underline{s}}^* \omega_{FS}$) where $\iota_{\underline{s}}$ is a balanced embedding.

There is an action of $SL(N)$ on the space of ordered bases of $H^0(Y, \mathcal{O}(1))$. Donaldson defines a symplectic form on the space of ordered bases of $H^0(Y, \mathcal{O}(1))$ which is invariant under the action of $SU(N)$. So there exists an equivariant moment map on this space such that its zeros are exactly balanced bases.

More precisely we define

$$\widetilde{\mathcal{Z}} = \{\underline{s} = (s_1, ..., s_N) | s_1, ..., s_N \text{ a basis of } H^0(Y, \mathcal{O}(1))\}/\mathbb{C}^*$$

and $\mathcal{Z} = \widetilde{\mathcal{Z}}/\mathbb{P}Aut(Y, \mathcal{O}(1))$. Donaldson defines a symplectic form Ω_D on $\mathcal{Z}$. There is a natural action of $SU(N)$ on $(\mathcal{Z}, \Omega_D)$ which preserves the symplectic form Ω_D. The moment map for this action is defined by

$$\mu_D(\underline{s}) = i[\langle s_\alpha, s_\beta \rangle_{h_{\underline{s}}} - \frac{V}{N}\delta_{\alpha,\beta}],$$

where $h_{\underline{s}}$ is the L^2- inner product with respect to the pull back of Fubini-Study metric and Fubini-Study Kähler form via the embedding $\iota_{\underline{s}}$. Also we identify $su(N)^*$ with $su(N)$ using the invariant inner product on $su(N)$, where $su(N)$ is the Lie algebra of the group $SU(N)$ and $su(N)^*$ is its dual. (For construction of Ω_D and more details see ([D3]) and ([PS2]) .)

Using Deligne's pairing, Phong and Sturm construct another symplectic form on $\mathcal{Z}$ as follows:

Let

$$\widetilde{\mathcal{Y}} = \{(x, \underline{s}) | x \in \mathbb{P}^{N-1}, \underline{s} = (s_1, ..., s_N), x \in \iota_{\underline{s}}(Y)\}$$

and $\mathcal{Y} = \widetilde{\mathcal{Y}}/\mathbb{P}Aut(Y, \mathcal{O}(1))$. One obtains a holomorphic fibration $\mathcal{Y} \to \mathcal{Z}$ where every fibre is isomorphic to Y. Let $p : \mathcal{Y} \to \mathbb{P}^{N-1}$ be the projection on the first factor. Then define a hermitian line bundle $\mathcal{M}$ on $\mathcal{Z}$ by

$$\mathcal{M} = \langle p^*\mathcal{O}_{\mathbb{P}^{N-1}}(1), ..., p^*\mathcal{O}_{\mathbb{P}^{N-1}}(1)\rangle(\frac{\mathcal{Y}}{\mathcal{Z}})$$

which is the Deligne's pairing of $(n+1)$ copies of $p^*\mathcal{O}_{\mathbb{P}^{N-1}}(1)$. Denote the curvature of this hermitian line bundle by $\Omega_{\mathcal{M}}$. It follows from properties of Deligne's pairing that

$$\Omega_{\mathcal{M}} = \int_{\mathcal{Y}/\mathcal{Z}} \omega_{\mathrm{FS}}^{n+1}. \tag{5.1}$$

Since $SU(N)$ is semisimple, there is a unique equivariant moment map $\mu_{\mathcal{M}} : \mathcal{Z} \to su(N)$ for the action of $SU(N)$ on $(\mathcal{Z}, \Omega_{\mathcal{M}})$.

Theorem 5.1.2. *([PS2, Theorem 1]) $\Omega_{\mathcal{M}} = \Omega_D$ and $\mu_{\mathcal{M}} = \mu_D$.*

Let ξ be an element of the Lie algebra $su(N)$. Since $SU(N)$ acts on $\mathcal{Z}$, the infinitesimal action of ξ defines a vector field $\sigma_{\mathcal{Z}}(\xi)$ on $\mathcal{Z}$. Fixing a point $z \in \mathcal{Z}$, we have a linear map $\sigma_z : su(N) \to T_z\mathcal{Z}$. Let σ_z^* be its adjoint with respect to the metric on $T\mathcal{Z}$ and the invariant metric on $su(N)$. Then we get the operator

$$Q_z = \sigma_z^* \sigma_z : su(N) \to su(N).$$

Define Λ_z^{-1} as the smallest eigenvalue of Q_z. In [D3], Donaldson proves the following.

Proposition 5.1.3. *([D3, Proposition 17]) Suppose given $z_0 \in \mathcal{Z}$ and real numbers λ, δ such that for all $z = e^{i\xi} z_0$ with $|\xi| \leq \delta$ and $\xi \in su(N)$, $\Lambda_z \leq \lambda$. Suppose that $\lambda|\mu(z_0)| \leq \delta$, then there exists $w = e^{i\eta}$ with $\mu(w) = 0$, where $|\eta| \leq \lambda|\mu(z_0)|$.*

5.2 Eigenvalue estimates

In this section, we obtain a lower bound for the derivative of the moment map μ_D. This is equivalent to an upper bound for the quantity Λ_z introduced in the previous section. In order to do this, we adapt the argument of Phong and Sturm to our setting. The main result is Theorem 5.2.4.

Let (Y, ω_0) and $\mathcal{O}(1) \to Y$ be as in the previous section. Let (L, h_∞) be a Hermitian line bundle over Y such that $\omega_\infty = Ric(h_\infty)$ is a semi positive $(1,1)$-form on Y. Define $\widetilde{\omega}_0 = \omega_0 + k\omega_\infty$. For the rest of this section and next section let m be the smallest integer such that $\omega_\infty^{m+1} = 0$. Also assume that $\omega_0^{n-m} \wedge \omega_\infty^m$ is a volume form and there exist positive constant n_1 and n_2 such that

$$N_k = \dim H^0(Y, \mathcal{O}(1) \otimes L^k) = n_1 k^m + O(k^{m-1}). \tag{5.2}$$

$$V_k = \int_Y (\omega_0 + k\omega_\infty)^n = n_2 k^m + O(k^{m-1}). \tag{5.3}$$

Notice that (5.3) is implied from the fact that $\omega_0^{n-m} \wedge \omega_\infty^m$ is a volume form and $\omega_\infty^{m+1} = 0$.

The case important for this paper is the following:

Example 5.2.1. *Let (X, ω_∞) be a compact Kähler manifold of dimension m and L be a very ample holomorphic line bundle on X such that $\omega_\infty \in 2\pi c_1(L)$. Let E be a holomorphic vector bundle on X of rank r such that the line bundle $\mathcal{O}_{\mathbb{P}E^*}(1) \to Y = \mathbb{P}E^*$ is an ample line bundle. We denote the pull back of ω_∞ to $\mathbb{P}E^*$ by ω_∞. Then $\omega_\infty^{m+1} = 0$ and by Riemann-Roch formula we have*

$$\dim H^0(Y, \mathcal{O}(1) \otimes L^k) = \dim H^0(X, E \otimes L^k) = \frac{r}{m!} \int_X c_1(L)^m k^m + O(k^{m-1}).$$

The following lemma is clear.

Lemma 5.2.2. *Let h_k be a sequence of hermitian metrics on $\mathcal{O}(1) \otimes L^k$ and let $\underline{s}^{(k)} = (s_1^{(k)}, ..., s_N^{(k)})$ be a sequence of ordered bases for $H^0(Y, \mathcal{O}(1) \otimes L^k)$. Suppose that for any k*

$$\sum |s_i^{(k)}|_{h_k}^2 = 1$$

and

$$\int_Y \langle s_i^{(k)}, s_j^{(k)} \rangle_{h_k} dvol_{h_k} = D^{(k)} \delta_{ij} + M_{ij}^{(k)},$$

where $D^{(k)}$ is a scalar and $M^{(k)}$ is a trace-free hermitian matrix. Then

$$D^{(k)} = \frac{V_k}{N_k} \to \frac{n_2}{n_1} \quad \text{as } k \to \infty,$$

where the constants n_1 and n_2 are defined by (5.2) and (5.3).

We start with the notion of R-boundedness introduced originally by Donaldson in [D3].

Definition 5.2.3. *Let R be a real number with $R > 1$ and $a \geq 4$ be a fixed integer and let $\underline{s} = (s_1, ..., s_N)$ be an ordered basis for $H^0(Y, \mathcal{O}(1) \otimes L^k)$. We say $\underline{s}$ has R-bounded geometry if the Kähler form $\widetilde{\omega} = \iota_{\underline{s}}^* \omega_{FS}$ satisfies the following conditions*

- $||\widetilde{\omega} - \widetilde{\omega}_0||_{C^a(\widetilde{\omega}_0)} \leq R$, *where* $\widetilde{\omega}_0 = \omega_0 + k\omega_\infty$.
- $\widetilde{\omega} \geq \frac{1}{R}\widetilde{\omega}_0$.

Recall the definition of Λ_z from the previous section. The main result of this section is the following.

Theorem 5.2.4. *Assume Y does not have any nonzero holomorphic vector fields. For any $R > 1$, there are positive constants C and $\epsilon \leq n_2/10n_1$ such that, for any k, if the basis $\underline{s} = (s_1, ..., s_N)$ of $H^0(Y, \mathcal{O}(1) \otimes L^k)$ has R-bounded geometry, and if $||\mu_D(\underline{s})||_{op} \leq \epsilon$, then*

$$\Lambda_{\underline{s}} \leq Ck^{2m+2}.$$

The rest of this section is devoted to the proof of Theorem 5.2.4. Notice that the estimate $\Lambda_z \leq Ck^{2m+2}$ is equivalent to the estimate

$$|\sigma_{\mathcal{Z}}(\xi)|^2 \geq ck^{-(2m+2)}||\xi||^2. \tag{5.4}$$

On the other hand (5.1) and Theorem 5.1.2 imply that

$$|\sigma_{\mathcal{Z}}(\xi)|^2 = \int_Y \iota_{Y_\xi, \overline{Y_\xi}} \omega_{\mathrm{FS}}^{n+1}. \tag{5.5}$$

Hence, in order to establish Theorem 5.2.4, we need to estimate the quantity $\int_Y \iota_{Y_\xi, \overline{Y_\xi}} \omega_{\mathrm{FS}}^{n+1}$ from below.

For the rest of this section, fix an ordered basis $\underline{s}^{(k)} = (s_1, ..., s_N)$ of $H^0(Y, \mathcal{O}(1) \otimes L^k)$ and let $M^{(k)} = -i\mu_D(\underline{s}^{(k)})$. It gives an embedding $\iota = \iota_{\underline{s}^{(k)}} : Y \to \mathbb{P}^{N-1}$, where $N = N_k = \dim H^0(Y, \mathcal{O}(1) \otimes L^k)$. For any $\xi \in su(N)$, we have a vector field Y_ξ on $\mathbb{P}^{N-1}$ generated by the infinitesimal action of ξ.

Every tangent vector to $\mathbb{P}^{N-1}$ is given by pairs (z, v) modulo an equivalence relation $\sim$. This relation is defined as follows:

$$(z, v) \sim (z', v') \text{ if } z' = \lambda z \text{ and } v' - \lambda v = \mu z \text{ for some } \lambda \in \mathcal{C}^* \text{ and } \mu \in \mathbb{C}.$$

For a tangent vector $[(z, v)]$, the Fubini-Study metric is given by

$$||[(z, v)]||^2 = \frac{v^* v z^* z - (z^* v)^2}{(z^* z)^2}.$$

Since the vector field Y_ξ is given by $[z, \xi z]$, we have

$$||Y_\xi(z)||^2 = \frac{-(z^* \xi z)^2 + (z^* \xi^2 z)(z^* z)}{(z^* z)^2}. \tag{5.6}$$

We have the following exact sequence of vector bundles over Y

$$0 \to TY \to \iota^* T\mathbb{P}^{N-1} \to Q \to 0.$$

Let $\mathcal{N} \subset \iota^* T\mathbb{P}^{N-1}$ be the orthogonal complement of TY. Then as smooth vector bundles, we have

$$\iota^* T\mathbb{P}^{N-1} = TY \oplus \mathcal{N}.$$

We denote the projections onto the first and second component by π_T and $\pi_{\mathcal{N}}$ respectively. Define

$$\sigma_t(z) = \exp(it\xi) z,$$

$$\varphi_t(z) = \log \frac{|\sigma_t(z)|}{|z|}.$$

Direct computation shows that

$$\frac{d}{dt}\Big|_{t=0} \varphi_t(z) = 2i \frac{z^* \xi z}{z^* z}, \tag{5.7}$$

$$\frac{d^2}{dt^2}\Big|_{t=0} \varphi_t(z) = 4 \frac{(z^* \xi z)^2 - (z^* \xi^2 z)(z^* z)}{(z^* z)^2}. \tag{5.8}$$

The following is straightforward.

Proposition 5.2.5. *For any $\xi \in su(N)$, we have*

$$||\pi_{\mathcal{N}} Y_\xi||^2_{L^2(Y,TY)} = \int_Y \iota_{Y_\xi, \overline{Y_\xi}} \omega_{FS}^{n+1}$$

Therefore, the estimate in Theorem 5.2.4 will follow from:

$$||\xi||^2 \leq c_R k^m ||Y_\xi||^2 \tag{5.9}$$

$$c_R^{'} ||\pi_T Y_\xi||^2 \leq k^{m+2} ||\pi_{\mathcal{N}} Y_\xi||^2 \tag{5.10}$$

$$||Y_\xi||^2 = ||\pi_T Y_\xi||^2 + ||\pi_{\mathcal{N}} Y_\xi||^2 \tag{5.11}$$

We will prove (5.9) in Proposition 5.2.8 and (5.10) in Proposition 5.2.11. Assuming these, we give the Proof of Theorem 5.2.4.

Proof of Theorem 5.2.4. By (5.5), we have

$$|\sigma_{\mathcal{Z}}(\xi)|^2 = \int_Y \iota_{Y_\xi, \overline{Y_\xi}} \omega_{\mathrm{FS}}^{n+1}.$$

Applying Proposition 5.2.5, we get

$$|\sigma_{\mathcal{Z}}(\xi)|^2 = ||\pi_{\mathcal{N}} Y_\xi||^2.$$

Thus, in order to prove Theorem 5.2.4, we need to show that

$$||\pi_{\mathcal{N}} Y_\xi||^2 \geq c_R k^{-(m+3)} ||\xi||^2.$$

By (5.9), we have

$$||\xi||^2 \leq c_R k^m ||Y_\xi||^2 = c_R k^m ||\pi_{\mathcal{N}} Y_\xi||^2 + c_R k^m ||\pi_T Y_\xi||^2.$$

Hence (5.10) implies that

$$\begin{aligned} ||\xi||^2 &\leq c_R k^m ||\pi_{\mathcal{N}} Y_\xi||^2 + c_R c_R' k^{2m+2} ||\pi_{\mathcal{N}} Y_\xi||^2 \\ &\leq c_R'' k^{2m+2} ||\pi_{\mathcal{N}} Y_\xi||^2. \end{aligned}$$

□

Lemma 5.2.6. *There exists a positive constant c independent of k such that for any $f \in C^\infty(Y)$, we have*

$$c \int_Y f^2 \widetilde{\omega}_0^n \leq k^m \int_Y \overline{\partial} f \wedge \partial f \wedge \widetilde{\omega}_0^{n-1} + k^{-m} \Big(\int_Y f \widetilde{\omega}_0^n \Big)^2$$

Proof. In the proof of this Lemma, we put $\omega_k = \omega_0 + k\omega_\infty$ and $\alpha = \omega_1 = \omega_0 + \omega_\infty$. For $k \geq 1$, we have

$$k^{-m} \omega_k^n \leq \alpha^n \leq \omega_k^n.$$

Assume that the statement is false. So, there exists a subsequence $k_j \to \infty$ and a sequence of functions f_j such that $\int_Y f_j^2 \omega_{k_j}^n = 1$ and

$$k^m \int_Y \overline{\partial} f_j \wedge \partial f_j \wedge \omega_{k_j}^{n-1} + k_j^{-m} \Big(\int_Y f_j \omega_{k_j}^n \Big)^2 \to 0$$

as $k \to \infty$. We define $||f||^2 = \int_Y f^2 \alpha^n$. Hence

$$||f_j||^2 = \int_Y f_j^2 \alpha^n \geq k_j^{-m} \int_Y f_j^2 \omega_{k_j}^n = k_j^{-m}.$$

Let $g_j = f_j/||f_j||$. We have

$$\begin{aligned}\int_Y |\partial g_j|^2_\alpha \alpha^n &= \int_Y \overline{\partial} g_j \wedge \partial g_j \wedge \alpha^{n-1} \\ &= ||f_j||^{-2} \int_Y \overline{\partial} f_j \wedge \partial f_j \wedge \alpha^{n-1} \\ &\leq k_j^m \int_Y \overline{\partial} f_j \wedge \partial f_j \wedge \omega_{k_j}^{n-1} \to 0 \quad \text{as} \quad k \to \infty.\end{aligned}$$

On the other hand $\int_Y g_j^2 \alpha^n = 1$ which implies that the sequence g_j is bounded in $L^2_1(\alpha^n)$. Hence, g_j has a subsequence which converges in $L^2(\alpha^n)$ and converges weakly in $L^2_1(\alpha^n)$ to a function $g \in L^2_1(\alpha^n)$. Without loss of generality, we can assume that the whole sequence converges. Since $\int_Y |\partial g_j|^2_\alpha \alpha^n \to 0$ as $k \to \infty$, it can be easily seen that g is a constant function. We have

$$\begin{aligned}k_j^{-m} \Big| \int_Y (g_j - g)\omega_{k_j}^n \Big| &\leq k_j^{-m} \int_Y |g_j - g| \omega_{k_j}^n \\ &\leq \int_Y |g_j - g| \alpha^n \\ &\leq C\Big(\int_Y |g_j - g|^2 \alpha^n\Big)^{\frac{1}{2}} \to 0,\end{aligned}$$

where $C^2 = \int_Y \alpha^n$ does not depend on k. Hence

$$k_j^{-m} \Big| \int_Y (g_j - g)\omega_{k_j}^n \Big| \to 0.$$

Since g is a constant function and $\int_Y \omega_{k_j}^n = n_2 k_j^m + O(k_j^{m-1})$, we get

$$k_j^{-m} \int_Y g_j \omega_{k_j}^n \to n_2 g,$$

where n_2 is defined by (5.3). On the other hand

$$\begin{aligned}\Big(k_j^{-m} \int_Y g_j \omega_{k_j}^n\Big)^2 &= k_j^{-2m} ||f_j||^{-2} \Big(\int_Y f_j \omega_{k_j}^n\Big)^2 \\ &\leq k_j^{-m} \Big(\int_Y f_j \omega_{k_j}^n\Big)^2 \to 0\end{aligned}$$

which implies $g \equiv 0$. It is a contradiction since $||g_j|| = 1$ and $g_j \to g$ in $L^2(\alpha^n)$.

$\square$

The proof of the following lemma can be found in ([PS2, p. 704]). For the sake of completeness, we give the details.

Lemma 5.2.7. *There exists a positive constant c_R independent of k such that for any Kähler form $\widetilde{\omega} \in c_1(\mathcal{O}(1) \otimes L^k)$ having R-bounded geometry and any $f \in C^\infty(Y)$, we have*

$$c_R \int_Y f^2 \widetilde{\omega}^n \leq k^m \int_Y \overline{\partial} f \wedge \partial f \wedge \widetilde{\omega}^{n-1} + k^{-m}\Big(\int_Y f\widetilde{\omega}^n\Big)^2.$$

Proof. Since $\widetilde{\omega}$ has R-bounded geometry, we have

$$R^{-1}\widetilde{\omega}_0 \leq \widetilde{\omega} \leq 2R\widetilde{\omega}_0.$$

Therefore,

$$c(2R)^{-n} \int_Y f^2 \widetilde{\omega}^n \leq c\int_Y f^2 \widetilde{\omega}_0^n \leq k^m \int_Y \overline{\partial} f \wedge \partial f \wedge \widetilde{\omega}_0^{n-1} + k^{-m}\Big(\int_Y f\widetilde{\omega}_0^n\Big)^2.$$

On the other hand, there exists a unique function ϕ such that $\widetilde{\omega} - \widetilde{\omega}_0 = \partial\overline{\partial}\phi$ and $\int_Y \phi\widetilde{\omega}_0^n = 0$. Hence,

$$\widetilde{\omega}^n - \widetilde{\omega}_0^n = \partial\overline{\partial}\phi \wedge \sum_{j=0}^{n-1} \widetilde{\omega}^j \wedge \widetilde{\omega}_0^{n-j-1}$$

We have,

$$\begin{aligned}
\Big|\int f(\widetilde{\omega}^n - \widetilde{\omega}_0^n)\Big| &= \Big|\int f\partial\overline{\partial}\phi \wedge \sum_{j=0}^{n-1} \widetilde{\omega}^j \wedge \widetilde{\omega}_0^{n-j-1}\Big| \\
&= \Big|\int \overline{\partial} f \wedge \partial\phi \wedge \sum_{j=0}^{n-1} \widetilde{\omega}^j \wedge \widetilde{\omega}_0^{n-j-1}\Big| \\
&\leq \sum_{j=0}^{n-1} \int |\overline{\partial} f|_{\widetilde{\omega}_0} |\partial\phi|_{\widetilde{\omega}_0} \big(\frac{\widetilde{\omega}}{\widetilde{\omega}_0}\big)^p \widetilde{\omega}_0^n \\
&\leq n(2R)^n \int |\overline{\partial} f|_{\widetilde{\omega}_0} |\partial\phi|_{\widetilde{\omega}_0} \widetilde{\omega}_0^n \\
&\leq C_1\Big(\int |\overline{\partial} f|^2_{\widetilde{\omega}_0} \widetilde{\omega}_0^n\Big)^{\frac{1}{2}} \Big(\int |\overline{\partial}\phi|^2_{\widetilde{\omega}_0} \widetilde{\omega}_0^n\Big)^{\frac{1}{2}} \\
&= C_1\Big(\int \overline{\partial} f \wedge \partial f \wedge \widetilde{\omega}_0^{n-1}\Big)^{\frac{1}{2}} \Big(\int |\overline{\partial}\phi|^2_{\widetilde{\omega}_0} \widetilde{\omega}_0^n\Big)^{\frac{1}{2}}.
\end{aligned}$$

We will show that

$$\int |\overline{\partial}\phi|^2_{\widetilde{\omega}_0}\widetilde{\omega}_0^n \le C_2 k^{2m}.$$

Since $\widetilde{\omega} - \widetilde{\omega}_0 = \partial\overline{\partial}\phi$ and $||\widetilde{\omega} - \widetilde{\omega}_0||_{C^a(\widetilde{\omega}_0)} \le R$, we have $||\partial\overline{\partial}\phi||_{C^a(\widetilde{\omega}_0)} \le R$. This implies that

$$||\triangle_{\widetilde{\omega}_0}\phi||_\infty \le R.$$

Applying Lemma 5.2.6 to ϕ, we get

$$c\int_Y \phi^2\widetilde{\omega}_0^n \le k^m \int_Y \overline{\partial}\phi \wedge \partial\phi \wedge \widetilde{\omega}_0^{n-1}$$

On the other hand

$$\begin{aligned}
\int_Y |\overline{\partial}\phi|^2_{\widetilde{\omega}_0}\widetilde{\omega}_0^n = \int_Y \overline{\partial}\phi \wedge \partial\phi \wedge \widetilde{\omega}_0^{n-1} &= \Big|\int_Y \phi\triangle_{\widetilde{\omega}_0}\phi\widetilde{\omega}_0^n\Big| \\
&\le \Big(\int_Y \phi^2\widetilde{\omega}_0^n\Big)^{\frac{1}{2}}\Big(\int_Y |\triangle_{\widetilde{\omega}_0}\phi|^2\widetilde{\omega}_0^n\Big)^{\frac{1}{2}} \\
&\le c^{\frac{-1}{2}}k^{\frac{m}{2}}\Big(\int_Y |\overline{\partial}\phi|^2_{\widetilde{\omega}_0}\widetilde{\omega}_0^n\Big)^{\frac{1}{2}}\Big(R^2\int_Y \widetilde{\omega}_0^n\Big)^{\frac{1}{2}} \\
&= Ck^m\Big(\int_Y |\overline{\partial}\phi|^2_{\widetilde{\omega}_0}\widetilde{\omega}_0^n\Big)^{\frac{1}{2}}
\end{aligned}$$

Therefore,

$$\int |\overline{\partial}\phi|^2_{\widetilde{\omega}_0}\widetilde{\omega}_0^n \le C_2 k^{2m}.$$

So, we get

$$\Big|\int f(\widetilde{\omega}^n - \widetilde{\omega}_0^n)\Big| \le Ck^m\Big(\int_Y \overline{\partial}f \wedge \partial f \wedge \widetilde{\omega}_0^{n-1}\Big)^{\frac{1}{2}}$$

On the other hand

$$\frac{1}{2}\Big(\int_Y f\widetilde{\omega}_0^n\Big)^2 \le \Big(\int_Y f\widetilde{\omega}^n\Big)^2 + \Big(\int_Y f\big(\widetilde{\omega}^n - \widetilde{\omega}_0^n\big)\Big)^2$$

Hence,

$$\begin{aligned}\widetilde{C}\int_Y f^2\widetilde{\omega}^n &\leq k^m\int_Y \overline{\partial} f\wedge\partial f\wedge\widetilde{\omega}_0^{n-1} + 2k^{-m}\Big(\Big(\int_Y f\widetilde{\omega}^n\Big)^2 + \Big(\int_Y f\big(\widetilde{\omega}^n - \widetilde{\omega}_0^n\big)\Big)^2\Big)\\ &\leq k^m\int_Y \overline{\partial} f\wedge\partial f\wedge\widetilde{\omega}_0^{n-1} + 2k^{-m}\Big(\int_Y f\widetilde{\omega}^n\Big)^2 + C_3k^m\int_Y\overline{\partial} f\wedge\partial f\wedge\widetilde{\omega}_0^{n-1}\\ &\leq C_4\Big(k^m\int_Y\overline{\partial} f\wedge\partial f\wedge\widetilde{\omega}^{n-1} + k^{-m}\Big(\int_Y f\widetilde{\omega}^n\Big)^2\Big).\end{aligned}$$

□

Proposition 5.2.8. *There exists a positive constant c_R such that for any $\xi\in su(N)$, we have*

$$||\xi||^2 \leq c_R k^m ||Y_\xi||^2,$$

where $||.||$ in the right hand side denotes the L^2- norm with respect to the Kähler form $\widetilde{\omega}$ on Y and Fubini-Study metric on the fibres.

Proof. By (5.6), we have

$$|Y_\xi|^2 = -4\frac{(z^*\xi z)^2 - (z^*\xi^2 z)(z^*z)}{(z^*z)^2}$$

This implies that

$$\begin{aligned}||Y_\xi||^2_{L^2(\widetilde{\omega})} &= tr\Big(\xi^*\xi\int\frac{zz^*}{z^*z}\widetilde{\omega}^n\Big) - \int\frac{(z^*\xi z)^2}{(z^*z)^2}\widetilde{\omega}^n\\ &= tr\Big(\xi^*\xi\int\frac{zz^*}{z^*z}\widetilde{\omega}^n\Big) - \int\dot{\varphi}^2\widetilde{\omega}^n.\end{aligned}$$

We can write

$$\int_Y\frac{zz^*}{z^*z}\widetilde{\omega}^n = D^{(k)}I + M^{(k)},$$

where $D^{(k)}\to n_2/n_1$ as $k\to\infty$ and $M^{(k)}$ is a trace free hermitian matrix with $||M^{(k)}||_{\text{op}}\leq\epsilon$. Therefore,

$$||Y_\xi||^2 = |\xi|^2 D^{(k)} + tr(\xi^*\xi M^{(k)}) - \int\dot{\varphi}^2\widetilde{\omega}^n.$$

Hence

$$|tr(\xi^*\xi M^{(k)})| = |tr(\xi M^{(k)})\xi| \leq ||\xi||^2||M^{(k)}||_{\mathrm{op}} \leq \epsilon||\xi||^2.$$

Since $D^{(k)} \to n_2/n_1$ as $k \to \infty$, there exists a positive constant c such that

$$||Y_\xi||^2 \geq c||\xi||^2 - \int \dot{\varphi}^2\widetilde{\omega}^n.$$

On the other hand

$$\begin{aligned}\Big|\int \dot{\varphi}\widetilde{\omega}^n\Big| &= |tr(\xi M^{(k)})| \leq \sqrt{N}||\xi||||M^{(k)}||_{\mathrm{op}} \\ &\leq ck^{\frac{m}{2}}||\xi||||M^{(k)}||_{\mathrm{op}}.\end{aligned}$$

Now applying Lemma 5.2.7, we get

$$\begin{aligned}C\int_Y \dot{\varphi}^2\widetilde{\omega}^n &\leq k^m\int_Y \overline{\partial}\dot{\varphi}\wedge\partial\dot{\varphi}\wedge\widetilde{\omega}^{n-1} + k^{-m}\Big(\int_Y \dot{\varphi}\widetilde{\omega}^n\Big)^2 \\ &\leq k^m\int_Y \overline{\partial}\dot{\varphi}\wedge\partial\dot{\varphi}\wedge\widetilde{\omega}^{n-1} + c_2||\xi||^2||M^{(k)}||^2_{\mathrm{op}}.\end{aligned}$$

This implies

$$(c_1 - C_2||M^{(k)}||^2_{\mathrm{op}})||\xi||^2 \leq ||Y_\xi||^2 + k^m\int_Y \overline{\partial}\dot{\varphi}\wedge\partial\dot{\varphi}\wedge\widetilde{\omega}^{n-1}.$$

Since $||M^{(k)}||_{\mathrm{op}} \leq \epsilon$ and ϵ is small enough, there exists a positive constant c such that

$$\begin{aligned}c||\xi||^2 &\leq ||Y_\xi||^2 + k^m\int_Y \overline{\partial}\dot{\varphi}\wedge\partial\dot{\varphi}\wedge\widetilde{\omega}^{n-1} \\ &= ||Y_\xi||^2 + k^m\int_Y |\overline{\partial}\dot{\varphi}|^2_{\widetilde{\omega}}\widetilde{\omega}^n\end{aligned}$$

We know that $\overline{\partial}\dot{\varphi}|_Y = \iota_{\pi_T Y_\xi}\widetilde{\omega}$ which implies

$$c||\xi||^2 \leq ||Y_\xi||^2 + k^m||\pi_T Y_\xi||^2.$$

$\square$

Lemma 5.2.9. *For $k \gg 0$, we have*

$$|S(\omega_0 + k\omega_\infty)| \leq C \log k,$$

where S is the scalar curvature.

Proof. We have

$$(\omega_0 + k\omega_\infty)^n = \sum_{j=0}^{m} \binom{n}{k} k^j \omega_0^{n-j} \wedge \omega_\infty^j = \big(1 + \sum_{j=1}^{m} k^j f_j\big)\omega_0^n,$$

for some smooth nonnegative functions f_j on Y. The function f_m is positive, since $\omega_0^{n-m} \wedge \omega_\infty^m$ is a volume form. Therefore there exists a positive constant l such that $f_m \geq l > 0$. We define

$$F = \sum_{j=1}^{m} k^{j-m} f_j.$$

There exits a constant C such that $||F||_{C^2} \leq C$, since F is bounded independent of k. We have

$$\nabla^2 \log(1 + k^m F) = \nabla\Big(\frac{k^m \nabla F}{1 + k^m F}\Big) = \frac{k^m \nabla^2 F}{1 + k^m F} - \frac{k^{2m}(\nabla F)^2}{(1 + k^m F)^2}.$$

Hence there exists a positive constant C such that

$$\big|\log(1 + k^m F)\big|_{C^2} \leq mC \log k + C,$$

since $||F||_{C^2}$ is bounded independent of k and $F \geq f_m \geq l > 0$. This implies that

$$\begin{aligned}
\big|\partial\overline{\partial} \log\det(\omega_0 + k\omega_\infty)\big|_{C^0} &\leq \big|\log\det(\omega_0 + k\omega_\infty)\big|_{C^2} \\
&= \big|\log(\omega_0 + k\omega_\infty)^n\big|_{C^2} \\
&\leq \big|\log \omega_0^n\big|_{C^2} + \big|\log(1 + k^m F)\big|_{C^2} \\
&\leq C_1 + C_2 m \log k.
\end{aligned}$$

Fix a point $p \in Y$ and a holomorphic local coordinate $z_1, ..., z_n$ around p such that

$$\omega_0(p) = i \sum dz_i \wedge d\overline{z_i},$$

$$\omega_\infty(p) = i \sum \lambda_i dz_i \wedge d\overline{z_i},$$

where λ_i's are some nonnegative real numbers. Therefore, we have

$$\begin{aligned}\left|S(\omega_0 + k\omega_\infty)(p)\right| &= \left|\sum (1 + k\lambda_i)^{-1} \partial_i \partial_{\bar{i}} \log \det(\omega_0 + k\omega_\infty)\right| \\ &\leq \sum (1 + k\lambda_i)^{-1} (C_1 + C_2 m \log k) \leq C_3 \log k,\end{aligned}$$

for $k \gg 0$.

□

Proposition 5.2.10. *For any holomorphic vector field V on $\mathbb{P}^{N-1}$, we have*

$$|\pi_\mathcal{N} V|^2 \geq C_R k^{-1} |\overline{\partial}(\pi_\mathcal{N} V)|^2.$$

Proof. The following is from ([PS2, pp. 705-708]). For the sake of completeness, we give the details of the proof. Fix $x \in Y$. Let $e_1, ..., e_n, f_1, ..., f_m$ be a local holomorphic frame for $\iota^* T\mathbb{P}^{N-1}$ around x such that

1. $e_1(x), ..., e_n(x), f_1(x), ..., f_m(x)$ form an orthonormal basis.

2. $e_1, ..., e_n$ is a local holomorphic basis for TY.

Then there exist holomorphic functions a_j and b_j's such that

$$V = \sum a_j e_j + \sum b_j f_j.$$

Notice that $\pi_\mathcal{N} f_j - f_j$ is tangent to Y since $\pi_\mathcal{N}(\pi_\mathcal{N} f_j - f_j) = 0$. Therefore, we can write

$$\pi_\mathcal{N} f_j - f_j = \sum \phi_{ij} e_j,$$

where ϕ_{ij}'s are smooth functions. Since $e_1(x), ..., e_n(x), f_1(x), ..., f_m(x)$ form an orthonormal basis, we have $\phi_{ij}(x) = 0$. Then

$$\pi_\mathcal{N} V = \sum_{j=1}^{m} b_j \big(f_j - \sum_i \phi_{ij} e_i\big).$$

It implies that

$$\overline{\partial}(\pi_{\mathcal{N}}V)=\sum_{j=1}^{m}b_j\big(-\sum_i(\overline{\partial}\phi_{ij})e_i\big).$$

So in order to establish 5.2.10, we need to prove that

$$\sum_{i=1}^{n}|\sum_{j=1}^{m}b_j\overline{\partial}\phi_{ij}|^2\le C_R^{-1}k\sum_{j=1}^{m}|b_j|^2.$$

Using the Cauchy-Schwartz inequality, it suffices to prove

$$\sum_{i=1}^{n}\sum_{j=1}^{m}|\overline{\partial}\phi_{ij}|^2\le C_2k,$$

where $C_2=C_2(R)$ is independent of k (depends on R.) Now the matrix $A^*=(\overline{\partial}\phi_{ij})$ is the dual of the second fundamental form A of TY in $\iota^*T\mathbb{P}^{N-1}$. Let $F_{\iota^*T\mathbb{P}^{N-1}}$ be the curvature tensor of the bundle $\iota^*T\mathbb{P}^{N-1}$ with respect to the Fubini-Study metric. $F_{\iota^*T\mathbb{P}^{N-1}}$ is a 2-form on Y withe values in $End(\iota^*T\mathbb{P}^{N-1})$. Thus $F_{\iota^*T\mathbb{P}^{N-1}}\big|_{TY}$ is a two form on Y with values in $Hom(TY,\iota^*T\mathbb{P}^{N-1})$. So, $\pi_T\circ(F_{\iota^*T\mathbb{P}^{N-1}}\big|_{TY})$ is a two form on Y with values in $End(TY)$. Also let F_{TY} be the curvature tensor of the bundle TY with respect to the pulled back Fubini-Study metric $\widetilde{\omega}=\iota^*\omega_{\mathrm{FS}}$. Now by computations in [PS2, 5.28], we have

$$\sum_{i=1}^{n}\sum_{j=1}^{m}|\overline{\partial}\phi_{ij}|^2=\Lambda_{\widetilde{\omega}}Tr\big[\pi_T\circ(F_{\iota^*T\mathbb{P}^{N-1}}\big|_{TY})-F_{TY}\big],$$

where $\Lambda_{\widetilde{\omega}}$ is the contraction with the Kähler form $\widetilde{\omega}$. The formula [PS2, 5.33] gives

$$\Lambda_{\widetilde{\omega}}Tr\big[\pi_T\circ(F_{\iota^*T\mathbb{P}^{N-1}}\big|_{TY})\big]=n+1.$$

On the other hand $\Lambda_{\widetilde{\omega}}Tr(F_{TY})$ is the scalar curvature of the metric $\widetilde{\omega}$ on Y. Since $\widetilde{\omega}$ has R-bounded geometry, we have

$$|S(\widetilde{\omega})-S(\widetilde{\omega}_0)|\le R.$$

Lemma 5.2.9 implies that $|S(\widetilde{\omega}_0)|\le C\log k\le Ck$.

$\square$

The only thing we need in addition is the following

Proposition 5.2.11. *Assume that there are no nonzero holomorphic vector fields on Y. Then there exists a constant $c_R^{'}$ such that for any $\xi \in su(N)$, we have*

$$c_R^{'}||\pi_T Y_\xi||^2 \leq k^{m+2}||\pi_{\mathcal{N}} Y_\xi||^2.$$

Proof. We define $\alpha = \omega_0 + \omega_\infty$. Since there are no holomorphic vector fields on Y, for any smooth smooth vector field W on Y, we have

$$c||W||^2_{L^2(\alpha)} \leq ||\overline{\partial} W||^2_{L^2(\alpha)}.$$

The trivial inequalities $k\alpha \geq \widetilde{\omega}_0$ and $k^{-m}\widetilde{\omega}_0^n \leq \alpha^n \leq \widetilde{\omega}_0^n$ imply that

$$\begin{aligned} c||W||^2_{L^2(\widetilde{\omega}_0)} = c\int |W|^2_{\widetilde{\omega}_0}\widetilde{\omega}_0^n &\leq ck^{m+1}\int |W|^2_\alpha \alpha^n \\ &\leq k^{m+1}\int |\overline{\partial} W|^2_\alpha \alpha^n \\ &\leq k^{m+1}\int |\overline{\partial} W|^2_{\widetilde{\omega}_0}\widetilde{\omega}_0^n \\ &= k^{m+1}||\overline{\partial} W||^2_{L^2(\widetilde{\omega}_0)}. \end{aligned}$$

Hence, there exists a positive constant c depends on R and independent of k, such that for any $\widetilde{\omega}_0$ having R-bounded geometry, we have

$$c||W||^2_{L^2(\widetilde{\omega})} \leq k^{m+1}||\overline{\partial} W||^2_{L^2(\widetilde{\omega})}.$$

Now, putting $W = \pi_T Y_\xi$, we get

$$c||\pi_{\mathcal{T}} Y_\xi||^2_{L^2(\widetilde{\omega})} \leq k^2||\overline{\partial}(\pi_{\mathcal{T}} Y_\xi)||^2_{L^2(\widetilde{\omega})}.$$

On the other hand

$$||\pi_{\mathcal{N}} V||^2 \geq C_R k^{-1}||\overline{\partial}(\pi_{\mathcal{N}} V)||^2,$$

which implies the desired inequality.

$\square$

5.3 Perturbing to a balanced metric

We continue with the notation of the previous section. The goal of this section is to prove Theorem 5.3.7 which gives a condition for when an almost balanced metric can be perturbed to a balanced one. In order to do this, first we need to establish Theorem 5.3.6. We need the following estimate.

Proposition 5.3.1. *There exist positive real numbers K_j depends only on h_0, g_∞ and j such that for any $s \in H^0(Y, \mathcal{O}(1) \otimes L^k)$, we have*

$$|\nabla^j s|^2_{C^0(\widetilde{\omega}_0)} \leq K_j k^{n+j} \int_Y |s|^2 \frac{\omega_0^n}{n!}.$$

In order to prove Proposition 5.3.1, we start with some complex analysis.

Let φ be a strictly plurisubharmonic function and ψ be a plurisubharmonic function on $B = B(2) \subset \mathbb{C}^n$ such that $\varphi(0) = \psi(0) = 0$. We can find a coordinate on $B(2)$ such that

$$\varphi(z) = |z|^2 + O(|z|^2) \quad \text{and} \quad \psi(z) = \sum \lambda_i |z_i|^2 + O(|z|^2),$$

where $\lambda_i \geq 0$. For any function $u : B \to \mathbb{C}$, we define $u^{(k)}(z) = u(\frac{z}{\sqrt{k}})$.

Theorem 5.3.2. *(Cauchy Estimate cf.[Ho, Theorem 2.2.3])There exist positive real numbers C_j such that for any holomorphic function $u : B \to \mathbb{C}$, we have*

$$|\nabla^j u|^2(0) \leq C_j \int_{|z|\leq 1} |u(z)|^2 dz \wedge d\overline{z}$$

Theorem 5.3.3. *There exist positive real numbers c_j depends only on j, φ, ψ and $d\mu$ such that for any holomorphic function $u : B \to \mathbb{C}$, we have*

$$|\nabla^j u|^2(0) \leq c_j k^{n+j} \int_{B(1)} |u|^2 e^{-\varphi - k\psi} d\mu,$$

where $d\mu$ is a fixed volume form on B.

Proof. Applying Cauchy estimate to $u^{(k)}$, we get

$$k^{-j}|\nabla^j u|^2(0) \le C_j \int_{|z|\le 1} |u^{(k)}(z)|^2 dz \wedge d\overline{z}$$
$$\le C \int_{|z|\le 1} |u^{(k)}(z)|^2 e^{-\sum(\lambda_i+1)|z_i|^2} dz \wedge d\overline{z},$$

since $e^{-\sum(\lambda_i+1)|z_i|^2}$ is bounded from below by a positive constant on the unit ball. Using the change of variable $w = \frac{z}{\sqrt{k}}$ we get

$$k^{-j}|\nabla^j u|^2(0) \le Ck^n \int_{|w|\le k^{-1/2}} |u(w)|^2 e^{-k\sum(\lambda_i+1)|w_i|^2} dw \wedge d\overline{w}$$
$$\le Ck^n \int_{|w|\le k^{-1/2}} |u(w)|^2 e^{-\sum(k\lambda_i+1)|w_i|^2} dw \wedge d\overline{w}.$$

On the other hand, we have

$$\varphi(z) + k\psi(z) = k\sum(\lambda_i+1)|z_i|^2 + \mu(z) + k\sigma(z),$$

where $\lim\limits_{z\to 0} \dfrac{\mu(z)}{|z|^2} = \lim\limits_{z\to 0} \dfrac{\sigma(z)}{|z|^2} = 0.$

Let $|w| \le k^{-1/2}$, we have

$$|k\sigma(w) + \mu(w)| \le c(k|w|^2 + |w|^2) \le 2c$$

for some constant c depending only on ψ and φ. Hence

$$k^{-j}|\nabla^j u|^2(0) \le Ck^n \int_{|w|\le k^{-1/2}} |u(w)|^2 e^{-\sum(k\lambda_i+1)|w_i|^2} dw \wedge d\overline{w}$$
$$= Ce^{2c}k^n \int_{|w|\le k^{-1/2}} |u(w)|^2 e^{-\sum(k\lambda_i+1)|w_i|^2 - 2c} dw \wedge d\overline{w}$$
$$\le C'k^n \int_{|w|\le k^{-1/2}} |u(w)|^2 e^{-\sum(k\lambda_i+1)|w_i|^2 - (\mu(w)+k\sigma(w))} dw \wedge d\overline{w}$$
$$= C'k^n \int_{|w|\le k^{-1/2}} |u(w)|^2 e^{-(\varphi(w)+k\psi(w))} dw \wedge d\overline{w}$$
$$\le C'k^n \int_{B(1)} |u|^2 e^{-\varphi-k\psi} dz \wedge d\overline{z}.$$

Hence,

$$|\nabla^j u|^2(0) \le c_j k^{n+j} \int_{B(1)} |u|^2 e^{-\varphi - k\psi} d\mu.$$

□

Proof of Proposition 5.3.1. Fix a point p in Y and a geodesic ball $B \subset Y$ centered at p. Let e_L be a holomorphic frame for L on B and e be a holomorphic frame for $\mathcal{O}(1)$ such that $||e_L||(p) = ||e||(p) = 1$. Any $s \in H^0(Y, \mathcal{O}(1) \otimes L^k)$ can be written as $s = ue \otimes e_L^{\otimes k}$ for some holomorphic function $u : B \to \mathbb{C}$. We have

$$\nabla^j s = \sum \binom{j}{i} \nabla^i u \otimes \nabla^{j-i}(e \otimes e_L^{\otimes k}).$$

Therefore,

$$|\nabla^j s|^2(p) \le C(\sum |\nabla^i u|^2(p) ||\nabla^{j-i}(e \otimes e_L^{\otimes k})||^2(p)).$$

On the other hand we have

$$||\nabla^\alpha(e \otimes e_L^{\otimes k})||^2(p) \le \sum_{i=0}^{\alpha} (||\nabla^i e||^2(p) + k^{\alpha-i} ||\nabla^{\alpha-i} e_L||^2(p)) \le C_\alpha k^\alpha.$$

Hence

$$|\nabla^j s|^2(p) \le C'(\sum |\nabla^i u|^2(p) k^{j-i}).$$

Applying Theorem 6.1.2 concluds the proof.

□

For the rest of this section, we fix a positive integer q. We continue with the notation $(Y, \omega_\infty, \omega_0, \widetilde{\omega_0}\ )$ of section 3. In the rest of this section, we fix the reference metric ω_0 on Y and recall the Definition 5.2.3.

Definition 5.3.4. *The sequence of hermitian metrics h_k on $\mathcal{O}(1) \otimes L^k$ and ordered bases $\underline{s}^{(k)} = (s_1^{(k)}, ..., s_N^{(k)})$ for $H^0(Y, \mathcal{O}(1) \otimes L^k)$ is called* almost balanced of order q *if for any k*

$$\sum |s_i^{(k)}|^2_{h_k} = 1$$

and

$$\int_Y \langle s_i^{(k)}, s_j^{(k)} \rangle h_k dvol_{h_k} = D^{(k)}\delta_{ij} + M_{ij}^{(k)},$$

where $D^{(k)}$ *is a scalar so that* $D^{(k)} \to n_2/n_1$ *as* $k \to \infty$ *(See* (5.2) *and* (5.3)*.), and* $M^{(k)}$ *is a trace-free hermitian matrix such that* $||M^{(k)}||_{op} = O(k^{-q-1})$*.*

We state the following lemma without proof. The proof is a straightforward calculation.

Lemma 5.3.5. *Let the sequence of hermitian metrics* h_k *on* $\mathcal{O}(1) \otimes L^k$ *and ordered bases* $\underline{s}^{(k)} = (s_1^{(k)}, ..., s_N^{(k)})$ *for* $H^0(Y, \mathcal{O}(1) \otimes L^k)$ *be almost balanced of order* q*. Suppose*

$$||\widetilde{\omega_k} - \widetilde{\omega_0}||_{C^a(\widetilde{\omega_0})} = O(k^{-1}), \tag{5.12}$$

where $\widetilde{\omega_k} = Ric(h_k)$*. Then for any* $\epsilon > 0$ *there exists a positive integer* k_0 *such that*

$$\widetilde{\omega_k} \geq (1-\epsilon)\widetilde{\omega_0} \ \textit{for} \ k \geq k_0.$$

Assume that there exist a sequence of almost balanced metrics h_k of order q and bases $\underline{s}^k = (s_1^{(k)}, ..., s_N^{(k)})$ for $H^0(Y, \mathcal{O}(1) \otimes L^k)$ which satisfies (5.12). As before $\widetilde{\omega_k} = \mathrm{Ric}(h_k)$. Then Lemma 6.1.4 implies that for $k \gg 0$, $\widetilde{\omega_k}$ has R-bounded geometry.

Fix k and let $B \in isu(N_k)$. Without loss of generality, we can assume that B is the diagonal matrix $\mathrm{diag}(\lambda_i)$, where $\lambda_i \in \mathbb{R}$ and $\sum \lambda_i = 0$. There exists a unique hermitian metric h_B on $\mathcal{O}_{\mathbb{P}E^*}(1) \otimes L^k$ such that

$$\sum e^{2\lambda_i} |s_i^{(k)}|_{h_B}^2 = 1.$$

Let $\widetilde{\omega}_B = Ric(h_B)$. In the next theorem, we will prove that there exist a constant c and open balls $U_k \subset isu(N_k)$ around the origin of radius $ck^{-(n+a+2)}$ so that if $B \in U_k$, then h_B is R-bounded. More precisely,

Theorem 5.3.6. *Suppose that* (5.12) *holds.*

- *There exist $c > 0$ and $k_0 > 0$ such that if $k \geq k_0$ and $B \in isu(N_k)$ satisfies*

$$||B||_{op} \leq ck^{-(n+a+2)}R,$$

then the metric $\widetilde{\omega_B}$ is R-bounded.

- *There exists $c > 0$ such that if $B \in isu(N_k)$ satisfies*

$$||B||_{op} \leq k^{-(n+a+3)},$$

then

$$||M^B||_{op} \leq ck^{-1},$$

where the matrix $M^B = (M_{ij}^B)$is defined by

$$M_{ij}^B = e^{\lambda_i+\lambda_j} \int_Y \langle s_i^{(k)}, s_j^{(k)} \rangle_{h_B} \frac{\widetilde{\omega}_B^n}{n!} - \frac{V_k}{N_k}\delta_{ij}.$$

Proof. Let $h_B = e^{\varphi_B} h_k$. So, we have

$$1 = \sum e^{2\lambda_i} |s_i^{(k)}|^2_{h_B} = e^{\varphi_B} \sum e^{2\lambda_i} |s_i^{(k)}|^2_{h_k}.$$

Hence

$$\varphi_B = -\log \sum e^{2\lambda_i} |s_i^{(k)}|^2_{h_k} = -\log \big(1 + \sum (e^{2\lambda_i} - 1)|s_i^{(k)}|^2_{h_k}\big).$$

If $||B||_{\text{op}}$ is small enough, there exists $C > 0$ so that

$$||\varphi_B||_{C^{a+2}(\widetilde{\omega}_0)} \leq C||B||_{\text{op}} \sum_{i=1}^{N_k} |\nabla^{a+2} s_i^{(k)}|^2_{C^0(\widetilde{\omega}_0)}$$

and therefore Proposition 5.3.1 implies that

$$\begin{aligned} ||\varphi_B||_{C^{a+2}(\widetilde{\omega}_0)} &\leq C||B||_{\text{op}} k^{n+a+2} \sum_{i=1}^{N_k} \int_Y |s_i^{(k)}|^2_{h_k} \frac{\omega_0^n}{n!} \\ &= C||B||_{\text{op}} k^{n+a+2} \int_Y \sum_{i=1}^{N_k} |s_i^{(k)}|^2_{h_k} \frac{\omega_0^n}{n!} \\ &= C||B||_{\text{op}} k^{n+a+2} \int_Y \frac{\omega_0^n}{n!} = c_1 ||B||_{\text{op}} k^{n+a+2} \end{aligned}$$

for some positive constant c_1. Now if $||B||_{\text{op}} \leq c_1^{-1}\frac{R-1}{2R}k^{-(n+a+2)}$, then

$$||\varphi_B||_{C^{a+2}(\widetilde{\omega}_0)} \leq \frac{R-1}{2R}. \tag{5.13}$$

Therefore,

$$||i\bar{\partial}\partial\varphi_B||_{C^0(\widetilde{\omega}_0)} \leq \frac{R-1}{2R},$$

which implies that

$$i\bar{\partial}\partial\varphi_B \geq -\frac{R-1}{2R}\widetilde{\omega_0}. \tag{5.14}$$

In order to show that $\widetilde{\omega_B}$ is R-bounded, we need to prove the following:

$$||\widetilde{\omega} - \widetilde{\omega}_0||_{C^a(\widetilde{\omega}_0)} \leq R, \tag{5.15}$$

$$\widetilde{\omega_B} \geq \frac{1}{R}\widetilde{\omega}_0. \tag{5.16}$$

To prove (5.15), (5.12) and (5.13) imply that for $k \gg 0$

$$\begin{aligned} ||\widetilde{\omega}_B - \widetilde{\omega}_0||_{C^a(\widetilde{\omega}_0)} &\leq ||\widetilde{\omega}_B - \widetilde{\omega}_k||_{C^a(\widetilde{\omega}_0)} + ||\widetilde{\omega}_k - \widetilde{\omega}_0||_{C^a(\widetilde{\omega}_0)} \\ &\leq ||\varphi_B||_{C^{a+2}(\widetilde{\omega}_0)} + k^{-1} \leq \frac{R-1}{2R} + k^{-1} \\ &\leq R. \end{aligned}$$

To prove (5.16), applying Lemma 6.1.4 with $\epsilon = \frac{R-1}{2R}$ gives

$$\widetilde{\omega_k} \geq \frac{R+1}{2R}\widetilde{\omega_0},$$

and therefore (5.14) implies

$$\widetilde{\omega_B} - \frac{1}{R}\widetilde{\omega}_0 = \widetilde{\omega_k} + i\bar{\partial}\partial\varphi_B - \frac{1}{R}\widetilde{\omega}_0 \geq \widetilde{\omega_k} - \frac{R+1}{2R}\widetilde{\omega_0} \geq 0,$$

for $k \gg 0$.

In order to prove the second part, by a unitary change of basis, we may assume without loss of generality that the matrix M^B is diagonal. By definition

$$M_{ij}^B = e^{\lambda_i+\lambda_j}\int_Y F\langle s_i, s_j\rangle \frac{\widetilde{\omega}_B^n}{n!} - \frac{V_k}{N_k}\delta_{ij},$$

where

$$F = e^{-\varphi_B}\frac{\widetilde{\omega_B}^n}{\widetilde{\omega_k}^n}.$$

We have

$$\begin{aligned}
M_{ii}^B &= e^{2\lambda_i}\int_Y F|s_i|_{h_k}^2 \frac{\widetilde{\omega}_k^n}{n!} - \frac{V_k}{N_k}\delta_{ij} \\
&= e^{2\lambda_i}\int_Y F|s_i|_{h_k}^2 \frac{\widetilde{\omega}_k^n}{n!} - \int_Y |s_i|_{h_k}^2 \frac{\widetilde{\omega}_k^n}{n!} + (M^{(k)})_{ii} \\
&= \int_Y (e^{2\lambda_i}F - 1)|s_i|_{h_k}^2 \frac{\widetilde{\omega}_k^n}{n!} + (M^{(k)})_{ii}.
\end{aligned}$$

Therefore,

$$|M_{ii}^B| \leq ||e^{2\lambda_i}F - 1||_\infty (\int_Y |s_i|_{h_k}^2 \frac{\widetilde{\omega}_k^n}{n!}) + |(M^{(k)})_{ii}| \leq C(||e^{2\lambda_i}F - 1||_\infty + k^{-q-1}).$$

Define $f = \frac{\widetilde{\omega_B}^n}{\widetilde{\omega_k}^n}$. If $||B||_{\mathrm{op}} \leq k^{-(n+a+3)}$, then

$$|f-1| = \Big|\frac{\widetilde{\omega_B}^n - \widetilde{\omega_k}^n}{\widetilde{\omega_k}^n}\Big| = O(k^{-1})$$

and

$$|(e^{2\lambda_i-\varphi_B} - 1)| = O(k^{-1}).$$

Therefore,

$$\begin{aligned}
||e^{2\lambda_i}F - 1|| = ||e^{2\lambda_i-\varphi_B}\frac{\widetilde{\omega_B}^n}{\widetilde{\omega_k}^n} - 1|| &= ||e^{2\lambda_i-\varphi_B}f - 1|| \\
&\leq ||(e^{2\lambda_i-\varphi_B} - 1)(f-1)|| + ||(f-1)|| \\
&\quad + ||(e^{2\lambda_i-\varphi_B} - 1)|| \\
&= O(k^{-1}),
\end{aligned}$$

which implies that

$$||M^B||_{\mathrm{op}} = O(k^{-1}).$$

□

Theorem 5.3.7. *Suppose that the sequence of metrics h_k on $\mathcal{O}(1) \otimes L^k$ and bases $\underline{s}^k = (s_1^k, ..., s_N^k)$ for $H^0(Y, \mathcal{O}(1) \otimes L^k)$ is almost balanced of order q. Suppose that* (5.12) *holds for*

$$\widetilde{\omega}_k = Ric(h_k) \quad \text{and } \omega_k = Ric(h_k) - k\omega_\infty.$$

If $q > \frac{5m}{2} + n + a + 5$, then $(Y, \mathcal{O}(1) \otimes L^k)$ admits balanced metric for $k \gg 0$.

Proof. Let $R > 1$ and k be a fixed large positive integer. Let $\sigma \in isu(N)$, where $N = N_k = \dim H^0(Y, \mathcal{O}(1) \otimes L^k)$. If $||\sigma||_{\text{op}} \leq \frac{c}{2} k^{-(n+a+3)} R$, then Theorem 5.3.6 implies that $e^\sigma \underline{s}$ has R-bounded geometry and $||M^\sigma||_{\text{op}} \leq \epsilon$ for $k \gg 0$, where ϵ is the constant in the statement of Theorem 5.2.4. Thus, Theorem 5.2.4 implies that $\Lambda(e^\sigma \underline{s}^{(k)}) \leq Ck^{2m+2} = \lambda$. With the notation of Proposition 5.1.3, we have $\mu(z_0) = M^{(k)}$. Therefore

$$|\mu(z_0)| = |M^{(k)}| \leq \sqrt{N_k} ||M^{(k)}||_{\text{op}} \leq C' k^{\frac{m}{2} - q}.$$

Letting $\delta = \frac{c}{2} k^{-(n+a+3)} R$, we have $\lambda |\mu(z_0)| < \delta$ if $q > \frac{5m}{2} + n + a + 5$ and $k \gg 0$. Therefore if $q > \frac{5m}{2} + n + a + 5$ and $k \gg 0$, we can apply Proposition 5.1.3 to get balanced metrics for $k \gg 0$.

$\square$

5.4 Asymptotic Expansion

The goal of this section is to prove Theorem 1.3.2. Theorem 1.3.2 gives an asymptotic expansion for the Bergman kernel of $(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L^k)$. We obtain such an expansion by using the Bergman kernel asymptotic expansion proved in ([C], [Z]). Also we compute the first nontrivial coefficient of the expansion. In the next section, we use this to construct sequence of almost balanced metrics. We start with some linear algebra.

Let (X, ω) be a Kähler manifold of dimension m and E be a holomorphic vector bundle on X of rank r. Let L be an ample line bundle on X endowed with a Hermitian metric σ such that $Ric(\sigma) = \omega$. For any hermitian metric h on E, we define the volume form

$$d\mu_g = \frac{\omega_g^{r-1}}{(r-1)!} \wedge \frac{\pi^*\omega^m}{m!},$$

where $g = \widehat{h}$,$\omega_g = Ric(g) = Ric(\widehat{h})$ and $\pi : \mathbb{P}E^* \to X$ is the projection map. The goal is to find an asymptotic expansion for the Bergman kernel of $\mathcal{O}_{\mathbb{P}E^*}(1) \otimes L^k \to \mathbb{P}E^*$ with respect to the L^2-metric defined on $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L^k)$. We define the L^2-metric using the fibre metric $g \otimes \sigma^{\otimes k}$ and the volume form $d\mu_{g,k}$ defined as follows

$$d\mu_{g,k} = k^{-m} \frac{(\omega_g + k\omega)^{m+r-1}}{(m+r-1)!} = \sum_{j=0}^{m} k^{j-m} \frac{\omega_g^{m+r-1-j}}{(m+r-j)!} \wedge \frac{\omega^j}{j!}. \tag{5.17}$$

In order to do that, we reduce the problem to the problem of Bergman kernel asymptotics on $E \otimes L^k \to X$. The first step is to use the volume form $d\mu_g$ which is a product volume form instead of the more complicated one $d\mu_{g,k}$. So, we replace the volume form $d\mu_{g,k}$ with $d\mu_g$ and the fibre metric $g \otimes \sigma^k$ with $g(k) \otimes \sigma^k$, where the metrics $g(k)$ are defined on $\mathcal{O}_{\mathbb{P}E^*}(1)$ by

$$g(k) = k^{-m} (\sum_{j=0}^{m} k^j f_j) g = (f_m + k^{-1} f_{m-1} + ... + k^{-m} f_0) g, \tag{5.18}$$

and

$$\frac{\omega_g^{m+r-1-j}}{(m+r-j)!} \wedge \frac{\omega^j}{j!} = f_j d\mu_g. \tag{5.19}$$

Clearly the L^2-inner products $L^2(g \otimes \sigma^k, d\mu_{g,k})$ and $L^2(g(k) \otimes \sigma^k, d\mu_g)$ on $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L^k)$ are the same. The second step is going from $\mathcal{O}_{\mathbb{P}E^*}(1) \to \mathbb{P}E^*$ to $E \to X$. In order to do this we somehow push forward the metric $g(k)$ to get a metric $\widetilde{g}(k)$ on E (See Definition 5.4.4). Then we can apply the result on the asymptotics of the Bergman kernel on E. The last step is to use this to get the result.

Definition 5.4.1. *Let $\widehat{s_1^k}, \ldots., \widehat{s_N^k}$ be an orthonormal basis for $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L^k)$ w.r.t. $L^2(g \otimes \sigma^k, d\mu_{k,g})$. We define*

$$\rho_k(g, \omega) = \sum_{i=1}^{N} |\widehat{s_i^k}|^2_{g \otimes \sigma^k}. \tag{5.20}$$

Definition 5.4.2. *For any (j,j)-form α on X, we define the contraction $\Lambda^j_\omega \alpha$ of α with respect to the Kähler form ω by*

$$\frac{m!}{(m-j)!} \alpha \wedge \omega^{m-j} = (\Lambda^j_\omega \alpha)\ \omega^m.$$

In this section we fix the Kähler form ω on X and therefore simply denote $\Lambda^j_\omega \alpha$ by $\Lambda^j \alpha$.

Lemma 5.4.3. *Let ν_0 be a fixed Kähler form on X. For any positive integer p there exists a constant C such that for any (j,j)-form γ, we have*

$$||\nabla^p(\Lambda^j \gamma)|| \leq \frac{C}{\inf_{x \in X} |\omega(x)^m|_{\nu_0(x)}} (||\gamma||_{C^p(\nu_0)} + ||\Lambda^j \gamma||_{C^{p-1}(\nu_0)})(\sum_{i=1}^{m} ||\omega||^i_{C^p(\nu_0)}).$$

Proof. Let γ be a (j,j)-form. By definition, we have

$$(\Lambda^j \gamma)\ \omega^m = \frac{m!}{(m-j)!} \gamma \wedge \omega^{m-j}.$$

Therefore for any positive integer p, we have

$$\nabla^p((\Lambda^j \gamma)\ \omega^m) = \frac{m!}{(m-j)!} \nabla^p(\gamma \wedge \omega^{m-j}).$$

Applying Leibnitz rule, we get

$$\sum_{i=0}^{p} \binom{p}{i} \nabla^i(\Lambda^j \gamma) \nabla^{p-i} \omega^m = \frac{m!}{(m-j)!} \sum_{i=0}^{p} \binom{p}{i} \nabla^i \gamma \wedge \nabla^{p-i} \omega^{m-i}.$$

Thus there exists a positive constant C' so that

$$||\nabla^p(\Lambda^j \gamma) \omega^m||_{C^0(\nu_0)} \leq C'(||\omega^m||_{C^p(\nu_0)} ||\Lambda^j \gamma||_{C^{p-1}(\nu_0)} + ||\gamma||_{C^p(\nu_0)} ||\omega^{m-j}||_{C^p(\nu_0)}).$$

On the other hand there exists constant $c_{p,j}$ such that for any any $0 \leq j \leq m-1$,

$$||\omega^{m-j}||_{C^p(\nu_0)} \leq c_{p,j}||\omega||^{m-j}_{C^p(\nu_0)} \leq c_{p,j}(\sum_{i=1}^{m}||\omega||^i_{C^p(\nu_0)}).$$

Hence there exists a constant C such that

$$||\nabla^p(\Lambda^j\gamma)|| \leq \frac{C}{\inf_{x\in X}|\omega(x)^m|_{\nu_0(x)}}(||\gamma||_{C^p(\nu_0)} + ||\Lambda^j\gamma||_{C^{p-1}(\nu_0)})(\sum_{i=1}^{m}||\omega||^i_{C^p(\nu_0)}).$$

$\square$

Definition 5.4.4. *For any hermitian form g on $\mathcal{O}_{\mathbb{P}E^*}(1)$, we define a hermitian form $\widetilde{g}$ on E as follow*

$$\widetilde{g}(s,t) = C_r^{-1}\int_{\mathbb{P}E_x^*} g(\widehat{s},\widehat{t})\frac{\omega_g^{r-1}}{(r-1)!}, \tag{5.21}$$

for $s,t \in E_x$. (See (2.7) *for definition of C_r.)*

Notice that if $g = \widehat{h}$ for some hermitian metric h on E, Lemma 2.5.2 implies that $\widetilde{g} = h$. Define hermitian metrics $\widetilde{g_j}$'s on E by

$$\widetilde{g_j}(s,t) = C_r^{-1}\int_{\mathbb{P}E_x^*} f_j g(\widehat{s},\widehat{t})\frac{\omega_g^{r-1}}{(r-1)!}, \tag{5.22}$$

for $s,t \in E_x$. Also we define $\Psi_j \in End(E)$ by

$$\widetilde{g_j} = \Psi_j h. \tag{5.23}$$

Proposition 5.4.5. *Let ν_0 be a fixed Kähler form on X as in Lemma 5.4.3. For any positive numbers l and l' and any positive integer p, there exists a positive number $C_{l,l',p}$ such that if*

$$||\omega||_{C^p(\nu_0)}, ||h||_{C^{p+2}(\nu_0)} \leq l$$

and

$$\inf_{x\in X}|\omega(x)^m|_{\nu_0(x)} \geq l',$$

then

$$||\Psi_i||_{C^p(\nu_0)} \leq C_{l,l',p} \textit{ for any } 1 \leq i \leq m.$$

Proof. Fix a point $p \in X$. Let $e_1, ..., e_r$ be a local holomorphic frame for E around p such that

$$\langle e_i, e_j \rangle_h(p) = \delta_{ij}, \qquad d\langle e_i, e_j \rangle_h(p) = 0$$

and

$$\frac{i}{2\pi} F_h(p) = \begin{pmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_r \end{pmatrix}.$$

Let $\lambda_1, ..., \lambda_r$ be the homogeneous coordinates on the fibre. At the fixed point p, we have

$$\omega_g = \omega_{\mathrm{FS},g} + \frac{\sum \omega_i |\lambda_i|^2}{\sum |\lambda_i|^2}.$$

Therefore,

$$\omega_g^{r+j-1} \wedge \omega^{m-j} = \binom{r+j-1}{r-1} \omega_{\mathrm{FS},g}^{r-1} \wedge \left(\frac{\sum \omega_i |\lambda_i|^2}{\sum |\lambda_i|^2}\right)^j \wedge \omega^{m-j}.$$

Definition of f_{m-j} gives

$$f_{m-j} \omega_g^{r-1} \wedge \omega^m = \binom{m}{j} \omega_g^{r-1} \wedge \Big(\left(\frac{\sum \omega_i |\lambda_i|^2}{\sum |\lambda_i|^2}\right)^j \wedge \omega^{m-j} \Big)$$

Hence

$$f_{m-j} \omega_{\mathrm{FS},g}^{r-1} \wedge \omega^m = \binom{m}{j} \omega_{\mathrm{FS},g}^{r-1} \wedge \Big(\left(\frac{\sum \omega_i |\lambda_i|^2}{\sum |\lambda_i|^2}\right)^j \wedge \omega^{m-j} \Big).$$

Therefore,

$$\omega_{\mathrm{FS},g}^{r-1} \wedge \Big(f_{m-j} \omega^m - \binom{m}{j} \left(\frac{\sum \omega_i |\lambda_i|^2}{\sum |\lambda_i|^2}\right)^j \wedge \omega^{m-j} \Big) = 0,$$

which implies

$$
\begin{aligned}
f_{m-j}\omega^m &= \binom{m}{j}\Big(\frac{\sum \omega_i |\lambda_i|^2}{\sum |\lambda_i|^2}\Big)^j \wedge \omega^{m-j} \\
&= \binom{m}{j}\frac{\sum_{j_1+\cdots+j_r=j}\binom{j}{j_1,\dots,j_r}\omega_1^{j_1}\wedge\cdots\wedge\omega_r^{j_r}|\lambda_1|^{2j_1}\dots|\lambda_r|^{2j_r}}{(\sum|\lambda_i|^2)^j}\wedge\omega^{m-j}.
\end{aligned}
$$

Simple calculation gives

$$
\int_{\mathbb{C}^{r-1}} \frac{|\lambda_\alpha|^2|\lambda_1|^{2j_1}\dots|\lambda_{r-1}|^{2j_{r-1}}d\lambda\wedge d\overline{\lambda}}{(1+\sum_{j=1}^{r-1}|\lambda_j|^2)^{r+j+1}} = \frac{C_r r! j_1!\dots j_r!(j_\alpha+1)}{(r+j)!},
$$

when $j_1+\cdots+j_r=j$ and $1\leq\alpha\leq r$. Hence

$$
\begin{aligned}
\widetilde{g}_{m-j}(e_\alpha,e_\alpha) &= C_r^{-1}\pi_*\big(f_{m-j}g(\widehat{e_\alpha},\widehat{e_\alpha})\frac{\omega_g^{r-1}}{(r-1)!}\big) \\
&= \frac{r!}{(r+j)!}\Lambda^j\big(\sum_{j_1+\cdots+j_r=j}(j_\alpha+1)\omega_1^{j_1}\wedge\cdots\wedge\omega_r^{j_r}\big).
\end{aligned} \tag{5.24}
$$

From the theory of symmetric functions, one can see that there exist polynomials $P_i(x_1,\dots,x_j)$ of degree i such that

$$
\Psi_{m-j} = \Lambda^j\Big(F_h^j + P_1(c_1(h),\dots,c_j(h))F_h^{j-1}+\cdots+P_j(c_1(h),\dots,c_j(h))\Big),
$$

where $c_i(h)$ is the i th chern form of h. Since $||h||_{C^{p+2}(\nu_0)}\leq l$, there exists a positive constant c' such that

$$
||F_h^j+\cdots+P_j(c_1(h),\dots,c_j(h))||_{C^p(\nu_0)}\leq c'(1+l)^j.
$$

Therefore Lemma 5.4.3 implies that

$$
||\nabla^p\Psi_{m-j}|| \leq \frac{C}{l'}(c'(1+l)^j+||\Psi_{m-j}||_{C^{p-1}(\nu_0)})(1+l)^m,
$$

since

$$
\inf_{x\in X}|\omega(x)^m|_{\nu_0(x)}\geq l'
$$

and

$$
\sum_{i=1}^m ||\omega||^i_{C^p(\nu_0)} \leq \sum_{i=1}^m l^i \leq (1+l)^m.
$$

On the other hand

$$\begin{aligned}||\Psi_{m-i}||_{C^p(\nu_0)} &= ||\nabla^p\Psi_{m-j}|| + ||\Psi_{m-i}||_{C^{p-1}(\nu_0)} \\ &\leq \frac{C}{l'}(c'(1+l)^j + ||\Psi_{m-j}||_{C^{p-1}(\nu_0)})(1+l)^m + ||\Psi_{m-i}||_{C^{p-1}(\nu_0)}.\end{aligned}$$

Now we can conclude the proof by induction on p.

□

Lemma 5.4.6. *We have the following*

1. $\Psi_m = I_E$.

2. $\Psi_{m-1} = \dfrac{i}{2\pi(r+1)}(Tr(\Lambda F_h)I_E + \Lambda F_h)$.

Proof. The first part is an immediate consequence of Lemma 2.5.2 and the definition of Ψ_m. For the second part, we use the notation used in the proof of Proposition 5.4.5. It is easy to see that for $\alpha \neq \beta$, we get $\widetilde{g_{m-1}}(e_\alpha, e_\beta) = 0$. On the other hand by plugging $j = 1$ in (5.24), we get

$$\widetilde{g_{m-1}}(e_\alpha, e_\alpha) = \frac{1}{(r+1)}(Tr(\Lambda F) + \Lambda\omega_\alpha).$$

□

The following lemmas are straightforward.

Lemma 5.4.7. $\widetilde{g \otimes \sigma^k} = \widetilde{g} \otimes \sigma^k$.

Lemma 5.4.8. *Let $s_1, ..., s_N$ be a basis for $H^0(X, E)$. Then*

$$\sum |\widehat{s_i}([v^*])|^2_{\widehat{h}} = Tr\big(B\lambda(v^*, h)\big),$$

where $B = \sum s_i \otimes s_i^{*h}$.

Proof of Theorem 1.3.2. We define the metric $h(k)$ on E by

$$h(k) = \sum_{j=0}^{m} k^{j-m}\widetilde{g}_j = (\sum_{j=0}^{m} k^{j-m}\Psi_j)h. \tag{5.25}$$

Let $B_k(h(k),\omega)$ be the Bergman kernel of $E \otimes L^k$ with respect to the L^2-metric defined by the hermitian metric $h(k) \otimes \sigma^k$ on $E \otimes L^k$ and the volume form $\frac{\omega^m}{m!}$ on X. Therefore, if $s_1, ..., s_N$ is an orthonormal basis for $H^0(X, E \otimes L^k)$ with respect to the $L^2(H(k) \otimes \sigma^k, \frac{\omega^m}{m!})$, then

$$B_k(h(k),\omega) = \sum s_i \otimes s_i^{*_{h(k)\otimes\sigma^k}}, \tag{5.26}$$

We define $\widetilde{B}_k(h,\omega)$ as follow

$$\widetilde{B}_k(h,\omega) = \sum s_i \otimes s_i^{*_{h\otimes\sigma^k}}. \tag{5.27}$$

Let $\widehat{s_1}, ..., \widehat{s_N}$ be the corresponding basis for $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes L^k)$. Hence,

$$\int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j} \rangle_{g\otimes\sigma^k} d\mu_{g,k} = \int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j} \rangle_{g\otimes\sigma^k} (\sum_{j=0}^{m} k^j f_j) d\mu_g$$
$$= \int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j} \rangle_{g(k)\otimes\sigma^k} d\mu_g = C_r \int_X \langle s_i, s_j \rangle_{h(k)\otimes\sigma^k} \frac{\omega^m}{m!} = C_r \delta_{ij}.$$

Therefore $\frac{1}{\sqrt{C_r}}\widehat{s_1}, ..., \frac{1}{\sqrt{C_r}}\widehat{s_N}$ is an orthonormal basis for $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes L^k)$ with respect to $L^2(g \otimes \sigma^k, d\mu_{k,g})$. Hence Lemma 5.4.8 implies

$$C_r \rho_k(g) = Tr\big(\lambda(v^*, h)\widetilde{B}_k(h,\omega)\big).$$

Now, in order to conclude the proof, it suffices to show that there exist smooth endomorphisms $A_i \in \Gamma(X, E)$ such that

$$\widetilde{B}_k(h,\omega) \sim k^m + A_1 k^{m-1} + ...$$

Let $B_k(h,\omega)$ be the Bergman kernel of $E \otimes L^k$ with respect to the $L^2(h \otimes \sigma^k)$. A fundamental result on the asymptotics of the Bergman kernel ([C], [Z]) states that there exists an asymptotic expansion

$$B_k(h,\omega) \sim k^m + B_1(h)k^{m-1} + ...,$$

where

$$B_1(h) = \frac{i}{2\pi}\Lambda F_{(E,h)} + \frac{1}{2}S(\omega)I_E.$$

(See also [BBS],[W2].) Moreover this expansion holds uniformly for any h in a bounded family. Therefore, we can Taylor expand the coefficients $B_i(h)$'s. We conclude that for endomorphisms $\Phi_1, ..., \Phi_M$,

$$B_k(h(I + \sum_{i=0}^{M} k^{-i}\Phi_i), \omega) \sim k^m + B_1(h)k^{m-1+...}$$

Note that $B_1(h)$ in the above expansion does not depend on Φ_i's and is given as before by

$$B_1(h) = \frac{i}{2\pi}\Lambda F_{(E,h)} + \frac{1}{2}S(\omega)I_E.$$

On the other hand

$$B_k(h(k), \omega) = \sum s_i \otimes s_i^{*_{\widetilde{g(k)}\otimes\sigma^k}} = (\sum s_i \otimes s_i^{*_{h\otimes\sigma^k}})(\sum_{j=0}^{m} k^{j-m}\Psi_j)$$

$$= \widetilde{B}_k(h, \omega)(\sum_{j=0}^{m} k^{j-m}\Psi_j).$$

Therefore,

$$\widetilde{B}_k(h, \omega) = B_k(h(k), \omega)(\sum_{j=0}^{m} k^{j-m}\Psi_j)^{-1} \sim k^m + (B_1(h) - \Psi_{m-1})k^{m-1} + ...$$

Notice that Proposition 5.4.5 implies that if h and ω vary in a bounded family and ω is bounded from below, then $\Psi_1, .., \Psi_m$ vary in a bounded family. Therefore the asymptotic expansion that we obtained for $\widetilde{B}_k(h, \omega)$ is uniform as long as h and ω vary in a bounded family and ω is bounded from below.

□

Proposition 5.4.9. *Suppose that $\omega_\infty \in 2\pi c_1(L)$ be a Kähler form with constant scalar curvature and h_{HE} be a Hermitian-Einstein metric on E, i.e.*

$$\Lambda_{\omega_\infty} F_{(E,h_{HE})} = \mu I_E,$$

where μ is the ω_∞−slope of the bundle E. We have

$$\begin{aligned} A_{1,1} &:= \frac{d}{dt}\Big|_{t=0} A_1(h_{HE}(I+t\phi), \omega_\infty + it\overline{\partial}\partial\eta) \\ &= \frac{r+1}{2r}\mathcal{D}^*\mathcal{D}\eta I_E + \frac{i}{2\pi}\big((\Lambda_{\omega_\infty}\overline{\partial}\partial\Phi + \Lambda^2_{\omega_\infty}(F_{h_{HE}} \wedge (i\overline{\partial}\partial\eta))) \\ &\quad - \frac{1}{r}tr(\Lambda_{\omega_\infty}\overline{\partial}\partial\Phi) + \Lambda^2_{\omega_\infty}(F_{h_{HE}} \wedge (i\overline{\partial}\partial\eta))\big), \end{aligned}$$

where $\mathcal{D}^*\mathcal{D}$ is Lichnerowicz operator (cf. [D3, Page 515]).

Proof. Define

$$f(t) = \Lambda_{\omega_\infty + it\overline{\partial}\partial\eta} F_{(h_{\mathrm{HE}}(I+t\phi))}$$

Therefore, we have

$$mF_{(h_{\mathrm{HE}}(I+t\phi))} \wedge (\omega_\infty + it\overline{\partial}\partial\eta)^{m-1} = f(t)(\omega_\infty + it\overline{\partial}\partial\eta)^m.$$

Differentiating with respect to t at $t = 0$, we obtain

$$m\overline{\partial}\partial\Phi \wedge \omega_\infty^{m-1} + m(m-1)F_{h_{\mathrm{HE}}} \wedge (i\overline{\partial}\partial\eta) \wedge \omega_\infty^{m-2} = f'(0)\omega_\infty^m + mf(0)(i\overline{\partial}\partial\eta) \wedge \omega_\infty^{m-1}.$$

Since $f(0) = \mu I_E$, we get

$$f'(0) = \Lambda_{\omega_\infty}\overline{\partial}\partial\Phi + \Lambda^2_{\omega_\infty}(F_{h_{\mathrm{HE}}} \wedge (i\overline{\partial}\partial\eta)) - \mu\Lambda_{\omega_\infty}(i\overline{\partial}\partial\eta)I_E.$$

On the other hand (cf. [D3, pp. 515, 516].)

$$\frac{d}{dt}\Big|_{t=0} S(\omega_\infty + it\overline{\partial}\partial\eta) = \mathcal{D}^*\mathcal{D}\eta.$$

□

Proof. First, notice that $\Gamma_0(End(E)) = \Gamma_{00}(End(E)) \bigoplus C_0^\infty(X)$, where $\Gamma_{00}(End(E))$ is the space of trace-free endomorphisms of E and $C_0^\infty(X)$ is the space of smooth functions η on X such that $\int_X \eta\omega_\infty^m = 0$. Assume that $A_{1,1}(\Phi,\eta) = 0$, where $\Phi \in \Gamma_{00}(End(E))$ and $\eta \in C_0^\infty(X)$. Hence

$$\frac{r+1}{2r}\mathcal{D}^*\mathcal{D}\eta = 0, \quad \text{and}$$
$$\frac{i}{2\pi}\big((\Lambda_{\omega_\infty}\overline{\partial}\partial\Phi + \Lambda^2_{\omega_\infty}(F_{h_{\mathrm{HE}}} \wedge (i\overline{\partial}\partial\eta))) - \frac{1}{r}tr(\Lambda_{\omega_\infty}\overline{\partial}\partial\Phi) + \Lambda^2_{\omega_\infty}(F_{h_{\mathrm{HE}}} \wedge (i\overline{\partial}\partial\eta))\big) = 0$$

Since $Aut(X,L)/\mathbb{C}^*$ is discrete, the first equation implies that η is constant and therefore $\eta = 0$. So, the second equation reduces to the following

$$\Lambda_{\omega_\infty}\overline{\partial}\partial\Phi - \frac{1}{r}tr(\Lambda_{\omega_\infty}\overline{\partial}\partial\Phi) = 0$$

It implies that

$$\Lambda_{\omega_\infty}\overline{\partial}\partial\Phi = 0,$$

since Φ is traceless. Hence simplicity of E implies that $\Phi = 0$ (cf. [K]).

In order to prove surjectivity let $\Psi \in \Gamma_0(End(E))$. We know that the map

$$\eta \in C_0^\infty \to \mathcal{D}^*\mathcal{D}\eta \in C_0^\infty$$

is surjective since $Aut(X,L)/\mathbb{C}^*$ is discrete (cf. [D3, pp. 515, 516]). Hence we can find η_0 such that $\mathcal{D}^*\mathcal{D}\eta_0 = tr(\Psi)$. On the other hand

$$\frac{i}{2\pi}\big(\Lambda^2_{\omega_\infty}(F_{h_\infty} \wedge (i\overline{\partial}\partial\eta_0)) - \frac{1}{r}tr(\Lambda^2_{\omega_\infty}(F_{h_\infty} \wedge (i\overline{\partial}\partial\eta_0))\big) + \Psi - \frac{1}{r}tr(\Psi) \in \Gamma_0(End(E)).$$

The map

$$\Phi \in \Gamma_0(End(E)) \to \frac{i}{2\pi}\Lambda_{\omega_\infty}\overline{\partial}\partial\Phi \in \Gamma_0(End(E))$$

is surjective since E is simple. Hence, we can find ϕ_0 such that $A_{1,1}(\phi_0,\eta_0) = \Psi$.

□

5.5 Constructing almost balanced metrics

Let h_∞ be a hermitian metric on L such that $\omega_\infty = Ric(h_\infty)$ be a Kähler form with constant scalar curvature and h_{HE} be the corresponding Hermitian-Einstein metric on E, i.e.

$$\Lambda_{\omega_\infty} F_{(E,h_{\mathrm{HE}})} = \mu I_E,$$

where μ is the slope of the bundle E. Let $\omega_0 = Ric(\widehat{h_{\mathrm{HE}}})$. After tensoring by high power of L, we can assume without loss of generality that ω_0 is a Kähler form on $\mathbb{P}E^*$. We fix an integer $a \geq 4$. In order to prove the following, we use ideas introduced by Donaldson in ([D3, Theorem 26])

Theorem 5.5.1. *Suppose $Aut(X,L)$ is discrete. There exist smooth functions $\eta_1, \eta_2, \ldots$ on X and smooth endomorphisms $\Phi_1, \Phi_2, \ldots$ of E such that for any positive integer q if*

$$\nu_{k,q} = \omega_\infty + i\overline{\partial}\partial\Big(\sum_{j=1}^{q} k^{-j}\eta_j\Big)$$

and

$$h_{k,q} = h_{HE}\Big(I_E + \sum_{j=1}^{q} k^{-j}\Phi_j\Big),$$

then

$$\widetilde{B}_k(h_{k,q}, \nu_{k,q}) = \frac{C_r N_k}{k^{-m} V_k}(I_E + \delta_q), \tag{5.28}$$

where $||\delta_q||_{C^{a+2}} = O(k^{-q-1})$ and $V_k = Vol(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes L^k)$ is a topological invariant.

Proof. The error term in the asymptotic expansion is uniformly bounded in C^{a+2} for all h and ω in a bounded family. Therefore there exists a positive integer s depends only on p and q such that

$$A_p(h(1+\Phi), \omega + i\overline{\partial}\partial\eta) = A_p(h,\omega) + \sum_{j=1}^{q} A_{p,j}(\Phi,\eta) + O(||(\Phi,\eta)||_{C^s}^{q+1}), \tag{5.29}$$

where $A_{p,j}$ are homogeneous polynomials of degree j , depending on h and ω, in Φ and η and its covariant derivatives. Let $\Phi_1, ..., \Phi_q$ be smooth endomorphisms of E and $\eta_1, ..., \eta_q$ be smooth functions on X. We have

$$\begin{aligned} A_p(h(1+\sum_{j=1}^{q} k^{-j}\Phi_j), \omega + i\overline{\partial}\partial(\sum_{j=1}^{q} k^{-j}\eta_j)) \\ = A_p(h,\omega) + \sum_{j=1}^{q} b_{p,j}k^{-j} + O(k^{-q-1}), \end{aligned} \tag{5.30}$$

where $b_{p,j}$'s are multi linear expression on Φ_i's and η_i's.

Hence

$$\begin{aligned} &\widetilde{B}_k(h(1+\sum_{j=1}^{q} k^{-j}\Phi_j), \omega + i\overline{\partial}\partial(\sum_{j=1}^{q} k^{-j}\eta_j)) \\ &= k^m + A_1(h,\omega)k^{m-1} + ... \\ &\quad + (A_q(h,\omega) + b_{q-1,1} + ... + b_{1,q-1})k^{m-q} + O(k^{m-q-1}). \end{aligned} \tag{5.31}$$

We need to choose Φ_j and η_j such that coefficients of $k^m, ...k^{m-q}$ in the right hand side of (5.31) are constant. Donaldson's key observation is that η_p and ϕ_p only appear in the coefficient of k^{m-p} in the form of $A_{1,1}(\phi_p, \eta_p)$. Hence, we can do this inductively. Assume that we choose $\eta_1, \eta_2, ...\eta_{p-1}$ and $\Phi_1, \Phi_2, ..., \Phi_{p-1}$ so that the coefficients of $k^m, ...k^{m-p+1}$ are constant. Now we need to choose η_p and Φ_p such that the coefficient of k^{m-p} is constant. This means that we need to solve the equation

$$A_{1,1}(\Phi_p, \eta_p) - c_p I_E = P_{p-1}, \tag{5.32}$$

for Φ_p, η_p and the constant c_p. In this equation P_{p-1} is determined by $\Phi_1, ..., \Phi_{p-1}$ and $\eta_1, ..., \eta_{p-1}$. Corollary **??** implies that we can always solve the equation (5.32).

□

Proof. Let $g_{k,q} = \widehat{h_{k,q}}$. By Theorem 5.5.1, We have

$$\begin{aligned}\rho_k(g_{k,q}, \nu_{k,q}) &= \frac{N_k}{k^{-m}V_k} Tr(\lambda(v^*, h_{k,q})(I_E + \delta_q)) \\ &= \frac{N_k}{k^{-m}V_k}(1 + Tr(\lambda(v^*, h_{k,q})\delta_q))).\end{aligned}$$

The first part of corollary is proved, since $h_{k,q}$ is bounded and $||\delta_{k,q}||_{C^{a+2}} = O(k^{-q-1})$.

Define $\widetilde{\omega_0} = \omega_0 + k\omega_\infty$. For the second part, we have

$$\begin{aligned}||\omega_{g_{k,q}} + k\nu_{k,q} - (\omega_0 + k\omega_\infty)||_{C^a(\widetilde{\omega_0})} &\leq ||\omega_{g_{k,q}} - \omega_0||_{C^a(\widetilde{\omega_0})} + k||\nu_{k,q} - \omega_\infty||_{C^a(\widetilde{\omega_0})} \\ &\leq ||\omega_{g_{k,q}} - \omega_0||_{C^a(\omega_0)} + k||\nu_{k,q} - \omega_\infty||_{C^a(k\omega_\infty)} \\ &= ||\omega_{g_{k,q}} - \omega_0||_{C^a(\omega_0)} + ||\nu_{k,q} - \omega_\infty||_{C^a(\omega_\infty)} \\ &= O(k^{-1}).\end{aligned}$$

Notice that by definition, we have

$$\begin{aligned}||\omega_{g_{k,q}} - \omega_0||_{C^a(\omega_0)} &= O(k^{-1}), \\ ||\nu_{k,q} - \omega_\infty||_{C^a(\omega_\infty)} &= O(k^{-1}).\end{aligned}$$

□

5.6 Proof of Theorem 1.3.1

In this section, we prove Theorem 1.3.2. In order to do that, we want to apply Theorem 5.3.7. Hence, we need to construct a sequence of almost balanced metrics on $\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes L^{\otimes k}$. Also, we need to show that $\mathbb{P}E^*$ has no nontrivial holomorphic vector fields.

Proposition 5.6.1. *Let E be a holomorphic vector bundle over a compact Kähler manifold X. Suppose that X has no nonzero holomorphic vector fields. If E is stable, then $\mathbb{P}E^*$ has no nontrivial holomorphic vector fields.*

Proof. Let TF be the sheaf of tangent vectors to the fibre of π. We have the following exact sequence over $\mathbb{P}E$:

$$0 \to TF \to T\mathbb{P}E^* \to \pi^* TX \to 0.$$

This gives the long exact sequence

$$0 \to H^0(\mathbb{P}E^*, TF) \to H^0(\mathbb{P}E^*, T\mathbb{P}E^*) \to H^0(\mathbb{P}E^*, \pi^* TX) \to \dots$$

Since $H^0(\mathbb{P}E^*, \pi^* TX) = 0$, we have

$$H^0(\mathbb{P}E^*, TF) \simeq H^0(\mathbb{P}E^*, T\mathbb{P}E^*)$$

On the other hand, $\pi_* TF$ may be identified with the sheaf of trace free endomorphisms of E. Therefore by simplicity of E (cf. [K])

$$H^0(\mathbb{P}E^*, TF) \simeq H^0(X, \pi_* TF) = 0.$$

$\square$

Proof of Theorem 1.3.1. Since Chow stability is equivalent to the existence of balanced metric, it suffices to show that $(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L^k)$ admits balanced metric for $k \gg 0$. Fix a positive integer q. From now on we drop all indexes q for simplicity. Let $\sigma_k = \sigma_{k,q}$ be a metric on L such that $Ric(\sigma_k) = \nu_k$, where $\nu_k = \nu_{k,q}$ is the one in the statement of Theorem 5.1.2. Let $t_1, ..., t_N$ be an orthonormal basis for $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes L^k)$ w.r.t. $L^2(g_k \otimes \sigma_k^{\otimes k}, \frac{(\omega_{g_k} + k\nu_k)^{m+r-1}}{(m+r-1)!})$. Thus, Corollary **??** implies

$$\sum |t_i|^2_{g_k \otimes \sigma_k^{\otimes k}} = \frac{N_k}{V_k}(1 + \epsilon_k).$$

Define $g_k' = \frac{V_k}{N_k}(1 + \epsilon_k)^{-1} g_k$. We have

$$\sum |t_i|^2_{g_k' \otimes \sigma_k^{\otimes k}} = 1.$$

This implies that the metric $g_k^{'}$ is the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}E^*}(1)\otimes L^k$ induced by the embedding $\iota_{\underline{t}} : \mathbb{P}E^* \to \mathbb{P}^{N-1}$, where $\underline{t} = (t_1, ..., t_N)$. We prove that this sequence of embedding is almost balanced of order q, i.e

$$\int_{\mathbb{P}E^*} \langle t_i, t_j\rangle_{g_k^{'}\otimes\sigma_k^{\otimes k}} \frac{(\omega_{g_k^{'}} + k\nu_k)^{m+r-1}}{(m+r-1)!} = D^{(k)}\delta_{ij} + M_{ij},$$

where $M^{(k)} = [M_{ij}]$ is a trace free hermitian matrix, $D^{(k)} \to C_r$ as $k \to \infty$ and $||M^{(k)}||_{\mathrm{op}} = O(k^{-q-1})$.

$$\begin{aligned} M_{ij}^{(k)} &= \int_{\mathbb{P}E^*} \langle t_i, t_j\rangle_{g_k^{'}\otimes\sigma_k^{\otimes k}} \frac{(\omega_{g_k^{'}} + k\nu_k)^{m+r-1}}{(m+r-1)!} \\ &\quad - \frac{V_k}{N_k}\int_{\mathbb{P}E^*} \langle t_i, t_j\rangle_{g_k\otimes\sigma_k^{\otimes k}} \frac{(\omega_{g_k} + k\nu_k)^{m+r-1}}{(m+r-1)!} \\ &= \frac{V_k}{N_k}\int_{\mathbb{P}E^*} \langle t_i, t_j\rangle_{g_k\otimes\sigma_k^{\otimes k}} (f_k(1+\epsilon_k)^{-1} - 1)\frac{(\omega_{g_k} + k\nu_k)^{m+r-1}}{(m+r-1)!}, \end{aligned}$$

where

$$(\omega_{g_k^{'}} + k\nu_k)^{m+r-1} = f_k(\omega_{g_k} + k\nu_k)^{m+r-1}.$$

By a unitary change of basis, we may assume without loss of generality that the matrix $M^{(k)}$ is diagonal. Thus

$$||M^{(k)}||_{\mathrm{op}} \leq \frac{V_k}{N_k}||f_k(1+\epsilon_k)^{-1} - 1||_{L^\infty}.$$

On the other hand,

$$\begin{aligned} ||\omega_{g_k^{'}} - \omega_{g_k}||_{C^0(\omega_0)} &= ||\overline{\partial}\partial\log(1+\epsilon_k)||_{C^0(\omega_0)} \\ &\leq ||\log(1+\epsilon_k)||_{C^2(\omega_0)}. \\ &\leq -\log(1 - C||\epsilon_k||_{C^2(\omega_0)}) \\ &= O(k^{-q-1}). \end{aligned}$$

Therefore,

$$\begin{aligned}||f_k - 1||_\infty &= \Big|\frac{\omega_{g_k'}^{m+r-1} - \omega_{g_k}^{m+r-1}}{\omega_{g_k}^{m+r-1}}\Big| = \Big|\frac{\omega_{g_k'}^{m+r-1} - \omega_{g_k}^{m+r-1}}{\omega_0^{m+r-1}}\frac{\omega_0^{m+r-1}}{\omega_{g_k}^{m+r-1}}\Big| \\ &\leq Ck^{-q-1}\Big|\frac{\omega_0^{m+r-1}}{\omega_{g_k}^{m+r-1}}\Big|.\end{aligned}$$

This implies that

$$||f_k - 1||_\infty \leq Ck^{-q-1},$$

since $\Big|\frac{\omega_0^{m+r-1}}{\omega_{g_k}^{m+r-1}}\Big|$ is bounded. Hence

$$||f_k(1+\epsilon_k)^{-1} - 1|| \leq C^{'}k^{-q-1}.$$

Therefore

$$||M^{(k)}||_{\text{op}} = O(k^{-q-1}).$$

Proposition 5.6.1 implies that $\mathbb{P}E^*$ has no nontrivial holomorphic vector fields. Therefore, applying Theorem 5.3.7 and (**??**) conclude the proof.

$\square$

Chapter 6

One Dimensional Case

6.1 Almost balanced metrics

In this section, we start with a sequence of balanced metrics h^k on $E \otimes L^k$. Then we prove that the sequence of metrics $\widehat{h^k}$ on $\mathcal{O}_{\mathbb{P}E^*}(1)) \otimes L^k$ are balanced up to an error term which is exponentially small as k becomes large. This is the content of Theorem 6.1.6.

As before let (X, ω_0) be a compact Kähler manifold of dimension n and (L, g) be an ample holomorphic hermitian line bundle over X such that $Ric(g) = \omega_0$. Let E be a holomorphic vector bundle of rank r and degree d over X. We also assume that E is very ample.

We have the following theorem due to Catlin and Zelditch.([C], [Z])

Theorem 6.1.1. *If $s_1^k, ...s_{N_k}^k$ be an ONB for $H^0(X, E \otimes L^k)$ with respect to the inner product*

$$\langle s, t \rangle = \int_X \langle s(x), t(x) \rangle_{h \otimes g^{\otimes k}} \frac{\omega_0^n}{n!},$$

then we have the following complete asymptotic of the Bergman kernel

$$B_k(h) = \sum s_i^k \otimes (s_i^k)^* = k^n + A_1 k^{n-1} + ...$$

The relation between stability and the existence of balanced metrics comes from the following theorem of Wang.

Theorem 6.1.2. *([?],[W2, Theorem 1.2]) The bundle E is Gieseker stable if and only if there exists balanced metrics h^k on $E \otimes L^k$ for $k \gg 0$. In addition if there exists a Hermitian metric h_∞ on E such that $h_k \to h_\infty$, then*

$$\frac{i}{2\pi} \Lambda F_{(E,h_\infty)} + \frac{1}{2} S(\omega_\infty) I_E = \Big(\frac{d}{Vr} + \frac{\overline{s}}{2} \Big) I_E,$$

where $h_k = h^k \otimes g_\infty^{\otimes(-k)}$, $S(\omega_\infty)$ is the scalar curvature of ω_∞ and $\overline{s} = \frac{1}{V} \int_X S(\omega_\infty) \frac{\omega_\infty^n}{n!}$. Conversely, if h_∞ solves the above equation, then $h_k \to h_\infty$.

One can say more in the following special case

Theorem 6.1.3. *Assume X is a compact Riemann surface and ω_∞ is a Kähler form of constant curvature on X. Also assume that*

$$\frac{i}{2\pi} F_{(E,h_\infty)} = \omega_\infty I_E.$$

Then

$$||h_k - h_\infty||_{C^2(h_\infty)} = O(k^{-\infty}).$$

The proof is straight forward from following lemmas.

Lemma 6.1.4. *Assume that $A_1, ...A_q$ are constant and E is stable. If q is big enough, then there exists a sequence of balanced metrics h^k on $E \otimes L^k$ for $k \gg 0$ such that*

$$||h - h^k \otimes g^{\otimes(-k)}||_{C^a(h)} = O(k^{3+\frac{13n}{2}+\frac{a}{2}-q}).$$

Proof. First we claim that

$$B_k(h) = \frac{\chi(k)}{rV}(I_E + \sigma_k),$$

where $||\sigma_k||_{C^a} = O(k^{n-q-1})$.

In order to prove this, we observe that there exists a smooth section $A(x)$ of $End(E)$ such that

$$B_k(h) = k^n + A_1 k^{n-1} + ... + A_q k^{n-q} + A(x) k^{n-q-1}.$$

The bundle E is stable and A_j's are constant sections of $End(E)$. Therefore there exist numbers $a_1, ..., a_q$ such that $A_j = a_j I_E$. On the other hand

$$\int_X tr(B_k(h) \frac{\omega_\infty^n}{n!} = \chi(k)V,$$

where $V = \int_X \frac{\omega_\infty^n}{n!}$. Therefore

$$B_k(h) - \frac{\chi(k)}{rV} I_E = \Big(A(x) - \frac{1}{rV}\int_X A(x) I_E\Big) k^{n-q-1}.$$

Define $h(k) = h$, we have

$$B_k(h(k)) = \frac{\chi(k)}{rV}(I + \sigma_k),$$

where $||\sigma_k||_{C^a} = O(k^{n-q-1})$. Now Wang's argument ([W2, page 276]) implies that

$$\| \frac{1}{k}\omega_k - \omega_\infty \|_{C^a(\omega_\infty)} = O(k^{\frac{-n}{2}+r+1-q}).$$

□

Lemma 6.1.5. *In the situation of Theorem 6.1.3, all coefficients A_i's are constant.*

Using canonical isomorphism $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1)) = H^0(X, E)$, any hermitian metric h on $E \otimes L^k$ gives a Hermitian metric $\widehat{h}$ on $\mathcal{O}_{\mathbb{E}^*}(1) \otimes L^k$. The main goal of this section is proving the following

Theorem 6.1.6. *Assume X is a compact Riemann surface and ω_∞ is a Kähler form of constant curvature on X. Also assume that*

$$\frac{i}{2\pi}F_{(E,h_\infty)} = \omega_\infty I_E.$$

Let $s_1^{(k)}, ..., s_{N_k}^{(k)}$ be a basis for $H^0(X, E \otimes L^{\otimes k})$ such that

$$\sum_{i=1}^{N_k} s_i^{(k)} \otimes (s_i^{(k)})^{*_{h^{(k)}}} = Id$$

$$\int_X \langle s_i^{(k)}, s_j^{(k)} \rangle_{h^{(k)}} \omega_\infty = \frac{rVol(X, \omega_\infty)}{N_k}\delta_{ij}.$$

Then

$$\int_{\mathbb{P}E^*} \langle \widehat{s_i^{(k)}}, \widehat{s_j^{(k)}} \rangle_{\widehat{h^{(k)}}} dVol_{\widehat{h^{(k)}}} = D_k I + M_k,$$

where $D_k \frac{N_k}{rV_k} \to 1$ as $k \to \infty$ and M_k is a traceless Hermitian matrix such that $||M_k||_{op} = o(k^{-\infty})$. Here $V_k = \int_{\mathbb{P}E^*} c_1(\mathcal{O}_{\mathbb{E}^*}(1) \otimes L^k)^r$.

Let V be a complex vector space of dimension r. Recall that there is a natural isomorphism $\hat{}: V \to H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(1))$, which sends $v \in V$ to $\hat{v} \in H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(1))$ so that for any $f \in V^*$, $\hat{v}(f) = f(v)$. Also any hermitian inner product h on Vinduces a hermitian metric $\hat{h}$ on $H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(1))$. We have the following.

Lemma 6.1.7. *Let h_0 and h be Hermitian inner products on V. If $||h - h_0||_{h_0} \leq \epsilon$, then $||\hat{h} - \hat{h}_0||_{C^2(\hat{h}_0)} \leq \epsilon$.*

Lemma 6.1.8. *Let X be a Kähler manifold of dimension n and ω_0 and ω be two Kähler forms on X. If $||\omega - \omega_0||_{C^0(\omega_0)} \leq \epsilon$, then $|\frac{\omega^n - \omega_0^n}{\omega_0^n}| \leq \epsilon^n$*

Proof. □

Proposition 6.1.9. *If $||h - h_0||_{h_0} \leq \epsilon$, then for any $v, w \in V$, we have*

$$\Big| \int_{\mathbb{P}V^*} \langle \hat{v}, \hat{w} \rangle_{\hat{h}} \frac{\omega_{FS,h_0}^{r-1}}{(r-1)!} - C_r \langle v, w \rangle_h \Big| \leq \epsilon^r V |v|_h |w|_h,$$

where V is the volume of the projective space with respect to the standard Fubini-Study volume form.

Proof.

$$\Big| \int_{\mathbb{P}V^*} \langle \hat{v}, \hat{w} \rangle_{\hat{h}} \frac{\omega_{FS,h_0}^{r-1}}{(r-1)!} - C_r < v, w >_h \Big|$$

$$= \Big| \int_{\mathbb{P}V^*} \langle \hat{v}, \hat{w} \rangle_{\hat{h}} \frac{\omega_{FS,h_0}^{r-1}}{(r-1)!} - \int_{\mathbb{P}V^*} \langle \hat{v}, \hat{w} \rangle_{\hat{h}} \frac{\omega_{FS,h}^{r-1}}{(r-1)!} \Big|$$

$$= \Big| \int_{\mathbb{P}V^*} \langle \hat{v}, \hat{w} \rangle_{\hat{h}} \Big(\frac{\omega_{FS,h_0}^{r-1}}{(r-1)!} - \frac{\omega_{FS,h}^{r-1}}{(r-1)!} \Big) \Big|$$

$$\leq \int_{\mathbb{P}V^*} \Big| \langle \hat{v}, \hat{w} \rangle_{\hat{h}} \Big| \Big| \frac{\omega_{FS,h_0}^{r-1}}{(r-1)!} - \frac{\omega_{FS,h}^{r-1}}{(r-1)!} \Big|$$

$$\leq \epsilon^r \int_{\mathbb{P}V^*} |\hat{v}|_{\hat{h}} |\hat{w}|_{\hat{h}} \frac{\omega_{FS,h_0}^{r-1}}{(r-1)!} \leq \epsilon^r |v|_h |w|_h V$$

□

Let (X, ω_∞) be a Kähler manifold of dimension n and E be a holomorphic vector bundle on X of rank r and degree d. The slope of E is defined by $\mu = d/r$.

Similar to the case of vector spaces, we have the natural isomorphism $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1)) = H^0(X, E)$. Also, for any Hermitian metric h on E, we have a Hermitian metric $\hat{h}$ on $\mathcal{O}_{\mathbb{P}E^*}(1)$.

Lemma 6.1.10. *Let h_0 and h be Hermitian metrics on E. If $||h - h_0||_{C^2(h_0)} \leq \epsilon$, then $||\hat{h} - \hat{h}_0||_{C^2(\hat{h}_0)} \leq \epsilon$.*

Proof. Without loss of generality, we can assume that $E = X \times \mathbb{C}^r$ and h_o is the standard metric on $\mathbb{C}^r$. Let $e_1, ..., e_r$ be the standard basis for $\mathbb{C}^r$ and $\xi_1, ..., \xi_r$ be the homogenous coordinate on $\mathbb{P}^{r-1}$. Let $h_{ij} = h(e_i, e_j)$ and $\epsilon_{ij} = h^{ij} - \delta_{ij}$. There exists a function φ on $\mathbb{P}E^*$ such that $\widehat{h} = e^\varphi \widehat{h_0}$. We have

$$\varphi = -\log \frac{\sum \xi_i \overline{\xi_j} h^{ij}}{\sum |\xi_i|^2} = -\log \Big(1 + \frac{\sum \xi_i \overline{\xi_j} \epsilon_{ij}}{\sum |\xi_i|^2} \Big).$$

Define the function $f(z, \xi) = \frac{\sum \xi_i \overline{\xi_j} \epsilon_{ij}(z)}{\sum |\xi_i|^2}$. Hence, there exists a constant C such that

$$|\varphi|_{C^\alpha} = |\log(1 + f)|_{C^\alpha} \leq \log(1 + C|f|_{C^\alpha}) \leq C'|f|_{C^\alpha}.$$

Now since $||h-h_0||_{C^2(h_0)} \leq \epsilon$, we have $|\epsilon_{ij}|_{C^2} \leq \epsilon$ which implies that $|f|_{C^2} \leq C^{'}\epsilon$.

□

Let h be a metric on E and $s_1, ..., s_N$ be a basis for $H^0(X, E)$. Let $\hat{s_1}, ..., \hat{s_N}$ be the corresponding basis for $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1))$. Assume that we have

$$\int_X \langle s_i, s_j \rangle_h \frac{\omega_\infty^n}{n!} = \frac{rVol(X, \omega_\infty)}{N}\delta_{ij}.$$

One would expect that the matrix $[\int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j} \rangle_{\widehat{h}} dvol_{\widehat{h}}]$ be close to a scalar matrix. Indeed, we can prove it in some special case.

Proposition 6.1.11. *Let h_∞ be a Hermitian metric on E such that $iF_{h_\infty} = \mu\omega_\infty Id$, where $\mu = d/r$. If $||h - h_\infty||_{C^2(h_\infty)} \leq \epsilon$, then*

$$\Big| \int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j} \rangle_{\widehat{h}} dvol_{\widehat{h}} - C' C_{r-1} \mu^n \frac{rVol(X, \omega_\infty)}{N} \delta_{ij} \Big| \leq$$

$$\big(\epsilon^{n+r-1} Vol(\mathbb{P}E^*) + C' \mu^n \epsilon^{r-1} V_{\mathbb{P}^{r-1}} Vol(X)\big) \max |s_i(z)|_h^2$$

where C' is a constant depends only on r.

Proof. Fix a point p in X. Let $e_1, ..., e_r$ be a holomorphic local frame for E such that at the point p, $h_\infty(e_i, e_j) = \delta_{ij}$ and $dh_\infty(e_i, e_j) = 0$. Any $e^* \in E_p^*$ can be written as $e^* = \sum \xi_i e_i^*$. So, we have a local coordinate $\xi_1, ...\xi_{r-1}$ on fibres. Direct computation yields

$$dvol_{\widehat{h_\infty}} = C' \mu^n \frac{d\xi_1 \wedge d\overline{\xi_1} \wedge .... \wedge d\xi_{r-1} \wedge d\overline{\xi_{r-1}}}{(1 + |\xi|^2)^r} \wedge \frac{\omega_\infty^n}{n!}$$

Put $C = C'\mu^n$. Therefore,

$$\int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j} \rangle_{\widehat{h_\infty}} dvol_{\widehat{h_\infty}} = CC_{r-1} \int_X \langle s_i, s_j \rangle_{h_\infty} dvol_{h_\infty}.$$

Now, we have

$$\Big| \int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j} \rangle_{\widehat{h}} dvol_{\widehat{h}} - CC_{r-1} \frac{rVol(X, \omega_\infty)}{N} \delta_{ij} \Big|$$

$$= \Big| \int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j} \rangle_{\widehat{h}} dvol_{\widehat{h}} - CC_{r-1} \int_X \langle s_i, s_j \rangle_h dvol_{h_\infty} \Big|$$

$$\leq \Big| \int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j} \rangle_{\widehat{h}} dvol_{\widehat{h}} - \int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j} \rangle_{\widehat{h}} dvol_{\widehat{h_\infty}} \Big|$$
$$+ \Big| \int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j} \rangle_{\widehat{h}} dvol_{\widehat{h_\infty}} - CC_{r-1} \int_X \langle s_i, s_j \rangle_h dvol_{h_\infty} \Big|$$
$$\leq \int_{\mathbb{P}E^*} \Big| \langle \widehat{s_i}, \widehat{s_j} \rangle_{\widehat{h}} \Big| \Big| dvol_{\widehat{h}} - dvol_{\widehat{h_\infty}} \Big|$$
$$+ \Big| C \int_{\mathbb{P}E^*} \langle \widehat{s_i}, \widehat{s_j} \rangle_{\widehat{h}} \frac{\omega_{FS,h_\infty}}{(r-1)!} dvol_{h_\infty} - CC_{r-1} \int_X \langle s_i, s_j \rangle_h dvol_{h_\infty} \Big|$$

$$= \int_{\mathbb{P}E^*} \Big| \langle \widehat{s_i}, \widehat{s_j} \rangle_{\widehat{h}} \Big| \Big| dvol_{\widehat{h}} - dvol_{\widehat{h_\infty}} \Big|$$
$$+ C \Big| \int_X \Big(\int_{Fibre} \langle \widehat{s_i}, \widehat{s_j} \rangle_{\widehat{h}} \frac{\omega_{FS,h_\infty}}{(r-1)!} - C_{r-1} \langle s_i, s_j \rangle_h \Big) dvol_{h_\infty} \Big|$$
$$\leq \epsilon^{n+r-1} \max |s_i(z)|^2 Vol(\mathbb{P}E^*) + C V_{\mathbb{P}^{r-1}} \epsilon^{r-1} \Big| \int_X |s_i|_h |s_j|_h dvol_{h_\infty} \Big|$$
$$\leq \big(\epsilon^{n+r-1} Vol(\mathbb{P}E^*) + C \epsilon^{r-1} V_{\mathbb{P}^{r-1}} Vol(X) \big) \max |s_i(z)|_h^2$$

□

Corollary 6.1.12. *Let $t_1, ..., t_M$ be a basis for $H^0(X, E \otimes L^{\otimes k})$ and h be a Hermitian metric on $E \otimes L^{\otimes k}$ such that*

$$\int_X \langle t_i, t_j \rangle_h \frac{\omega_\infty^n}{n!} = \frac{rVol(X, \omega_\infty)}{M} \delta_{ij}.$$

Let $\hat{t_1}, ..., \hat{t_m}$ be the corresponding basis for $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes L^{\otimes k})$. If $||h \otimes g^{\otimes(-k)} - h_\infty||_{C^2(h_\infty)} \leq \epsilon$, then

$$\Big| \int_{\mathbb{P}E^*} \langle \widehat{t_i}, \widehat{t_j} \rangle_{\widehat{h}} dvol_{\widehat{h}} - CC_{r-1} \frac{rVol(X, \omega_\infty)}{M} \delta_{ij} \Big| \leq$$
$$\big(\epsilon^{n+r-1} Vol(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes L^{\otimes k}) + C \epsilon^{r-1} V_{\mathbb{P}^{r-1}} Vol(X) \big) \max |t_i(z)|_h^2$$

Proof. Put $\widetilde{\omega_\infty} = (k+1)\omega_\infty$ and $\widetilde{h_\infty} = h_\infty \otimes g^{\otimes k}$. We have,

$$iF_{\widetilde{h_\infty}} = iF_{h_\infty} + k\omega_\infty I = \frac{\mu + k}{k}\widetilde{\omega_\infty} Id = \mu' \widetilde{\omega_\infty} Id.$$

On the other hand, $||h \otimes g^{\otimes(-k)} - h_\infty||_{C^2(h_\infty)} \leq \epsilon$ implies that $||h - \widetilde{h_\infty}||_{C^2(\widetilde{h_\infty})} \leq \epsilon$. Applying the previous proposition gives the estimate.

□

Proof of Theorem 6.1.6. We define $\omega^{(k)} = Ric(\widehat{h^{(k)}})$, $\omega_k = Ric(\widehat{h}_k)$ and $\omega = Ric(\widehat{h}_\infty)$. Let $\widehat{s_1^{(k)}}, ..., \widehat{s_{N_k}^{(k)}}$ be the corresponding basis for $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes L^{\otimes k})$. The equation

$$\sum_{i=1}^{N_k} s_i^{(k)} \otimes (s_i^{(k)})^{*_{h^{(k)}}} = Id$$

implies that

$$\sum |\widehat{s_i^{(k)}}|^2_{\widehat{h^{(k)}}} = 1$$

By Theorem 6.1.3, we have $||h_k - h_\infty||_{C^2(h_\infty)} = O(k^{-\infty})$, since ω_∞ has constant curvature. So Corollary 6.1.12 implies

$$\Big| \int_{\mathbb{P}E^*} \langle \widehat{s_i^{(k)}}, \widehat{s_j^{(k)}} \rangle_{\widehat{h^{(k)}}} dvol_{\widehat{h^{(k)}}} - \frac{Vol(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes L^{\otimes k})}{N_k} \delta_{ij} \Big| = O(k^{-\infty}).$$

□

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