TRANSLATING GRAPHS BY MEAN CURVATURE FLOW

by

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Abstract

The aim of this work is to study translating graphs by mean curvature flow. Mean curvature flow is an example of a geometric flow, where a hypersurface $\mathcal{M}$ deforms according to some geometric principles. Precisely, at each point p the hypersurface moves in the direction of its unit normal vector with the speed equal to the mean curvature of $\mathcal{M}$ at p. There are special surfaces, called solitons, that move under mean curvature flow by rigid motion of the ambient space. In this work we look at translating surfaces, a type of soliton where the surface shifts in a fixed direction at a constant speed. Specifically, we restrict our attention to surfaces that can be viewed as the graph of a function over a domain in $\mathbb{R}^n$. We begin by studying the capillary equation. This is motivated by a paper of Altschuler and Wu, who used the capillary equation to prove existence of complete translating graphs in R^3. In joint work with Maria Calle, we obtain a gradient bound for the solution of the capillary problem in $\mathcal{M}^n \times \mathbb{R}$, where $\mathcal{M}^n$ is a Riemannian submanifold in $\mathbb{R}^{n+1}$. In the rest of the work, we prove non-existence of complete translating graphs over bounded domains in $\mathbb{R}^3$. Furthermore, we show that there are only three types of complete translating graphs in $\mathbb{R}^3$; entire graphs, graphs between two vertical planes, and graphs in one side of a plane. In the last two types, graphs are asymptotic to planes next to their boundaries. We also prove stability of translating graphs and then we obtain a pointwise curvature bound for translating graphs in $\mathbb{R}^3$. In addition, we obtain a monotonicity formula for convex translating graphs in $\mathbb{R}^{n+1}$.

Advisor: William P. Minicozzi II

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Chapter 1

Introduction

Our main area of study in this work is mean curvature flow, where a surface moves in the direction of its mean curvature vector. This is a gradient flow for volume, and so a surface undergoing mean curvature flow has its area decrease. In general, this leads to solutions of mean curvature flow developing singularities in a finite amount of time. Understanding the nature of these singularities is quite challenging, but it is known that surfaces which move in a self-similar manner under the mean curvature play a key role in the singularity theory of the flow. The most important ones are self-similar shirkers and translating surfaces, which model the so-called type I and type II singularities, respectively. In this work, we study translating graphs by mean curvature flow in $\mathbb{R}^{n+1}$, which are a type of translating surface that can be viewed as the graph of a function over a domain in $\mathbb{R}^n$. First, we obtain local and global gradient bounds for the solution to the capillary equation on $\mathcal{M}^n \times \mathbb{R}$, where $\mathcal{M}^n$

is n-dimensional Riemannian manifold, and capillary equation is a prescribed mean curvature equation over a bounded domain with given contact angle in the boundary of the domain. Then, we prove the mean value theorem and monotonicity formula for convex translating graphs in $\mathbb{R}^{n+1}$. We also classify complete translating graphs in $\mathbb{R}^3$. Furthermore, we obtain a curvature estimate and a compactness theorem for complete translating graphs with polynomial volume growth in $\mathbb{R}^3$ by proving a stability theorem for translating graphs.

1.1 Background

We start by providing some required background materials on mean curvature flow. First of all, we explain the mean curvature flow and its singularities. Afterwards, we define translating graphs by mean curvature flow. Finally, we describe the capillary problem.

1.1.1 Mean curvature flow

Mean curvature flow takes place in the description of the interface evolution in some physical models. The enthusiasm for the study of the mean curvature flow arises from geometric applications. One can use the flow as a tool to construct minimal surfaces, or to classify surfaces satisfying certain curvature conditions. The numerical implementation of the mean curvature flow has been developed by Osher and Sethian

(27) for curve evolution via level sets. Recently, Kazhdan (21) showed that a minor modification to the mean curvature flow removes the numerical instability, providing simpler expressions for both the discrete and continuous formulations of the flow. Moreover, mean curvature flow has been used for image analysis, fairing and denoising surface meshes (4; 10), material design (29; 30), and surface processing (37).

Mean curvature flow evolves hypersurfaces in the unit normal direction with speed equal to the mean curvature at each point. It is the steepest descent flow for the area functional. In particular, minimal hypersurfaces are stationary solutions. In other words, a family of smoothly embedded hypersurfaces $(\mathcal{M}_t)_{t\in I}$ moves by mean curvature if

$$\frac{\partial x}{\partial t} = \vec{H}(x), \tag{1.1.1}$$

for $x \in \mathcal{M}_t$ and $t \in I$, $I \subset \mathbb{R}$ an open interval. Here $\vec{H}(x)$ is the mean curvature vector at $x \in \mathcal{M}_t$.

By looking at the equation (1.1.1), one can easily see that during the mean curvature flow minimal surfaces are not moving at all, because their mean curvature H is zero. The general definition of mean curvature flow for hypersurfaces (1.1.1) implies the following definition for graphs evolving by mean curvature flow. If the smooth hypersurfaces $(\mathcal{M}_t)_{t\in I}$ moving by mean curvature flow are locally graphs over some open set $\Omega \subset \mathbb{R}^n$, i.e. there is a function $u : \Omega \to \mathbb{R}$ so that $x(t) = (x_1, x_2, \cdots, x_n, u(x_1, \cdots, x_n))$ then we have following equation

$$\frac{\partial u}{\partial t} = \sqrt{1+|\nabla u|^2}\mathrm{div}\left(\frac{\nabla u}{1+|\nabla u|^2}\right). \tag{1.1.2}$$

On the other hand, if there is a function u satisfying the above parabolic equation, then its graph is a hypersurface moving by mean curvature flow.

1.1.2 Singularities of mean curvature flow

The evolution equation (1.1.1) can develop singularities in finite time T, which are classified into two types according to the rate at which the maximal curvature, $\max_{\mathcal{M}_t} |A(t)|$, tends to infinity for $t \to T$. Here $|A(t)|$ is the second fundamental form of $\mathcal{M}_t$.

Definition 1.1.1 *If* $\max_{\mathcal{M}_t} |A(t)| \leq c(T-t)^{-1/2}$ *for some* $c < \infty$, *then* $\mathcal{M}_t$ *is said to have a type-I singularity, otherwise* $\mathcal{M}_t$ *has a type-II singularity.*

Roughly speaking, singularities are of type-I if we have a good control of the geometry. By proving the monotonicity formula, Huisken (18) showed that the flow is asymptotically self-similar near a given type-I singularity and, thus, is modeled by self-shrinking solutions of the flow.

The self shrinking hypersurfaces cannot live forever during the mean curvature flow; at some time $T > 0$ they become singular. In two cases this phenomenon happens. The first one is when there is a time t so that the manifold $\mathcal{M}_t$ is not smooth, which usually occurs when some derivatives of x are not bounded as t approaches T. The second case is when for some t, $\mathcal{M}_t$ is not an immersion. In these both cases the curvature of the evolving hypersurface has to become unbounded, because if the second fundamental form $|A|$ stays uniformly bounded until time T, then T cannot be the maximal time of existence of a smooth mean curvature flow.

The examples of convergence in (2; 3) indicate that type-II singularities are modeled by translating surfaces. Huisken and Sinestrari (19) proved that if the initial surface has non-negative mean curvature, the family of evolving surfaces, appropriately rescaled, converges to a n-dimensional, strictly convex translating surface, or $\mathcal{M}^k \times \mathbb{R}^{n-k}$, where $\mathcal{M}^k$ is a lower dimensional, strictly convex translating surface. The proof of this theorem used an important theorem of Hamilton (15), which states that any strictly convex eternal solution to the mean curvature flow, where the mean curvature assumes its maximum value at a point in space-time must be a translating solution.

For example let the hypersurfaces $\mathcal{M}_0 = S_R$ be the sphere of radius R centered at the origin. By the uniqueness theorem, during the mean curvature flow at every time the hypersurface $\mathcal{M}_t$ is the sphere of radius $R(t)$ centered at origin (i.e. $\mathcal{M}_t = S_{R(t)}$), as illustrated in Figure 1.1. Also at each point $p \in S_R$, the mean curvature is $H(p) = \frac{n}{R}$. We choose the inward unit normal, the mean curvature equation (1.1.1) implies the evolution equation for the radius of the sphere, which is $R'(t) = \frac{n}{R(t)}$ with $R(0) = R$. By solving this simple ODE, we obtain $R(t) = \sqrt{R^2 - 2nt}$. That means at the time $T = \frac{R^2}{2n}$ the sphere shrinks to a point so the flow becomes singular. Therefore, the time T is the maximal time of existence. By computing the norm of

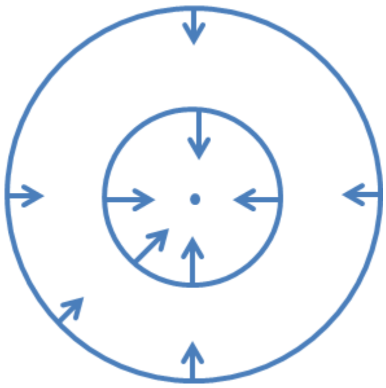

Figure 1.1: Evolution of a circle under the mean curvature flow: an example of self-similar shrinker

the second fundamental during the flow, we obtain

$$|A|(t) = \frac{\sqrt{n}}{R(t)} = \frac{1}{\sqrt{2(T-t)}}. \tag{1.1.3}$$

This implies that spheres are examples of type I singularities (self similar shrinkers) for the mean curvature flow.

As we mentioned before, solitons which move in a selfsimilar manner under the mean curvature play a key role in the singularity theory of the flow. Now for obtaining a formula for solitons, let M_t be a solution to the equation (1.1.1) which moves along the integral curves of a vector field V. If M_0 satisfies the following equation

$$\vec{H} = V^{\perp}, \tag{1.1.4}$$

then the mean curvature flow with initial data M_0 moves along the integral curves of V. When $V = -\frac{x}{2}$ or $V = e_{n+1}$, we will respectively get self similar shrinkers and vertically translating surfaces in $\mathbb{R}^{n+1}$.

1.1.3 Translating graphs

There are some surfaces which are translating by the mean curvature flow at constant speed without changing during the flow. These surfaces are called translating surfaces. That means for this surfaces, we have

$$\frac{\partial x}{\partial t} = \vec{H}(x) = \vec{C} + \vec{v}, \tag{1.1.5}$$

where $\vec{C}$ is the constant velocity vector of the translation and $\vec{v}$ is a vector field tangent to the surface $\mathcal{M}_t$. Without loss of generality, we can fix $\vec{C} = Ce_{n+1}$ to be the $n+1$ coordinate unit vector in $\mathbb{R}^{n+1}$, and taking the inner product with the normal vector, we have

$$H = \langle e_{n+1}, \upsilon \rangle. \tag{1.1.6}$$

One example of a translating surface is a graph of a function translating by mean curvature flow. According to Huisken and Sinestrari (19) studying type II singularities of the equation (1.1.2) for initial mean convex graph is the same as studying translating graphs by mean curvature flow. Translating graphs are graphs which, under mean curvature flow, move in a certain direction without changing their geometry. So $u = u(x)$ is a translating graph by the mean curvature flow if the function $v(x,t) = u(x) + Ct$ solves the equation (1.1.2). Since $H = \operatorname{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}})$, v is a vertically translating graph with constant speed C if and only if u is a solution to the following equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{C}{\sqrt{1+|\nabla u|^2}}. \tag{1.1.7}$$

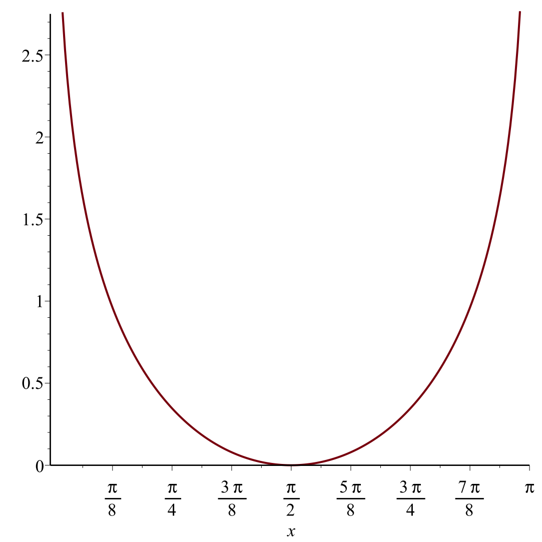

Figure 1.2: Grim reaper

The famous translating graph in $\mathbb{R}^2$ is the grim reaper, which is the solution to the mean curvature flow given by a graph with the following function for each t:

$$u(x,t) = t - \log(\sin(x)),$$

where $x \in (0,\pi)$ and $t \in [0,\infty)$. For $t = 0$ the graph of grim reaper is shown in Figure 1.2.

Now we bring the argument stated in (25) to show that, for every fixed vector $v \in \mathbb{R}^{n+1}$, there is a unique rotationally symmetric, strictly convex hypersurface (which is actually an entire graph) moving by translation under the mean curvature flow. Without loss of generality, we prove existence of such a convex graph over a domain in $\mathbb{R}^n$, translating in the $v = e_{n+1}$ direction with unit speed. So we need to find a convex function $u : \Omega \to \mathbb{R}$ such that

$$\triangle u - \frac{Hessu(\nabla u, \nabla u)}{1 + |\nabla u|^2} = 1, \tag{1.1.8}$$

and $u(0) = \nabla u(0) = 0$, where $Hessu$ is the $Hessian$ of u. For obtaining rotational symmetry around the origin let $u(x) = u(r)$ with $r = |x|$, thus the equation (1.1.8) becomes

$$u_{rr} + \frac{(n-1)u_r}{r} - \frac{u_{rr}u_r^2}{1 + u_r^2} = 1, \tag{1.1.9}$$

which is

$$u_{rr} = (1 + u_r^2)\left(1 - \frac{(n-1)u_r}{r}\right), \tag{1.1.10}$$

with the condition $\lim_{r \to 0} u(r) = \lim_{r \to 0} u'(r) = 0$.

When $n = 1$ the solution of this ODE gives the grim reaper; when $n > 1$ there is only one solution, defined on entire $\mathbb{R}^+$ growing quadratically at infinity. This solution provides the only rotationally symmetric, convex, translating hypersurface moving by mean curvature.

1.1.4 Capillary problem

Capillary problems arise from the physical phenomenon that occurs whenever two different materials are situated adjacent to each other and do not mix. If one (at least) of the materials is a fluid, which forms with another fluid (or gas) a free surface interface, then the interfaces will be referred to as a capillary surface. A great deal of work has been devoted to capillarity phenomena since the initial works of Young and

Laplace in the early nineteenth century (see the book of Finn (12) for an account on the subject).

The capillary problem can be formulated as a PDE equation which is

$$\begin{aligned} &\text{(a)} \quad div\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \Psi(x,u) \qquad (x \in \Omega) \\ &\text{(b)} \quad \upsilon \cdot \gamma = \Phi(x,u) \qquad\qquad\qquad (x \in \partial\Omega) \end{aligned} \tag{1.1.11}$$

where Ω is a bounded domain in n-dimensional manifold $\mathcal{M} \subset \mathbb{R}^{n+1}$ with Riemannian metric σ, Ψ and Φ are given functions on $\mathcal{M} \times \mathbb{R}$ and $\partial\Omega \times \mathbb{R}$ respectively, υ is the downward unit normal to the graph of u and γ is the inner normal to $\partial\Omega \times \mathbb{R}$.

Notice that the capillary problem is the same as the prescribed mean curvature equation with given contact angle in the boundary. Recently the topics of existence of minimal and constant mean curvature surfaces in $\mathcal{M} \times \mathbb{R}$, where $\mathcal{M}$ is a Riemannian manifold, have gathered great interest. For example; B. Nelli and H. Rosenberg considered minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$; particularly, surfaces which are vertical graphs over domains in $\mathbb{H}^2$ (26). Also H. Rosenberg discussed minimal surfaces in $\mathcal{M} \times \mathbb{R}$, where $\mathcal{M}$ is the 2-sphere (with the constant curvature one metric) or a complete Riemannian surface with a metric of non-negative curvature, or $\mathcal{M}$ is the hyperbolic plane (28). L. Hauswirth, H. Rosenberg, and J. Spruck proved existence of constant mean curvature graphs in $\mathcal{M} \times \mathbb{R}$ where $\mathcal{M} = \mathbb{H}^2$ or $\mathbb{S}^2$ the hyperbolic plane of curvature -1 or the 2-sphere of curvature 1 (17). Also J. Spruck established a priori interior gradient estimates and existence theorems for n-dimensional graphs of constant mean curvature $H > 0$ in $\mathcal{M}^n \times \mathbb{R}$ where $\mathcal{M}^n$ is simply connected and complete and Ω is a bounded domain in $\mathcal{M}$ (35).

1.2 Overview of results

First, we obtain a gradient bound for the solution to the capillary equation (1.1.11).

Theorem 1.2.1 *Let $\Omega \subset \mathcal{M}^n$ be a bounded domain with C^3 boundary $\partial\Omega$ such that the diameter of Ω is less than the injectivity radius of $\mathcal{M}^n$. If for each $K_1 < \infty$, there exists $K_2 < \infty$ so that*

$$
\begin{aligned}
&\text{(i)} && n + |\Psi| + |\Psi_x| < K_2 \\
&\text{(ii)} && \Psi_z > 0 \\
&\text{(iii)} && 1 - |\Phi| \geq K_2^{-1} \\
&\text{(iv)} && \Phi_z \geq 0 \\
&\text{(v)} && |\Phi|_{C^2} < K_2
\end{aligned}
$$

on some open set containing $\overline{\Omega} \times [-K_1, K_1]$, then for the function $u \in C^2(\Omega) \cap C^1_{loc}(\overline{\Omega})$ that is a solution to the capillary problem (1.1.11) in Ω, there is an $M = M(K_1, K_2, \partial\Omega)$ such that $|\nabla u|(x) \leq M$ for all $x \in \overline{\Omega}$.

To obtain this gradient bound we show that Korevaar's technique in (22) works in $\mathcal{M}^n \times \mathbb{R}$. Then we state a counterexample founded by Dr. Spruck for one part of the proof of Altschuler and Wu, in which they used capillary equation to prove existence of complete translating graphs in $\mathbb{R}^3$.

In the second part of the work, we study translating graphs in $\mathbb{R}^n$. We prove the following mean value theorem to obtain a monotonicity formula for convex translating graphs in $\mathbb{R}^{n+1}$.

Theorem 1.2.2 ***(Mean Value Theorem)*** *Let g be a smooth function on a smooth translating graph Σ in $B_{\rho_0}(x_0) \subset \mathbb{R}^{n+1}$. Then the formula*

$$\begin{aligned} \frac{d}{dr}\Big(\frac{1}{r^n}\int_{\Sigma\cap B_r(x_0)} g\Big) &= \frac{d}{dr}\int_{\Sigma\cap B_r(x_0)} g\frac{|(x-x_0)^\perp|^2}{|x-x_0|^{n+2}} \\ &+ \frac{1}{2r^{n+1}}\int_{\Sigma\cap B_r(x_0)} (r^2-|x-x_0|^2)\triangle_\Sigma g \\ &+ \frac{1}{r^{n+1}}\int_{\Sigma\cap B_r(x_0)} \frac{Cg}{w}\upsilon\cdot(x-x_0), \end{aligned} \tag{1.2.1}$$

holds in the distribution sense for $r \in (0, \rho_0)$ (and also classically for almost every $r \in (0, \rho_0)$). If Σ is convex, $x_0 \in \Sigma$, $g \geq 0$ and g is subharmonic on Σ, then

$$g(x_0) \leq \frac{1}{\omega_n \rho^n}\int_{\Sigma\cap B_\rho(x_0)} g, \tag{1.2.2}$$

for all $\rho \in (0, \rho_0)$. Here ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

This theorem implies the following monotonicity formula:

Corollary 1.2.3 ***(Monotonicity Formula)*** *If Σ is a smooth convex translating graph in $\mathbb{R}^{n+1}$ and $x_0 \in \Sigma$, then*

$$\mathcal{H}^n(\Sigma \cap B_\rho(x_0)) \geq \omega_n \rho^n. \tag{1.2.3}$$

Finally, using the definition of a translating graph, we obtain a bound for the volume of a ball on a translating graph $\Sigma \subset \mathbb{R}^{n+1}$.

Afterward, for proving the compactness theorem for complete translating graphs with polynomial volume growth in $\mathbb{R}^3$, we define the second order operator L by

$$Lu = \Delta u + |A|^2 u + \langle e_{n+1}, \nabla u \rangle,$$

where A is the second fundamental form.

Definition 1.2.4 *We say a translating surface is L-stable if for any compactly supported function η we have*

$$\int_\Sigma \eta L \eta e^{x_{n+1}} \leq 0. \tag{1.2.4}$$

We then obtain following important theorem.

Theorem 1.2.5 *Translating graphs are L-stable.*

Note that the above theorem, together with the compactness theorem for stable minimal graphs in $\mathbb{R}^{n+1}$, implies a compactness theorem for translating graphs in $\mathbb{R}^{n+1}$ with polynomial volume growth $(1 \leq n \leq 5)$.

Also, we obtain a point-wise curvature estimate for translating graphs in $\mathbb{R}^3$. Specifically, for every point $p \in \Sigma$, we obtain a bound for $|A|(p)$:

Theorem 1.2.6 *Let $\Sigma^2 \subset \mathbb{R}^3$ be a complete translating graph in mean curvature flow, if $B^\Sigma_{r_0 e}(p) \subset (\Sigma \cap B_1(p)) \setminus \partial(\Sigma \cap B_1(p))$, and $r_0 e^{1/2} < \rho_1(\pi e^{-1}, e)$, then for some C and all $0 < \sigma \leq r_0$,*

$$\sup_{B^\Sigma_{r_0-\sigma}} |A|^2 \leq C\sigma^{-2}. \tag{1.2.5}$$

Finally, we show translating graphs over a domain $\Omega \subset \mathbb{R}^n$ are asymptotic to a minimal surface next to the boundary of Ω. That implies theorem 1.2.7 (i.e. non-existence of translating graphs over a bounded domain).

Theorem 1.2.7 *There is no complete translating graph $\Sigma \subset \mathbb{R}^{n+1}$ $(n > 1)$ with non-zero constant speed C over a smooth bounded connected domain $\Omega \subset \mathbb{R}^n$.*

As a corollary, we can conclude that complete translating graphs in $\mathbb{R}^3$ can only come in three forms: they can be on one side of a vertical plane, between two parallel vertical planes, or entire graphs over $\mathbb{R}^2$.

Corollary 1.2.8 *If Σ is a complete translating graph over a domain $\Omega \subset \mathbb{R}^2$, then next to the boundary of Ω, Σ is asymptotic to a plane. So a translating graph over $\mathbb{R}^2$ can only be between 2 parallel planes or in one side of a plane or an entire graph.*

Chapter 2

Capillary equation

In this chapter we discuss the capillary problem, which is

$$\begin{aligned} &(a) \quad div\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \Psi(x,u) \qquad (x \in \Omega) \\ &(b) \quad \upsilon \cdot \gamma = \Phi(x,u) \qquad\qquad\qquad (x \in \partial\Omega) \end{aligned} \tag{2.0.1}$$

where Ω is a bounded domain in n-dimensional manifold $\mathcal{M} \subset \mathbb{R}^{n+1}$ with Riemannian metric σ, Ψ and Φ are given functions on $\mathcal{M} \times \mathbb{R}$ and $\partial\Omega \times \mathbb{R}$ respectively, υ is the downward unit normal to the graph of u and γ is the inner normal to $\partial\Omega \times \mathbb{R}$.

First of all, we obtain local and global gradient estimates for the solution of the capillary equation. Then we bring the counterexample founded by Dr. Spruck for one part of the proof stated by Altschuler and Wu in (1). They used capillary equation to prove solutions of the the mean curvature flow over convex regions $\Omega \subset \mathbb{R}^2$ with given contact angle of the surface to the boundary cylinder converge to translating graphs.

For reaching the gradient bound, we show that Korevaar's technique in (22) works in $\mathcal{M}^n \times \mathbb{R}$. First, we describe how gradient bounds follow from the construction of suitable "barrier" comparison surfaces. For later reference, we collect the estimate which must hold to obtain a gradient bound. Then, we derive a priori local and global gradient bounds for nonparametric capillary surfaces above smooth domains.

2.1 Maximum Principle

In this section, we set our notation and then we show that how gradient bounds follow from the construction of suitable "barrier" comparison surfaces.

Let $\mathcal{M}^n \subset \mathbb{R}^{n+1}$ be a Riemannian Manifold. We rescale $\mathcal{M}$ so that $|x-y|_{\mathcal{M}} < 2|x-y|$ for any two points $x, y \in \mathcal{M}$.

We consider the (signed) distance function

$$d(x) = \begin{cases} \min\limits_{y\in\partial\Omega} |x-y|_{\mathcal{M}} & \text{if } x \in \Omega; \\ -\min\limits_{y\in\partial\Omega} |x-y|_{\mathcal{M}} & \text{if } x \in \mathcal{M}\setminus\Omega. \end{cases}$$

near $\partial\Omega$, and the inner normal $\gamma = \frac{\nabla d}{|\nabla d|}$. There exists a neighborhood of radius $\mu > 0$ of points within (unsigned) distance μ of $\partial\Omega$, on which d is C^3 and γ is C^2.

Define the Euclidean ball of radius R and center x in $\mathcal{M}$ by $B_R(x) = \{y \in \mathcal{M} : |x-y| < R\}$. If x is suppressed it is assumed to be zero.

We embed $\mathcal{M} \subset \mathcal{M}\times\mathbb{R}$ in the usual way: $\mathcal{M} = \{(x,z) \in \mathcal{M}\times\mathbb{R} : z = 0\}$. The capillary tube above $\partial\Omega$ is defined to be $\partial\Omega\times\mathbb{R} = \{(x,z) : x \in \partial\Omega, z \in \mathbb{R}\}$. We often extend functions (or vector fields) defined on $U \subset \mathcal{M}$ to $U \times \mathbb{R} \subset \mathcal{M} \times \mathbb{R}$ by making them constant in the z-direction. In particular, we so extend d and γ and they represent the distance function and normal vector field associated to $\partial\Omega \times \mathbb{R}$.

For a function u on $\mathcal{M}$, define $S = graph(u)$ to be $\{(x, z) \in \mathcal{M} \times \mathbb{R} : z = u(x)\}$. Let $x^1, ..., x^n$ be a system of local coordinate for $\mathcal{M}$ with corresponding metric σ_{ij}. Subscripts on functions generally denote partial derivatives, e.g. $f_i = \frac{\partial f}{\partial x^i}$, whereas superscripts refer to components of vectors. For total derivative, we use D for a function of x and z, that is

$$Df = Df(x, u(x)) = f_x + f_z Du = (D_1 f, ..., D_n f).$$

The downward unit normal to S is given by

$$\upsilon = (\upsilon^1, ..., \upsilon^{n+1}) = \frac{1}{\sqrt{1 + |Du|^2}}(u_1, ..., u_n, -1),$$

where $u_i = \sigma^{ij} D_j u$ and $|Du| = |Du|_{\mathcal{M}}$.

We extend υ and u away from S and Ω by making them constant in the vertical direction. Measure the steepness of S by

$$V = (\upsilon^{n+1})^{-1} = \sqrt{1 + |Du|^2}.$$

2.1.1 Barrier Technique

In this section, we set up a technique to construct suitable "barrier" comparison surfaces to obtain the gradient estimate in the third section using the maximum principle Lemma (2.1.3). We construct a family of surfaces $\Sigma(t) = graph(u^t(x))$ for sufficiently small nonnegative t, with $\Sigma(0) = \Sigma \subset\subset S = graph(u)$. Denote the interior of Σ by Σ^o, and its boundary by $\partial\Sigma$. The $\Sigma(t)$ are constructed by deforming Σ smoothly along a vector field Z. Although one can modify the $\Sigma(t)$ so that they are actually barriers (i.e. lying in a useful way entirely above or below $graph(u)$), we use them directly as comparison surfaces: for small t, the height separation $s(t)$ between S and $\Sigma(t)$ will be seen to be about $t(Z \cdot \upsilon)V$. For suitable Z we can use the contact angle boundary condition to show that $(Z \cdot \upsilon)V$ is bounded at any (relatively large) maximum value of $s(t)$ which occurs on the intersection of Σ with the capillary tube. $(Z \cdot \upsilon)V$ will be bounded by construction on the part of $\partial\Sigma$ that is inside the tube, $\partial\Sigma \cap S^o$. Finally, we will be able to use the prescribed mean curvature equation to show that $(Z \cdot \upsilon)V$ is bounded at any maximum of $s(t)$ which occurs on Σ^o. We will therefore conclude a bound for $(Z \cdot \upsilon)V$ on Σ, i.e. a local gradient estimate.

Proceeding with our construction, we assume there exists an open subset $O \subset \mathcal{M} \times \mathbb{R}$ with $\Sigma \subset O$, on which the deformation vector field Z is defined, with $|Z|_{C^2(O)} < \infty$. For $P \in \Sigma$ and small t, define $\widetilde{P}_P(t)$ by solving the ODE

$$\frac{d}{dt}\widetilde{P}_P = Z(\widetilde{P}_P) \tag{2.1.1}$$

$$\widetilde{P}_P(0) = P, \tag{2.1.2}$$

and define the resulting perturbed surface by

$$\Sigma(t) = \{\widetilde{P}_P(t) : P \in \Sigma\}. \tag{2.1.3}$$

It follows from ODE theory that $\Sigma(t)$ is the graph of a C^2 function $u^t(x)$, with domain nearly that of $u = u^0$. If we make the further requirement that Z be tangential:

$$Z(Q) \cdot \gamma(Q) = 0 \ for \ all \ Q \in \partial\Omega \times \mathbb{R} \cap O, \tag{2.1.4}$$

(and that $\partial\Omega \times \mathbb{R} \cap O$ is C^1), then the ODE (2.1.1) preserves $\partial\Omega \times \mathbb{R}$, a fact which implies that the domain of u^t is contained in $\overline{\Omega}$, which will be useful later in boundary (contact angle) calculations. We define the quantities to be estimated later. Writing $\widetilde{P}_P(t) = (\widetilde{x}, u^t(\widetilde{x}))$, denote the point in $S = graph(u)$ directly above (or below) it by $\widehat{P}_P(t) = (\widetilde{x}, u(\widetilde{x}))$. Let $s(P,t)$ be the (signed) vertical distance from $\widetilde{P}_P(t)$ to S, $s(P,t) = u(\widetilde{x}) - u^t(\widetilde{x})$. Let $\upsilon(P,t)$, $\Pi(P,t)$ and $H(P,t)$ be the normal, tangential plane and mean curvature of $\Sigma(t)$ at $\widetilde{P}_P(t)$, respectively. Whenever t is suppressed its value is zero. Hereafter t should be small enough such that $\widetilde{P}_P(t)$ and $\widehat{P}_P(t)$ are in the injectivity ball of P in $\mathcal{M} \times \mathbb{R}$.

For a fixed point $P \in \Sigma$, we consider a unitary frame $\{f_1, f_2, ..., f_{n+1}\}$ of $\mathcal{M} \times \mathbb{R}$ with the following properties:

1. For each $Q \in \Sigma$, $f_i \in \Pi(Q)$ for $i = 1, .., n$ and $f_{n+1} = -\upsilon(Q)$.

2. At P, they are orthonormal. Moreover, the vectors $f_1, .., f_{n-1}$ are horizontal, that is, they have the last component (in $\mathcal{M} \times \mathbb{R}$) equal to 0, and f_n is in the direction of steepest ascent in $\Pi(P)$.

Let g be the Riemannian metric equivalent to $\sigma + dz^2$ of $\mathcal{M} \times \mathbb{R}$ corresponding to this new frame.

Any vector field X on S can be written as $X = X^\alpha f_\alpha = X^i f_i - \chi f_{n+1}$. (Here and in the sequel we use the summation convention, summing from 1 to $n+1$ if the repeated indices are Greek and from 1 to n if they are Latin.) For any function χ, we can define natural tangent-plane analogs to the gradient ∇ and $\triangle$ of $(\mathcal{M} \times \mathbb{R}, g)$:

$$\begin{aligned} \nabla_\Pi \chi &= \nabla \chi - (\chi_\upsilon)\upsilon \\ \triangle_\Pi \chi &= \triangle \chi - \chi_{\upsilon\upsilon} \\ (\chi_\upsilon &= \nabla \chi \cdot \upsilon). \end{aligned} \tag{2.1.5}$$

For $|y|$ small less than the injectivity radius of $\mathcal{M} \times \mathbb{R}$ at P, S may be given near P by the exponential mapping $y \to exp_P(y^i f_i + U(y) f_{n+1})$, for some C^3 function U. Let $[U_{ij}(y)]$ denote its Hessian matrix, and for $y = 0$ write $U_{ij}U_{ij} = |A|^2$. ($[U_{ij}(0)]$ is the matrix for the second fundamental form of S at P, with respect to the $\{f_i\}$ frame on S).

Similarly, we can write a point $\widetilde{P}_Q(t) \in \Sigma(t)$ close enough to P as $\widetilde{P}_Q(t) = exp_P(\widetilde{y}^i f_i + U^t(\widetilde{y}) f_{n+1})$, for t small enough. We use the notation $\widetilde{P}_Q(t) = (\widetilde{y}, U^t(\widetilde{y}))$ and $Q = (y, U(y))$.

Lemma 2.1.1 *Let $P \in \Sigma$ and $t = 0$. Let the vector field Z satisfy (2.1.1) and let $u \in C^3(O)$. Express $Z = Z^i f_i - \zeta f_{n+1}$ in the P-based coordinate system. Then the* *surfaces $\Sigma(t)$ which result from the vector flow (2.1.1) evolve so that at $t = 0$*

$$\frac{\partial}{\partial t}s(P,t) = \zeta V$$
$$\frac{\partial}{\partial t}\upsilon(P,t) = -\nabla_{\Pi}\zeta$$
$$\frac{\partial}{\partial t}H(P,t) = -(2Z_i^k U_{ki} + \triangle_{\Pi}\zeta - \zeta_{\upsilon}H) \tag{2.1.6}$$

in the strong sense that for $L = s, \upsilon, H$ we have

$$L(P,t) = L(P,0) + t\frac{\partial L}{\partial t} + o(t), \tag{2.1.7}$$

with the error term $o(t)$ uniform for $P \in \Sigma$. In the special case that Z is a normal-perturbation vector field ($Z = \eta\upsilon$, with the function $\eta \in C^2(O)$ and υ), the evolution formulae are given by

$$\frac{\partial}{\partial t}s(P,t) = \eta V$$
$$\frac{\partial}{\partial t}\upsilon(P,t) = -\nabla_{\Pi}\eta$$
$$\frac{\partial}{\partial t}H(P,t) = -(2\eta|A|^2 + \triangle_{\Pi}\eta - \eta_{\upsilon}H) \tag{2.1.8}$$

Proof.

Consider the curve $\alpha(t) = \widehat{P}_P(t)$, the vertical projection onto Σ of the curve $\widetilde{P}_P(t)$ that solves the ODE (2.1.1). Then we can write:

$$\alpha(t) = exp_{\widetilde{P}_P(t)}(s(P,t)e_{n+1}),$$

and therefore we have the equation:

$$\begin{aligned}\frac{d\alpha}{dt} &= \frac{d}{dt} exp_{\widetilde{P}_P(t)}(s(P,t)e_{n+1}) = \\ &= d(exp_{\widetilde{P}_P(t)})_{s(P,t)e_{n+1}} \left(\frac{d\widetilde{P}_P(t)}{dt} + \frac{ds(P,t)}{dt} e_{n+1} \right) \\ &= d(exp_{\widetilde{P}_P(t)})_{s(P,t)e_{n+1}} \left(Z(\widetilde{P}_P(t)) + \frac{ds(P,t)}{dt} e_{n+1} \right).\end{aligned}$$

At $t = 0$, we have $\widetilde{P}_P(0) = P$ and $s(P,0) = 0$, therefore:

$$\frac{d\alpha}{dt}(0) = d(exp_P)_0 \left(Z(P) + \frac{ds(P,0)}{dt} e_{n+1} \right) = Z(P) + \frac{ds(P,0)}{dt} e_{n+1}.$$

Since $\alpha(t)$ is a curve in Σ, we have that $\frac{d\alpha}{dt}(0) \in \Pi(P)$, and therefore its product with the normal vector is 0:

$$0 = \frac{d\alpha}{dt}(0) \cdot \upsilon(P) = \left(Z(P) + \frac{ds(P,0)}{dt} e_{n+1} \right) \cdot \upsilon(P),$$

and from this and the fact that $e_{n+1} \cdot \upsilon(P) = V^{-1}$ we get the result for $\frac{ds}{dt}$.

To calculate $\upsilon(P,t)$ and $H(P,t)$, we consider the curve from $Q = (y, U(y)) \in \Sigma$ to $(\widetilde{y}, U^t(\widetilde{y})) \approxeq \widetilde{P}_Q(t)$ that solves the ODE (2.1.1). By the Taylor expansion for *exp* map we have

$$\begin{aligned}\widetilde{y}^i(t) &= y^i + \int_0^t Z^i(\widetilde{P}_Q(s))ds + o(t) \\ U^t(\widetilde{y}) &= U(y) - \int_0^t \zeta(\widetilde{P}_Q(s))ds + o(t).\end{aligned} \tag{2.1.9}$$

The first equation in (2.1.9) expresses $\widetilde{y}$ as a function of y. We use it to estimate the Jacobian of the inverse transformation, which expresses y as a function of $\widetilde{y}$. With this Jacobian we can use the chain rule and the second equation to estimate the first two derivatives of $U^t(\widetilde{y})$, with respect to $\widetilde{y}$, yielding estimates for $\upsilon(P,t)$ and $H(P,t)$. We note that the $o(t)$-error terms below depend at most on second derivatives of Z and third derivatives of U.

$$\frac{\partial \widetilde{y}^i}{\partial y^m} = \delta^i_m + tD_m Z^i(y, U(y)) + o(t)$$
$$\frac{\partial y^k}{\partial \widetilde{y}^i} = \delta^i_k - tD_i Z^k(Q) + o(t)$$
$$\frac{\partial U^t(\widetilde{y})}{\partial \widetilde{y}^i} = U_i(y) - t(D_i\zeta + U_k D_i Z^k) + o(t)$$
$$\frac{\partial^2 U^t(\widetilde{y})}{\partial \widetilde{y}^i \partial \widetilde{y}^j} = U_{ij} - t(U_{ik} D_j Z^k + U_{kj} D_i Z^k + D_j D_i \zeta + U_k D_j D_i Z^k) + o(t)$$

The terms involving DU are zero at the value of $\widetilde{y}$ corresponding to $y = 0$, and the derivative estimates there simplify to

$$\frac{\partial U^t(\widetilde{y})}{\partial \widetilde{y}^i} = -t\zeta_i + o(t)$$
$$\frac{\partial^2 U^t(\widetilde{y})}{\partial \widetilde{y}^i \partial \widetilde{y}^j} = U_{ij} - t(U_{ik} Z^k_j + U_{kj} Z^k_i + \zeta_{ij} + \zeta_{n+1} U_{ij}) + o(t). \qquad (2.1.10)$$

Estimating the normal and mean curvature of $\Sigma(t)$ at $\widetilde{P}_P(t)$ with the aid of (2.1.10) yields the evolution formulas for υ and H.

In the case $Z = \eta\upsilon$ is normal perturbation, we have

$$Z^k(Q) = \eta(Q)(\upsilon \cdot f_k) \quad (1 \le k \le n)$$
$$\zeta(Q) = \eta(Q)(\upsilon \cdot f_{n+1}). \qquad (2.1.11)$$

In computing the derivative of Z at P we use the facts that the gradient of U is zero there and that because $\upsilon^{n+1} = -1$ is a minimum value, so is the gradient of

υ^{n+1}. These computation yield

$$\zeta(P) = \eta, \qquad \zeta_i(P) = \eta_i, \qquad \zeta_{ii} = \eta_{ii} - \eta U_{ik} U_{ik} \tag{2.1.12}$$

$$\zeta_\upsilon(P) = \eta_\upsilon, \quad Z_i^k(P) = \eta U_{ik}. \tag{2.1.13}$$

Substituting the above expirations into the estimate (2.1.6) yields (2.1.8). ■

2.1.2 Maximum Principle Lemma

The maximality of $s(P,t)$ has two possible geometric consequences, depending on the location of P, we can obtain inequalities which are implicit in comparison principles for surfaces of related mean curvature and contact angle:

Lemma 2.1.2 *Let a positive maximum of $s(Q,t)$ (over Σ) occur at $P \in \Sigma$.*

1. *If $P \in (\partial\Sigma \bigcap \partial S)^o$, then $\upsilon(\widetilde{P}_P(t)) \cdot \gamma(\widetilde{P}_P(t)) \geq \upsilon(\widehat{P}_P(t)) \cdot \gamma(\widehat{P}_P(t))$.*

2. *If $P \in \Sigma^o$, then $H_{\Sigma(t)}(\widetilde{P}_P(t)) \geq H_\Sigma(\widehat{P}_P(t))$.*

Proof. These inequalities follow directly from calculus and the ellipticity of the contact angle and mean curvature operators. In both cases the function $s = u - u^t$ has a local maximum at $\widetilde{P}_P(t)$. In case (1) it follows that the gradient Ds points in the exterior normal direction, $-\gamma$, implying the contact angle inequality. In case (2) it follows that Ds is zero and $D^2 s$ is negative semi-definite, implying the mean curvature inequality. ∎

Now let $P \in \Sigma^o$ be a point where s is maximum, we write $\widetilde{P}_P(t) = (x, u^t(x))$ and $\widehat{P}_P(t) = (x, u(x))$, so that $s = u(x) - u^t(x)$. Since it is a maximum, we have that $Ds(x) = 0$, so that $Du(x) = Du^t(x)$ and therefore the tangent plane to $\Sigma(t)$ at $\widetilde{P}_P(t)$ is parallel to the tangent plane to Σ at $\widehat{P}_P(t)$. This also implies that, if we write $\widetilde{P}_P(t) = (\widetilde{y}, U^t(\widetilde{y}))$ and $\widehat{P}_P(t) = (y, U(y))$ in the f_i basis, we have that $DU(y) = DU^t(\widetilde{y})$.

Write $\xi(t)$ for the Σ-secant vector such that $\widehat{P}_P(t) = exp_P(\xi(t))$, and recall that $\alpha(t) = \widehat{P}_P(t)$. Using the calculations above for $\alpha(t)$, we can compute:

$$\frac{d}{dt}|_{t=0} exp_P(\xi(t)) = \frac{d\alpha(0)}{dt} = Z(P) + \frac{ds(P,0)}{dt} e_{n+1},$$

which is a vector in $\Pi(P)$. On the other hand,

$$\frac{d}{dt}|_{t=0} exp_P(\xi(t)) = d(exp_P)_0 \left(\frac{d\xi}{dt}|_{t=0} \right) = \frac{d\xi}{dt}|_{t=0} = \dot{\xi}(0),$$

so that $\xi(t) = \xi(0) + t\dot{\xi}(0) + o(t) = t\left(Z(P) + \frac{ds(P,0)}{dt} e_{n+1}\right) + o(t)$. We can then write $y = t\dot{\xi}(0) + o(t)$ and $U(y) = o(t)$.

Now we compute the Taylor expansion of $U_i(y)$ at 0:

$$U_i(y) = U_i(0) + DU_i(0) \cdot y + O(|y|^2) = DU_i(0) \cdot t\dot{\xi}(0) + o(t) = t(\dot{\xi}(0)^k U_{ik}) + o(t), \quad (2.1.14)$$

where we have used that $DU(0) = 0$. On the other hand, by (2.1.6) we have that $U_i^t(\widetilde{y}) = -t\zeta_i + o(t)$. By equating $DU(y)$ and $DU^t(\widetilde{y})$ and dividing by t, we obtain:

$$\dot{\xi}(0)^k U_{ik} = -\zeta_i + o(1). \quad (2.1.15)$$

We know that $Z = Z^m f_m + \zeta \upsilon$, for $m = 1..n$. On the other hand, by the definition of the basis (using the fact that $f_1, .., f_{n-1}$ are horizontal and f_n is in the direction of Du), we can see that $e_{n+1} \cdot f_n = \frac{|Du|}{V}$ (in fact, $f_n = (\frac{1}{V|Du|} Du + \frac{|Du|}{V} e_{n+1})$), so that we have

$$\dot{\xi}(0)^k = Z^k(P) + \zeta(P)|Du|\delta_{kn}. \qquad (2.1.16)$$

That implies:

$$\zeta|Du|U_{ni} = -\zeta_i - Z^k U_{ki} + o(1) \ for \ 1 \leq i \leq n. \qquad (2.1.17)$$

Now we will prove the key Lemma (2.1.3) to get the gradient bound in the next section.

Lemma 2.1.3 *Let $u \in C^3(\overline{\Omega})$ solve the capillary problem in Ω. Consider $\Sigma \subset\subset S = graph(u)$, $\Sigma \subset O$, and a C^2 tangential deformation vector field $Z = \eta\upsilon + X$. Assume that $\partial\Sigma$ is the union of $\overline{\partial\Sigma \cap S^o}$ and $(\partial\Sigma \cap \partial S)^o$. Fixing any point $P \in \Sigma$, denote the decomposition of X with respect to the tilted basis $\{f_\alpha\}$ at P by $X = X^i f_i - \chi f_{n+1}$. Then we can conclude the gradient estimate*

$$(Z \cdot \upsilon)V = (\eta + \chi)V \leq M$$

on Σ, for some $0 < M < \infty$, if we can verify the following three inequalities, for some $\delta > 0$:

1. $(Z \cdot \upsilon)V \leq M - \delta$, *($P \in \overline{\partial\Sigma \cap S^o}$).*

2. $\nabla_\Pi(\eta + \chi) \cdot \gamma > \delta - \frac{\partial}{\partial t}\Phi(\widetilde{P}_P(t)) + \upsilon \cdot \frac{\partial}{\partial t}\gamma(\widetilde{P}_P(t))$, *(at $t = 0$ for $P \in (\partial\Sigma \cap \partial S)^o$).*

3. $\eta|A|^2 + 2X_i^k U_{ki} + \triangle_\Pi(\eta+\chi) - (\eta+\chi)_\upsilon H + \dot{\xi}(0)\cdot\nabla\Psi > \delta,$ *(for* $P \in \Sigma^o$*).*

Proof. For small $t > 0$, we consider the maximum on Σ of the function $s(Q,t)$, and let that maximum occur at $P \in \Sigma$. If $s(P,t) \leq Mt$ for small t, since $s = t(\eta+\chi)V + o(t)$, that is enough to prove the gradient estimate. Consider the following cases for P:

1. If $P \in \overline{\partial\Sigma \cap S^o}$, then inequality (1) shows that the gradient estimate holds.

2. If $P \in (\partial\Sigma\cap\partial S)^o$: then using Capillary equation we have $\Phi(\widetilde{P}_P(0)) = \gamma(\widetilde{P}_P(0))\cdot \upsilon(\widetilde{P}_P(0))$, since $\widetilde{P}_P(0) = P$. Also we have $\frac{\partial\upsilon(P,t)}{\partial t} = -\nabla_\Pi(\eta+\chi)$ Thus inequality (2) will imply at $t = 0$:

$$-\frac{\partial\upsilon}{\partial t}(P,t)\cdot\gamma > \delta - \frac{\partial}{\partial t}\Phi(\widetilde{P}_P(t)) + \upsilon\cdot\frac{\partial}{\partial t}\gamma(\widetilde{P}_P(t))$$
$$\frac{\partial}{\partial t}\Phi(\widetilde{P}_P(t)) > \delta + \frac{\partial}{\partial t}(\upsilon\cdot\gamma(\widetilde{P}_P(t))). \tag{2.1.18}$$

Since Z is tangential then we have $\widehat{x} \in \partial\Omega$, so using Taylor expansion for $\upsilon\cdot\gamma$ and Φ also by knowing $\widetilde{P}_P + se_{n+1} = \widehat{P}_P$ then we have:

$$\upsilon\cdot\gamma(\widetilde{P}_P(t)) = \upsilon\cdot\gamma(\widetilde{P}_P(0)) + t\frac{\partial}{\partial t}(\upsilon\cdot\gamma(\widetilde{P}_P(t)))|_{t=0} + o(t)$$
$$\upsilon\cdot\gamma(\widetilde{P}_P(t)) - \upsilon\cdot\gamma(\widetilde{P}_P(0)) < -\delta t + t\frac{\partial}{\partial t}\Phi(\widetilde{P}_P(t))|_{t=0} + o(t)$$
$$\upsilon\cdot\gamma(\widehat{P}_P(0)) - \upsilon\cdot\gamma(P) < -\delta t + t\frac{\partial}{\partial t}\Phi(\widetilde{P}_P(t)) + o(t)$$
$$\frac{\partial}{\partial t}\Phi(\widehat{P}_P(t))|_{t=0} < Ct - \delta + \frac{\partial}{\partial t}\Phi(\widetilde{P}_P(t))|_{t=0} + o(t).$$

Thus we have $(\eta+\chi)V = \frac{\partial}{\partial t}s|_{t=0} < M$.

3. If $P \in \Sigma^o$, using Capillary equation we have $H(P,0) = H_\Sigma(P) = \Psi(P) = \Psi(\widetilde{P}_P(0))$, $H_\Sigma(\widehat{P}_P(t)) = \Psi(\widehat{P}_P(t))$ and from (3), (2.1.6) and (2.1.8) we have:

$$-\frac{\partial H}{\partial t}(P,t)|_{t=0} + \dot{\xi}(0)\cdot\nabla\Psi > \delta. \tag{2.1.19}$$

By Taylor expansion for H and Ψ and Lemma (2.1.2) we have:

$$\begin{aligned} H(P,0) - H(P,t) + o(t) + t\dot{\xi}(0)\cdot\nabla\Psi &> t\delta \\ H_\Sigma(P) - H_\Sigma(\widehat{P}_P(t)) + \Psi(\widehat{P}_P(t)) - \Psi(P) + o(t) &> t\delta \\ o(t) &> t\delta. \end{aligned}$$

For t small enough, $t\delta + o(t) > 0$, so that we get a contradiction. Therefore, the maximum of s cannot occur at an interior point, and we are in one of the other two cases. Therefore we have $(\eta+\chi)V < M$.

■

2.2 Gradient bounds in smooth domains

We prove three a-priori gradient estimates for solutions to the capillary problem: local interior and boundary estimates when there is positive gravity, and global estimates when there is not.

2.2.1 Local gradient bound

We will prove local interior and boundary gradient estimates assuming positive gravity using Lemma (2.1.3).

Theorem 2.2.1 *Let $u \in C^3(\Omega)$ solve the prescribed mean curvature equation, with positive gravity $\Psi_z > k > 0$. If R be less than injectivity radius of $\mathcal{M} \times \mathbb{R}$ and $\overline{B_R}(0) \subset \Omega$, then there exists a finite $M = M(R, K_1, K_2, k)$ so that*

$$V(x) \leq M \frac{R^2}{R^2 - |x|^2},$$

for all $x \in B_R$.

Proof. In the formalism of §2, define the subset $\Sigma \subset\subset S = graph(u)$ and the deformation vector field Z by

$$\Sigma = S \cap (\overline{B}_R \times \mathbb{R}),\ Z = \eta v,\ \eta(x, z) = 1 - \frac{|x|^2}{R^2}. \tag{2.2.1}$$

We only need to find $0 < M < \infty$ with which to satisfy (2.1.3)(3) in order to get interior estimate ($\Sigma \subset S^o$ and $\eta = 0$ on $\partial\Sigma$). Fixing a point $P \in \Sigma^o$ and the resulting vector $\dot{\xi}(0)$ (2.1.16) and definition of f_n we have

$$\dot{\xi}(0) \cdot \nabla\Psi = ((Z^k + \eta|Du|\delta_n^k)f_k) \cdot (\Psi_{x_k} e_k + \Psi_z e_{n+1})$$

$$\dot{\xi}(0) \cdot \nabla\Psi \geq \eta k \frac{|Du|^2}{V} - C. \tag{2.2.2}$$

Thus by definition of η there is a constant C so that

$$\eta|A|^2 + \triangle_{\Pi}\eta - \eta_v H + \dot{\xi}(0) \cdot \nabla\Psi > \eta k \frac{|Du|^2}{V} - C,$$

and (2.1.3)(3) can be verified for sufficiently large M. ∎

Remark 2.2.2 *Let $u \in C^3(\Omega)$ solve the prescribed mean curvature equation with positive gravity k. Let Ω satisfy a uniform interior sphere condition of radius $R > 0$ (i.e. each $P \in \Omega$ is contained in a sub ball of Ω having radius at least R). Then it follows immediately from pervious theorem and the definition of the distance function d, that there is an M so that $V(x) \leq \frac{M}{d(x)}$ for any $x \in \Omega$.*

Now we will prove a-priori boundary gradient estimate when there is positive gravity.

Theorem 2.2.3 *For Ω as in the capillary problem, let $u \in C^3(\overline{\Omega})$ solve the capillary problem with positive gravity k. Then for $r > 0$ and $y \in \mathcal{M}$, $B_{3r}(y)$, there exists an $M = M(r, k, K_1, K_2, \partial\Omega \cap B_{3r}(y))$, such that $V(x) \leq M$ for each $x \in B_r(y) \cap \overline{\Omega}$.*

Proof. Without loss of generality, we may assume $y = 0$. Modify the distance function d outside the μ-neighborhood of $\partial\Omega \cap B_{2r}$ on which it is C^3. Make it a C^3 function on all of B_{2r} in such a way that this modified d always has magnitude less than the actual (non-negative) distance to $\partial\Omega$, and so that its gradient is bounded in norm by 1. Extend γ to the gradient of the d in $B_{2r}(y)$, making it a C^2 function in the entire ball. It follows from remark (2.2.2) that we have the preliminary estimate

$$V(x) \leq Cd^{-1} \text{ in } \Omega \cap B_{2r}. \tag{2.2.3}$$

In analogy with (2.2.1), we define

$$\Sigma = S \cap \overline{B_{2r}},\ w(x,z) = 4r^2 - |x|^2. \tag{2.2.4}$$

Let $0 < \epsilon < 1$ and $N > 0$, we define the vector field $Z = \eta\upsilon + X$ by

$$\eta = \epsilon w + Nd,\ X = -\epsilon\Phi(w\gamma - d\nabla w), \tag{2.2.5}$$

and now we want to show the three conditions of (2.1.3) hold for sufficiently large M. We estimate the terms of (2.1.3)(2), for $P \in (\partial\Sigma \cap \partial S)^o$, since $1 - |\Phi|^2$ is bounded above zero we have:

$$\begin{aligned}
&\nabla_\Pi(\epsilon w + Nd + \chi)\cdot\gamma + \frac{\partial}{\partial t}\Phi(\widetilde{P}_P(t)) - \upsilon\cdot\frac{\partial}{\partial t}\gamma(\widetilde{P}_P(t)) > \\
&\nabla_\Pi(Nd)\cdot\gamma - \epsilon\nabla_\Pi(\Phi w\gamma\cdot\upsilon)\cdot\gamma + D\Phi\cdot Z - \upsilon\cdot(D(\nabla d)Z) - C > \qquad (2.2.6)\\
&\nabla_\Pi(Nd)\cdot\gamma - C = N(\gamma - (\gamma\cdot\upsilon)\upsilon)\cdot\gamma - C = N(|1 - |\Phi|^2) - C.
\end{aligned}$$

This implies that we can satisfy (2.1.3)(2) for large N, independently of M. For such N now we will show that (2.1.3)(1) can be verified for large M: since $w = 0$ on $\overline{\partial\Sigma \cap S^o}$ and because of the estimate (2.2.3), we have

$$(\eta + \chi)V = (\epsilon w + Nd + \chi)V \leq (Nd + Cd)Cd^{-1} \leq C\ (x \in \overline{\partial\Sigma \cap S^o}). \tag{2.2.7}$$

From (2.2.5) in the $\{f_\alpha\}$ coordinate we have $|(X^\alpha)_\beta + (X^\beta)_\alpha| \leq C(\epsilon w + d)$, so from symmetry of $[U_{ij}]$ we get:

$$\eta|A|^2 + 2X_i^k U_{ki} \geq \eta|A|^2 - C\eta|A| \geq -C. \tag{2.2.8}$$

From this inequality and an inequality analogous to (2.2.2) we estimate

$$\eta|A|^2 + 2X_i^k U_{ki} + \triangle_\Pi(\eta+\chi) - (\eta+\chi)_v H + \dot{\xi}(0)\cdot\nabla\Psi > C. \tag{2.2.9}$$

Taken together, (2.2.6), (2.2.7) and (2.2.9) show that there is a large M verify the three conditions of (2.1.3), hence ζV is bounded above by M on Σ.

$$\zeta = \epsilon w + Nd + X\cdot v \geq \epsilon w(1-|\Phi|) + (N-\epsilon C)d \geq \epsilon w(1-|\Phi|) > \frac{\epsilon}{C}. \tag{2.2.10}$$

Thus we have (2.2.3) for $x \in B_r$ and ϵ sufficiently small. ■

2.2.2 Global gradient estimate

We now obtain an a priori global gradient estimate. Notice that we will get this global gradient bound without assuming positive gravity (however, we will need the positive gravity to prove existence).

Theorem 2.2.4 *Let $u \in C^3(\overline{\Omega})$ solve the capillary problem. Then there is an $M = M(K_1, K_2, \partial\Omega)$ such that $V(x) \leq M$ for all $x \in \overline{\Omega}$.*

Proof. Recall the neighborhood of radius μ about $\partial\Omega$ on which d is C^3. Extend d to be a C^3 function in all of $\overline{\Omega}$, with $|\nabla d| \leq 1$, and extend γ as ∇d. For the positive parameter N construct an increasing C^2 function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(0) = 1$, $f'(0) = N$ and $f(t) = 2$ for $t \geq \mu$.

Introducing another positive parameter L, we define Σ and $Z = \eta v + X$ by

$$\Sigma = S = graph(u),\ \eta = f(d)e^{Lz},\ X = -\Phi e^{Lz}\gamma. \tag{2.2.11}$$

Since Φ is bounded by construction we have

$$K_2^{-1}e^{Lz} \leq \zeta \leq Ce^{Lz}. \tag{2.2.12}$$

We seek to verify (2.1.3)(2),(3) for sufficiently large M. We estimate:

$$\begin{aligned}\nabla_\Pi(\eta+\chi)\cdot\gamma + \frac{\partial}{\partial t}\Phi(\widetilde{P}_P(t)) &- \upsilon\cdot\gamma(\widetilde{P}_P(t)) \geq e^{Lz}(\nabla_\Pi f\cdot\gamma) + Lfe^{Lz}(\nabla_\Pi z\cdot\gamma)\cdot\gamma \\ &- \nabla_\Pi(|\Phi|^2 e^{Lz}) + D\Phi(\widetilde{P}_P(t))Z - \upsilon\cdot(D\nabla d(\widetilde{P}_P(t))Z) \\ &\geq e^{Lz}(N(1-|\Phi|^2) - C - CLV^{-1}), \end{aligned} \tag{2.2.13}$$

whenever $P \in (\partial S \cap \partial\Sigma)^o$. If ζV is sufficiently large (depending on L), then the second inequality of (2.2.12) and $|u| < K_1$ imply that $CL\frac{e^{Lz}}{V}$ in (2.2.13) is small. Hence we fix N large enough to verify (2.1.3)(2) for large M (depending on L).

To verify (2.1.3), we make the preliminary estimate

$$\eta|A|^2 + 2X_i^k U_{ki} \geq -Ce^{Lz}, \tag{2.2.14}$$

whenever ζV is large enough, because we have $|X_i^k| \leq Ce^{Lz}$ for $i < n$ and $|X_n^k| \leq CLe^{Lz}$, also $|U_{kn}| \leq (\zeta V)^{-1}C(1+|A|)$ for t small and $V > 1$. Using $(\zeta V)^{-1}$ to compensate for the L in estimating $X_n^k U_{kn}$, and then applying Cauchy-Schwartz, we will get (2.2.14) for ζV sufficiently large (depending on L). Using (2.2.14) and the fact that $|\nabla_\Pi z|^2 = 1 - V^{-2}$, we have

$$\begin{aligned}&\eta|A|^2 + 2X_i^k U_{ki} + \triangle_\Pi(\eta+\chi) - (\eta+\chi)_\upsilon H + \dot{\xi}(0)\cdot\nabla\Psi \geq \\ &\triangle_\Pi\eta - e^{Lz}(C + CLV^{-1}) \geq e^{Lz}(L^2(1-V^{-2}) - C - CL),\end{aligned} \tag{2.2.15}$$

whenever $P \in \Sigma \cap S^o$ and ζV is large enough. Now fixing L large enough, we use (2.2.15) and sufficiently large M to verify (2.1.3)(3). Since (2.1.3)(1) is true, all three conditions of Lemma (2.1.3) can be verified for a fixed N and L. We get ζV is uniformly bounded on Σ, using the first inequality of (2.2.12) we conclude the uniform bound for V on $\overline{\Omega}$. ■

2.3 Counterexample for Altschuler and Wu's paper

In (1), Altschuler and Wu study surfaces over convex regions $\Omega \subset \mathbb{R}^2$ which are evolving by the mean curvature flow. They specify the angle of contact of the surface to the boundary cylinder. They claim that solutions converge to ones moving only by translation.

Let $\Omega \subset \mathbb{R}^2$ be a compact domain with smooth boundary $\partial\Omega$. Let $k > 0$ be the curvature of $\partial\Omega$. We denote by γ the inward unit normal vector to $\partial\Omega$, and by υ the upward unit normal to the graph $u : \Omega \to \mathbb{R}$.

The non-parametric mean curvature flow with specified contact angle in the boundary is a well-known equation, that for $a^{ij} = \delta^{ij} - \frac{\nabla_i u \nabla_j u}{w^2}$ and $w = \sqrt{1 + |\nabla u|^2}$

is given by

$$u_t = w \operatorname{div}\left(\frac{\nabla u}{w}\right) = a^{ij}\nabla_i\nabla_j u \quad in \quad Q_T \tag{2.3.1}$$

$$\upsilon \cdot \gamma = \cos(\alpha) \quad on \quad \Gamma_T$$

$$u(\cdot, 0) = u_0(\cdot) \quad on \quad \Omega_0,$$

where $u_0 \in C^\infty(\overline{\Omega})$, $Q_T = \Omega \times [0, T]$, $\Gamma_T = \partial\Omega \times [0, T]$ and $\Omega_t = \Omega \times \{t\}$.

The elliptic version of the mean curvature equation (2.3.1) is

$$\operatorname{div}\left(\frac{\nabla u}{w}\right) = \frac{C}{w} \quad in \quad \Omega \tag{2.3.2}$$

$$\upsilon \cdot \gamma = \cos(\alpha) \quad on \quad \partial\Omega,$$

where using divergence theorem we can compute

$$C = \frac{\int_{\partial\Omega} \cos(\alpha) ds}{\int_\Omega \frac{1}{w} dx}. \tag{2.3.3}$$

Notice that if $u(x)$ is a solution of (2.3.2) then $\tilde{u} = u - Ct$ is the solution of the mean curvature equation (2.3.1) which is translating upward with speed C.

In this paper they claimed that they proved the following theorem.

Theorem 2.3.1 *Equation (2.3.2) has a unique solution, for a unique value of C given by (2.3.3).*

For proving this theorem, for $\epsilon > 0$, they defined the equation

$$\operatorname{div}\left(\frac{\nabla u}{w}\right) = \frac{\epsilon u}{w} \quad in \quad \Omega \tag{2.3.4}$$

$$\upsilon \cdot \gamma = \Phi(x, u) \quad on \quad \partial\Omega. \tag{2.3.5}$$

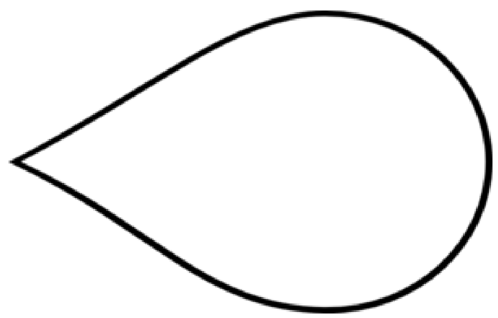

Figure 2.1: Convex domain Ω: the capillary equation does not have solution over Ω

They claimed the equation (2.3.4) has solution. However professor Spruck showed that for the specific convex domain Ω (Figure 2.1) the equation (2.3.4) does not have any solution for some α's. This domain Ω is the same domain introduced by Finn (12), for proving non-existence of solution to the capillary problem.

Also in chapter 5, we prove non-existence of complete translating graph over a bounded domain in $\mathbb{R}^2$.

Chapter 3

Properties of translating graphs

In this chapter, we study properties of translating graphs. We obtain a monotonicity formula for smooth convex translating graphs in $\mathbb{R}^{n+1}$ and some estimations for the volume of a ball on a translating surface.

3.1 Monotonicity formula

Let the function u over open set $\Omega \subset \mathbb{R}^n$ be a solution to the equation (2.3.2), and denote the graph of u by Σ. We want to get a monotonicity formula for smooth convex translating graphs in $\mathbb{R}^{n+1}$. In this section, the proof of the monotonicity formula is similar to the one that Ecker stated for smooth minimal hypersurfaces in (11).

Note that the divergence theorem for smooth properly embedded hypersurfaces $\mathcal{M}$

with smooth boundary is $\int_{\mathcal{M}} \text{div}_{\mathcal{M}} X = -\int_{\mathcal{M}} \vec{H} \cdot X + \int_{\partial \mathcal{M}} X \cdot \gamma$, where γ denotes the outer unit normal field to $\partial \mathcal{M}$ and X is a smooth vector field on $\mathcal{M}$.

Theorem 3.1.1 ***(Mean Value Theorem)*** *Let g be a smooth function on a smooth translating graph Σ in $B_{\rho_0}(x_0) \subset \mathbb{R}^{n+1}$. Then the formula*

$$\begin{aligned} \frac{d}{dr}\left(\frac{1}{r^n}\int_{\Sigma \cap B_r(x_0)} g\right) &= \frac{d}{dr}\int_{\Sigma \cap B_r(x_0)} g\frac{|(x-x_0)^{\perp}|^2}{|x-x_0|^{n+2}} \\ &+ \frac{1}{2r^{n+1}}\int_{\Sigma \cap B_r(x_0)} (r^2 - |x-x_0|^2)\triangle_{\Sigma} g \\ &+ \frac{1}{r^{n+1}}\int_{\Sigma \cap B_r(x_0)} \frac{Cg}{w}\upsilon \cdot (x-x_0) \end{aligned} \tag{3.1.1}$$

holds in the distribution sense for $r \in (0, \rho_0)$ (and also classically for almost every $r \in (0, \rho_0)$). If Σ is convex, $x_0 \in \Sigma$, $g \geq 0$ and g is subharmonic on Σ, then

$$g(x_0) \leq \frac{1}{\omega_n \rho^n}\int_{\Sigma \cap B_\rho(x_0)} g, \tag{3.1.2}$$

for all $\rho \in (0, \rho_0)$. Here ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

Proof. For simplicity we set $x_0 = 0$ and $B_r = B_r(0)$.

By the coarea formula i.e. $(\frac{d}{dr}\int_{\Sigma \cap B_r} g|\nabla_\Sigma |x||d\mathcal{H}^n = \int_{\Sigma \cap \partial B_r} gd\mathcal{H}^{n-1})$, we have

$$\frac{d}{dr}\int_{\Sigma \cap B_r} g = \int_{\Sigma \cap \partial B_r} \frac{g}{|\nabla_\Sigma |x||}, \tag{3.1.3}$$

for almost every $r \in (0, R)$. By Sard's theorem, we may assume that $\Sigma \cap \partial B_r$ is a smooth hypersurface inside Σ. By applying the divergence theorem to $\Sigma \cap B_r$ and using $H = \frac{C}{w}$ on Σ imply

$$\int_{\Sigma \cap B_r} \text{div}_\Sigma\left(\frac{x}{r}g\right) = -\int_{\Sigma \cap B_r} \frac{Cg}{rw}\upsilon \cdot x + \int_{\Sigma \cap \partial B_r} g\frac{x}{r} \cdot \gamma, \tag{3.1.4}$$

where $\gamma = \frac{\nabla_\Sigma |x|}{|\nabla_\Sigma |x||} = \frac{x^T}{|x^T|}$. Since $\frac{x}{r} \cdot \gamma = |\nabla_\Sigma |x||$, by combining (3.1.3) and (3.1.4) we have

$$\frac{d}{dr}\int_{\Sigma\cap B_r} g = \int_{\Sigma\cap\partial B_r} g\frac{1-|\nabla_\Sigma |x||^2}{|\nabla_\Sigma |x||} + \int_{\Sigma\cap B_r} \mathrm{div}_\Sigma(\frac{x}{r}g) + \int_{\Sigma\cap B_r} \frac{Cg}{rw}\upsilon\cdot x.$$

Using again coarea formula and the identities

$$1-|\nabla_\Sigma |x||^2 = \frac{|x^\perp|^2}{|x|^2},$$

and

$$\mathrm{div}_\Sigma(\frac{x}{r}g) = \frac{n}{r}g + \frac{x}{r}\cdot\nabla_\Sigma g,$$

we get

$$\begin{aligned}\frac{d}{dr}\int_{\Sigma\cap B_r} g &= \frac{d}{dr}\int_{\Sigma\cap B_r} g\frac{|x^\perp|^2}{|x|^2} + \frac{n}{r}\int_{\Sigma\cap B_r} g \\ &+ \int_{\Sigma\cap B_r} \frac{x^T}{r}\cdot\nabla_\Sigma g + \int_{\Sigma\cap B_r}\frac{Cg}{rw}\upsilon\cdot x.\end{aligned}$$

Since $x^T = -\frac{1}{2}\nabla_\Sigma(r^2-|x|^2)$, using integration by part, we obtain

$$\begin{aligned}\frac{d}{dr}\int_{\Sigma\cap B_r} g &= \frac{d}{dr}\int_{\Sigma\cap B_r} g\frac{|x^\perp|^2}{|x|^2} + \frac{n}{r}\int_{\Sigma\cap B_r} g \\ &+ \frac{1}{2r}\int_{\Sigma\cap B_r}(r^2-|x|^2)\triangle_\Sigma g + \int_{\Sigma\cap B_r}\frac{Cg}{rw}\upsilon\cdot x.\end{aligned}$$

Multiplying by r^{-n} and it follows from coarea formula that

$$\begin{aligned}\frac{d}{dr}(\frac{1}{r^n}\int_{\Sigma\cap B_r} g) &= \frac{d}{dr}\int_{\Sigma\cap B_r} g\frac{|x^\perp|^2}{|x|^{n+2}} \\ &+ \frac{1}{2r^{n+1}}\int_{\Sigma\cap B_r}(r^2-|x|^2)\triangle_\Sigma g + \frac{1}{r^{n+1}}\int_{\Sigma\cap B_r}\frac{Cg}{w}\upsilon\cdot x.\end{aligned}$$

Since Σ is convex $x_0 \in \Sigma$ implies that $\upsilon \cdot (x - x_0) \geq 0$. Thus, if $g \geq 0$, $\triangle_\Sigma g \geq 0$ and $x_0 \in \Sigma$ then

$$\frac{d}{dr}\left(\frac{1}{r^n}\int_{\Sigma\cap B_r} g\right) \geq 0. \tag{3.1.5}$$

Therefore

$$\lim_{r\to 0}\left(\frac{1}{\omega_n r^n}\int_{\Sigma\cap B_r(x_0)} g\right) \tag{3.1.6}$$

exists. Then the continuity of g and smoothness of Σ imply that this limits equals $g(x_0)$. This implies the mean value inequality. ■

For $g = 1$, Theorem 3.1.1 reduces to the important monotonicity formula.

Corollary 3.1.2 ***(Monotonicity Formula)*** *If Σ is a smooth convex translating graph in $\mathbb{R}^{n+1}$ and $x_0 \in \Sigma$, then*

$$\mathcal{H}^n(\Sigma \cap B_\rho(x_0)) \geq \omega_n \rho^n. \tag{3.1.7}$$

3.2 Bounds for volume of a ball

In this section we try to obtain some estimations for the volume of a ball on a translating surface $\Sigma \subset \mathbb{R}^{n+1}$. We get two different estimates, one estimation using isometry inequality and one bound using the definition of a translating graph.

Proposition 3.2.1 ***Isoperimetric Inequality*** *Let Σ be a smooth properly embedded hypersurface in $\mathbb{R}^{n+1}$. Then there exists a constant c_0 depending only on n such* *that for any $x_0 \in \mathbb{R}^{n+1}$ and almost every $r > 0$,*

$$\mathcal{H}^n(\Sigma \cap B_r(x_0))^{\frac{n-1}{n}} \leq c_0(\mathcal{H}^{n-1}(\Sigma \cap \partial B_r(x_0)) + \int_{\Sigma \cap B_r(x_0)} |H| d\mathcal{H}^n). \tag{3.2.1}$$

Proof. Appendix E in (11). ∎

Corollary 3.2.2 *Let Σ be a smooth vertically translating graph in $\mathbb{R}^{n+1}$ with constant speed C. Then there exists a constant c_0 depending only on n such that for any $x_0 \in \mathbb{R}^{n+1}$ and almost every $r > 0$,*

$$\mathcal{H}^n(\Sigma \cap B_r(x_0))\left(\frac{1}{\sqrt[n]{\mathcal{H}^n(\Sigma \cap B_r(x_0))}} - c_0 C\right) \leq c_0 \mathcal{H}^{n-1}(\Sigma \cap \partial B_r(x_0)). \tag{3.2.2}$$

Let $\Omega_r \subset \Omega$ and $M_r := \sup_{\Omega_r} |u(x)|$, then we have following Lemma.

Lemma 3.2.3

$$\mathcal{H}^n(\Sigma \cap (B_r \times \mathbb{R})) \leq \omega_{n-1} r^{n-1}(r(1 + CM_r) + 2M_r). \tag{3.2.3}$$

Proof. The proof is similar to the proof of the Lemma 2.2 in (36). Using the pull back of υ induced by the projection $\pi : \Omega \times \mathbb{R} \to \Omega$, we extend the vector field υ on the cylinder $\Omega \times \mathbb{R}$. Let ω be the n-form on $\Omega \times \mathbb{R}$, given by that for $f_1, ...f_n \in \mathbb{R}^{n+1}$,

$$\omega(f_1, \cdots, f_n) = \det(f_1, \cdots, f_n, \upsilon).$$

Therefore in standard coordinates $(x_1, ..., x_{n+1})$, we have

$$\omega = \frac{dx_1 \wedge \cdots \wedge dx_{n+1} + \Sigma_{i=1}^n (-1)^{n+i} u_{x_i} dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n}{w}. \tag{3.2.4}$$

Therefore

$$d\omega = (-1)^{n+1}\mathrm{div}(\frac{\nabla u}{w})dx_1 \wedge \cdots \wedge dx_{n+1} = (-1)^{n+1}\frac{C}{w}dx_1 \wedge \cdots \wedge dx_{n+1}. \qquad (3.2.5)$$

Let O be the region enclosed by Σ, $\partial\Omega_r \times \mathbb{R}$ and $\Omega_r \times \{-M_r\}$. Hence, by Stokes' Theorem we get

$$\mathcal{H}^n(\Sigma \cap (\Omega_r \times \mathbb{R})) = \int_{\partial O \cap \Sigma} \omega = -\int_{\partial O \setminus \Sigma} \omega + \int_O dw.$$

Thus

$$\begin{aligned} \mathcal{H}^n(\Sigma \cap (\Omega_r \times \mathbb{R})) &\leq \mathcal{H}^n(\partial\Omega_r \times [-M_r, M_r]) \\ &+ \mathcal{H}^n(\Omega_r \times \{-M_r\}) + C\mathcal{H}^{n+1}(\Omega_r \times [-M_r, M_r]). \quad (3.2.6) \end{aligned}$$

If $\Omega_r = B_r$ then we have

$$\begin{aligned} \mathcal{H}^n(\Sigma \cap (B_r \times \mathbb{R})) &\leq 2\omega_{n-1}r^{n-1}M_r + n^{-1}\omega_{n-1}r^n + Cn^{-1}\omega_{n-1}r^nM_r \\ &\leq \omega_{n-1}r^{n-1}(r(1 + CM_r) + 2M_r). \quad (3.2.7) \end{aligned}$$

■

Chapter 4

Curvature estimate for translating graphs

In this chapter, we define the notion of stability for translating graphs. We then prove translating graphs in $\mathbb{R}^{n+1}$ are stable. Afterwards, using stability of translating graphs in $\mathbb{R}^{n+1}$ we obtain a curvature estimate for translating graphs in $\mathbb{R}^3$.

4.1 Stability of translating graphs in $\mathbb{R}^3$

From now on without loss of generality we assume that the speed $C = 1$. Define the functional F on a hypersurface $\Sigma \subset \mathbb{R}^3$ by

$$F(\Sigma) = \int_\Sigma e^{x_3} d\mu. \tag{4.1.1}$$

In the following lemma, we prove translating surfaces are minimal hypersurfaces in $\mathbb{R}^3$ with respect to the conformal metric $g_{ij} = e^{x_3}\delta_{ij}$.

Lemma 4.1.1 *If $x' = \beta\upsilon$ is a compactly supported normal variation of a hypersurface $\Sigma \subset \mathbb{R}^3$ and s is the variation parameter, then*

$$\frac{\partial}{\partial s}F(\Sigma_s) = \int_{\Sigma} \beta\,(H + \langle e_3, \upsilon\rangle)\, d\mu.$$

Proof. By the first variation formula, we have $(d\mu)' = \beta H d\mu$. Also the s derivative of $\log(e^{x_3})$ is given by $\langle e_3, \upsilon\rangle$. Thus we have the lemma. ∎

Let the graph $\widetilde{\Sigma}$ be the graph Σ respect to the conformal metric g. We know that the hypersurface $\widetilde{\Sigma}$ subset of Riemannian manifold $(\mathbb{R}^3, g)$ is stable minimal hypersurface if and only if for every smooth compact support function η over $\widetilde{\Sigma}$ we get

$$\int_{\widetilde{\Sigma}} \langle \tilde{L}\eta, \eta\rangle \le 0, \tag{4.1.2}$$

where

$$\widetilde{L}\eta = \widetilde{\Delta}\eta + |\widetilde{A}|^2\eta + \widetilde{Ric}_{\mathbb{R}^3}(\upsilon, \upsilon)\eta. \tag{4.1.3}$$

Note that the Ricci curvature with respect to the conformal metric g is

$$\begin{aligned}
\widetilde{Ric}_{\mathbb{R}^3}(i,j) &= Ric_{\mathbb{R}^3}(i,j) - \frac{1}{2}\nabla_i\partial_j x_3 + \frac{1}{4}(\partial_i x_3)(\partial_j x_3) + (\frac{1}{2}\Delta x_3 - \frac{1}{4}|\nabla x_3|^2)\delta_{ij} \\
&= 0. \qquad (4.1.4)
\end{aligned}$$

Also, we have

$$\widetilde{\Delta}\eta = e^{-x_3}(\Delta\eta - \nabla^k x_3 \nabla_k \eta) = e^{-x_3}(\Delta\eta + \langle e_3, \nabla\eta\rangle), \tag{4.1.5}$$

and the norm of second fundamental form respect to the conformal metric g is

$$|\widetilde{A}|^2 = e^{-x_3}(|A|^2 - \frac{H^2}{2}), \tag{4.1.6}$$

because $H = \langle e_3, \upsilon\rangle$ and

$$\begin{aligned}\tilde{h}_{ij} &= e^{-\frac{x_3}{2}} h_{ij} - e^{-x_3}\frac{\partial}{\partial\widetilde{\upsilon}} e^{\frac{x_3}{2}}\delta_{ij} \\ &= e^{-\frac{x_3}{2}}\left(h_{ij} - \frac{H}{2}\delta_{ij}\right),\end{aligned} \tag{4.1.7}$$

where $\widetilde{\upsilon}$ is the upward unit normal of $\widetilde{\Sigma}$.

Now we are defining the second order operator L by

$$Lu = \Delta u + |A|^2 u + \langle e_3, \nabla u\rangle.$$

Definition 4.1.2 *We say a translating surface is L-stable, if for any compactly supported function η we have*

$$\int_\Sigma \eta L\eta e^{x_3} \leq 0. \tag{4.1.8}$$

The linear operator L is associated to normal perturbations of $H + \langle e_3, \upsilon\rangle$. The function $H + \langle \vec{e}_3, \upsilon\rangle$ is invariant under translations in $\mathbb{R}^2$, therefore $\langle \mathbf{v}, \upsilon\rangle$ is in the kernel of L for any constant vector $\mathbf{v}$.

Proposition 4.1.3 *If the translating graph Σ in $\mathbb{R}^3$ with respect to the Euclidean metric is L-stable, then it is a stable minimal hypersurface in $\mathbb{R}^3$ with respect to the conformal metric $g_{ij} = e^{x_3}\delta_{ij}$.*

Proof. Let η be a smooth compactly supported function over Σ equations (4.1.3), (4.1.4), (4.1.5) and (4.1.6) gives

$$\widetilde{L}\eta = e^{-x_3}\left(\Delta\eta + |A|^2\eta - \frac{H^2}{2}\eta + \langle e_3, \nabla\eta\rangle\right). \tag{4.1.9}$$

This implies

$$\begin{aligned} \int_{\tilde{\Sigma}} \langle \tilde{L}\eta, \eta\rangle &= \int_{\Sigma} \eta\widetilde{L}\eta e^{2x_3} \\ &= \int_{\Sigma} e^{x_3}\left(\eta\Delta\eta + |A|^2\eta^2 - \frac{H^2}{2}\eta^2 + \eta\langle e_3, \nabla\eta\rangle\right) \\ &\leq \int_{\Sigma} e^{x_3}\eta\left(\Delta\eta + |A|^2\eta + \langle e_3, \nabla\eta\rangle\right) \\ &= \int_{\Sigma} \eta L\eta e^{x_3}. \end{aligned} \tag{4.1.10}$$

Now the result follows from (4.1.10), the definition of stability for minimal surfaces (4.1.2) and definition of L-stability (4.1.8). ∎

Lemma 4.1.4 *For every constant vector* $\mathbf{v}$*, we have* $L\langle \mathbf{v}, \upsilon\rangle = 0$.

Proof. We give a computational proof. Let γ_i be an orthonormal frame for Σ and set $\xi = \langle \mathbf{v}, \upsilon\rangle$. Working at a fixed point P and choosing the frame γ_i, so that $\nabla^T_{\gamma_i}\gamma_j(P) = 0$, differentiating gives at P that

$$\nabla_{\gamma_i}\xi = \langle \mathbf{v}, \nabla_{\gamma_i}\upsilon\rangle = -a_{ij}\langle \mathbf{v}, \gamma_j\rangle. \tag{4.1.11}$$

Using Codazzi equation at P, we have

$$\nabla_{\gamma_k}\nabla_{\gamma_i}\xi = -a_{ik,j}\langle \mathbf{v}, \gamma_j\rangle - a_{ij}\langle \mathbf{v}, a_{jk}\upsilon\rangle.$$

Taking the trace gives

$$\Delta\xi = \langle \mathbf{v}, \nabla H\rangle - |A|^2\xi. \tag{4.1.12}$$

Notice that

$$\begin{aligned}\nabla_{\gamma_i} H &= -\nabla_{\gamma_i}\langle e_3, \upsilon\rangle \\ &= -\langle \nabla_{\gamma_i} e_3, \upsilon\rangle - \langle e_3, \nabla_{\gamma_i}\upsilon\rangle \\ &= a_{ij}\langle e_3, \gamma_j\rangle.\end{aligned}$$

Therefore, using (4.1.11) we have

$$\langle \nabla H, \mathbf{v}\rangle = a_{ij}\langle e_3, \gamma_j\rangle\langle \gamma_i, \mathbf{v}\rangle = -\langle e_3, \nabla\xi\rangle. \tag{4.1.13}$$

Thus, by (4.1.12) and (4.1.13) we get

$$L\xi = \Delta\xi + \langle \nabla\xi, e_3\rangle + |A|^2\xi = 0. \tag{4.1.14}$$

■

Theorem 4.1.5 *Translating graphs in* $\mathbb{R}^3$ *are L-stable.*

Proof. Since Σ is a graph there is a unit vector $\mathbf{v}$ in $\mathbb{R}^3$ so that $\langle \mathbf{v}, \upsilon(x)\rangle \neq 0$ for all $x \in \Sigma$. We define the function ξ on Σ by

$$\xi(x) = \langle \mathbf{v}, \upsilon(x)\rangle.$$

It follows that $0 < \xi \leq 1$ and, by Lemma 4.1.4, that $L\xi = 0$. Given any smooth compactly supported function η on Σ, the function $\phi = \eta\xi$ satisfies

$$\begin{aligned} L(\phi) &= \eta L\xi + \xi\left(\Delta\eta + \langle e_3, \nabla\eta\rangle\right) + 2\langle\nabla\eta, \nabla\xi\rangle \\ &= \xi\left(\Delta\eta + \langle e_3, \nabla\eta\rangle\right) + 2\langle\nabla\eta, \nabla\xi\rangle. \end{aligned} \tag{4.1.15}$$

Using Stokes' theorem with $\frac{1}{2}\mathrm{div}\left(\xi^2\nabla\eta^2 e^{x_3}\right)$, we obtain

$$\int \frac{1}{2}\langle\nabla\eta^2, \nabla\xi^2\rangle e^{x_3} = -\int \xi^2\left(\eta\Delta\eta + |\nabla\eta|^2 + \eta\langle e_3, \nabla\eta\rangle\right) e^{x_3}. \tag{4.1.16}$$

Applying (4.1.15) and (4.1.16), we obtain

$$\int \phi L(\phi) e^{x_3} = -\int \xi^2 |\nabla\eta|^2 e^{x_3} \leq 0. \tag{4.1.17}$$

When Σ is graphical, we have a direction ω for which $\xi > 0$ for all $x \in \Sigma$. Given a smooth compactly supported function ϕ, we take $\eta(x) := \phi(x)/\xi(x)$. This means that (4.1.17) is true for any compactly supported function ϕ, which is the definition of L-stability. ∎

Remark 4.1.6 *Since translating graphs are L-stable, compactness theorem for stable minimal graphs in $\mathbb{M}^3$, where $\mathbb{M}^3$ is a three dimensional Riemannian manifold, implies compactness theorem for translating graphs in $\mathbb{R}^3$ with polynomial volume growth.*

4.2 Curvature estimate

In this section we obtain a pointwise curvature bound for translating graphs by mean curvature in $\mathbb{R}^3$. First, we state Theorem 2.10 in (6) which is Schoen curvature estimate for two dimensional minimal hypersurfaces Σ immersed in Reiemannian Manifold M^3 with sectional curvature K_M(32). For $x \in M$, $B_s(x)$ denotes the extrinsic geodesic ball with radius s and center x. Similarly, For $x \in \Sigma$, $B_s^\Sigma(x) \subset \Sigma$ denotes the intrinsic geodesic ball and r the intrinsic distance to x.

Theorem 4.2.1 *(Schoen Curvature estimate (32), Colding-Minicozzi (6)) If* $\Sigma^2 \subset M^3$ *is an immersed stable minimal surface with trivial normal bundle and* $B_{r_0} = B_{r_0}^\Sigma(x) \subset \Sigma \setminus \partial\Sigma$, *where* $|K_M| \leq k^2$ *and* $r_0 < \rho_1(\pi/k, k)$ *(with* $\rho_1 < min\{\pi/k, k\}$*), then for some* $C = C(k)$ *and all* $0 < \sigma \leq r_0$,

$$\sup_{B_{r_0-\sigma}^\Sigma} |A|^2 \leq C\sigma^{-2}. \tag{4.2.1}$$

For applying this theorem to obtain the curvature estimate, we need to compute the sectional curvature of $\mathbb{R}^3$ respect to the conformal metric g. By doing some simple computations, we get for every $1 \leq i, j \leq 2$, $K_{ij} = \frac{R_{ijij}}{g_{ij}g_{ij}} = R_{ijij}e^{-2x_3} = -\frac{1}{4}e^{-x_3}$ and $K_{i3} = K_{3i} = 0$, where

$$R_{ijij} = e^{x_3}\left(\sum_l \Gamma_{ii}^l \Gamma_{jl}^j - \sum_l \Gamma_{ji}^l \Gamma_{il}^j + \frac{\partial}{\partial x_j}\Gamma_{ii}^j - \frac{\partial}{\partial x_i}\Gamma_{ji}^j\right), \tag{4.2.2}$$

and Γ_{ij}^l are Christoffel symbols of $(\mathbb{R}^3, g)$.

Theorem 4.2.2 *Let* $\Sigma^2 \subset \mathbb{R}^3$ *be a complete translating graph in mean curvature flow, if* $B_{r_0e}^\Sigma(p) \subset (\Sigma \cap B_1(p)) \setminus \partial(\Sigma \cap B_1(p))$, *and* $r_0e^{1/2} < \rho_1(\pi e^{-1}, e)$, *then for some* C *and all* $0 < \sigma \leq r_0$,

$$\sup_{B_{r_0-\sigma}^\Sigma} |A|^2 \leq C\sigma^{-2}. \tag{4.2.3}$$

Proof. For point $p \in \Sigma$, let B_1 be the Euclidean unit ball of radius 1 and center p in $\mathbb{R}^3$. Define $\hat{\Sigma} = \Sigma \cap B_1$, note that $\hat{\Sigma}$ is immersed submanifold in $B_1 = B_1(0)$ and $|\hat{A}|(p) = |A|(p)$. Now let $\tilde{B}_1$ be B_1 with respect to the metric $g_{ij} = e^{x_3 - p_3}\delta_{ij}$. From theorem 4.1.5, $\hat{\Sigma}$ is stable minimal hypersurface in $\tilde{B}_1$. Note that we only multiplied the metric g_{ij} in the theorem by a constant e^{-p_3}, which wont change the results of theorem.

Let the distance d be the distance corresponding to Euclidean metric and $\tilde{d}$ the distance corresponding to the conformal metric. Note that $B^{\tilde{\Sigma}}_{r_0} = \{x \in \Sigma : \ \tilde{d}(x,p) < r_0\}$, where $\tilde{d}(x,p)$ is the infimum of the length of geodesic curves connecting two points p and x in $\tilde{\Sigma}$. For $x = (x_1, x_2, x_3) \in \Sigma \cap B_1$, define minimizing geodesic $\gamma : [0,1] \to \Sigma \cap B_1$ in Euclidean metric, connecting p and x in Σ, such that $\gamma(0) = p$ and $\gamma(1) = x$. Using Cauchy Schwartz inequality, we have

$$\begin{aligned}
\tilde{d}(x,0) &\leq \int_0^1 ||\gamma'(t)||dt \\
&= \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t)\rangle} dt \\
&= \int_0^1 e^{\frac{\gamma_3(t) - p_3}{2}} |\gamma'(t)| dt \\
&\leq \left(\int_0^1 e^{\gamma_3(t) - p_3} dt\right)^{\frac{1}{2}} \left(\int_0^1 \gamma'^2(t) dt\right)^{\frac{1}{2}} \\
&= e^{1/2} d(x,0). \qquad (4.2.4)
\end{aligned}$$

This implies if $x \in B^{\Sigma}_r(p) \subset B_1$, then $x \in B^{\tilde{\Sigma}}_{re^{1/2}}(p)$. Now define minimizing geodesic $\tilde{\gamma} : [0,1] \to \tilde{\Sigma} \cap \tilde{B}_1$ connecting p and x in $\tilde{\Sigma} \cap \tilde{B}_1$, such that $\gamma(0) = 0$ and

$\gamma(1) = x$. Using Cauchy Schwartz inequality, we have

$$\begin{aligned} e^{-1/2} d(x,0) &\leq \int_0^1 e^{-1/2} |\tilde{\gamma}'(t)| dt \\ &\leq \int_0^1 e^{(\tilde{\gamma}_3(t) - p_3)/2} |\tilde{\gamma}'(t)| dt \\ &= \int_0^1 \sqrt{\langle \tilde{\gamma}'(t), \tilde{\gamma}'(t) \rangle} dt \\ &= \int_0^1 ||\gamma'(t)|| dt = \tilde{d}(x,0). \end{aligned} \tag{4.2.5}$$

This implies if $x \in B_r^{\tilde{\Sigma}}(p) \subset \tilde{B}_1$, then $x \in B^{\Sigma}_{re^{1/2}}(p)$. Note that if $x \in B^{\tilde{\Sigma}}_{r_0 e^{1/2}}(p)$, then $X \in B^{\Sigma}_{r_0 e}(p) \subset (\Sigma \cap B_1(p)) \setminus \partial(\Sigma \cap B_1(p))$ which implies that $x \in \hat{\Sigma} \setminus \partial \hat{\Sigma}$. Also if $x \in B^{\Sigma}_{r_0 - \sigma}$, then $x \in B^{\tilde{\Sigma}}_{e^{1/2}(r_0 - \sigma)}$.

Since sectional curvature of $\tilde{B}_1$ is bounded ($|K_{\tilde{B}_1}| < e$), formula (4.1.6) and theorem 4.2.1 imply that for $r_0 e^{1/2} < \rho_1(\pi e^{-1}, e)$ and $B^{\tilde{\Sigma}}_{r_0 e^{1/2}}(p) \subset \hat{\Sigma} \setminus \partial \hat{\Sigma}$, for some C we obtain for all $x \in B^{\Sigma}_{r_0 - \sigma}(p)$,

$$|A|^2(x) \leq e^{x_3 - p_3} |\tilde{A}|^2(x) \leq Ce(e^{1/2}\sigma)^{-2} = C\sigma^{-2}. \tag{4.2.6}$$

■

Chapter 5

Classification of translating graphs in $\mathbb{R}^3$

In this chapter, we prove complete translating graphs in $\mathbb{R}^3$ can only be an entire graph over $\mathbb{R}^2$ or can be in one side of a vertical plane or between two vertical parallel planes. In the last two types, graphs are asymptotic to planes next to their boundaries.

5.1 Complete translating graphs in $\mathbb{R}^3$

In this section, we prove translating graphs over a smooth connected domain $\Omega \subset \mathbb{R}^2$ are asymptotic to a minimal surface next to the $\partial\Omega$, which implies non-existence of translating graphs over a bounded domain. For reaching this goal, we need to prove the following lemma.

Lemma 5.1.1 *If $\Sigma \subset \mathbb{R}^3$ is a translating graph, then there is a $\delta > 0$ such that for every $p \in \Sigma$, Σ is a graph over the disk $D_\delta(p) \subset T_p\Sigma$ of radius δ centered at p.*

Proof. We prove this lemma in two different ways.

First, we prove by contradiction, assume that there is a sequence of points $p_n \in \Sigma$ so that Σ is a graph over the disk $D_{\delta_n}(p_n) \subset T_{p_n}\Sigma$, and $\delta_n \to 0$. For fix $R > 0$, we define $f_n = B_r(p_n) \cap \Sigma$. Now we translate each f_n so that p_n goes to the origin, we call this graph g_n. Each g_n is a translating graph in the ball $B_r(0) \subset \mathbb{R}^3$, we choose a subsequence of g_n so that the tangent planes of g_n at origin converges to some plane at origin. Compactness theorem for translating graphs implies that there is a subsequence of g_n which are converging to a translating surface g_∞. For $0 \in g_\infty$, there is a $\delta > 0$ so that g_∞ is a graph over the disk $D_\delta(0) \subset T_0 g_\infty$. Thus, there is an N large so that g_n is a graph over the disk $D_\delta(0) \subset T_0 g_n$, for every $n > N$.

Second way:

Let p and q be two different point in Σ. There is a geodesic $\gamma : [0,1] \to \Sigma$, so that $\gamma(0) = p$ and $\gamma(1) = q$. Now for υ normal vector to the Σ we have

$$\begin{aligned} |\upsilon(p) - \upsilon(q)| &\leq \int_0^1 |\nabla_{\gamma'} \upsilon(\gamma(t))| dt \\ &= \int_0^1 |A(\gamma(t))| dt. \end{aligned} \tag{5.1.1}$$

Hence the theorem 4.2.2 implies the lemma. ∎

Theorem 5.1.2 *There is no complete translating graph $\Sigma \subset \mathbb{R}^3$ with nonzero constant speed C over a bounded connected domain $\Omega \subset \mathbb{R}^2$ with smooth boundary.*

Proof. The proof inspired by the one used in (16). Suppose Σ is a complete immersed translating graph over a domain $\Omega \subset \mathbb{R}^2$. Lemma 5.1.1 implies there exists $\delta > 0$ such that for each $p \in \Sigma$, Σ is a graph in exponential coordinates over the disk $D_\delta(p) \subset T_p\Sigma$ of radius δ, centered at the p. We denote this graph by $G(p) \subset \Sigma$, has bounded geometry. Note that δ is independent of p and the bound on the geometry of $G(p)$ is uniform as well.

We define $F(p)$; the surface $G(p)$ translated to height zero $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\}$, i.e, let α_p be the isometry of $\mathbb{R}^3$ which takes p to $\pi(p)$, we denote $F(p) = \alpha_p(\Sigma)$.

Now, let $p \in \Sigma$, since Σ is a graph over Ω, there is a function $u : \Omega \to \mathbb{R}^3$ so that Σ is the graph of u. If Σ is not an entire graph then $\partial\Omega \neq \emptyset$. Since Σ is a translating graph by mean curvature flow, u has bounded gradient on relatively compact subsets of Ω. Let $q \in \partial\Omega$ be such that u does not extend to any neighborhood of q.

Let q_n be a sequence in Ω converging to q, and let $p_n = (q_n, u(q_n)) \in \Sigma$ be images of q_n in Σ. Let F_n denote the image of $G(p_n)$ under the vertical translation taking p_n to q_n. Observe that $T_{q_n}(F_n)$ converges to the vertical plane P, for any subsequence of the q_n. Otherwise the graph of bounded geometry $G(p_n)$, would extend to a vertical graph beyond q, for q_n close enough to q. Hence f would extend; a contradiction.

For $q \in \Omega$, we define $L_\delta(q)$ a line of length 2δ centered at q. Let $L_\delta(q)$ be the line whose normal vector has the same direction as the normal vector of limit normal vectors of F_n. Since each F_n is a graph over $D_\delta(p_n) \subset T_{p_n}(F_n)$, the surfaces F_n are bounded horizontal graphs over $L_\delta(q) \times [-\delta, \delta]$ for n large. The compactness theorem for translating graphs implies that there is a subsequence of F_n's which are converging to a translating surface F. The surface F is tangent to $L_\delta(q) \times [-\delta, \delta]$ at q. Note that $F = L_\delta(q) \times [-\delta, \delta]$. Because if it is not the case there is a small $\epsilon > 0$ so that $F(q - \epsilon\vec{n}(q))$ has two positive and negative values, where $\vec{n}$ is the unit normal to the $L_\delta(q)$. Therefore for n large F_n is not a graph, which is contradiction.

The plane $P = L_\delta(q) \times [-\delta, \delta]$, because both planes P and $L_\delta(q) \times [-\delta, \delta]$ are passing through the point $q \in \partial\Omega$ and their normal vectors are the same.

In this point, we prove $u(q_n) \to +\infty$ or $u(q_n) \to -\infty$. Let l be a line of length ϵ inside Ω, starting at q, orthogonal to $\partial\Omega$ at q. Let f be the graph of u over l. At points near q, l has no horizontal tangents, because tangent planes of u at these points are converging to P. So we assume u is increasing along l as one converges to q. If u is bounded above, then f would have a finite limit point (q, l_q) and f would have finite length up till (q, l_q). Since Σ is complete, $(q, l_q) \in \Sigma$, which contradicts by Σ has a vertical tangent plane at (q, l_q).

Note that from Lemma 4.1.4, $1/w$ satisfies an elliptic partial differential equation. Thus by the Harnack inequality, for any sequence $q_n \in \Omega$ converging to q we have $w(q_n) \to +\infty$. That means $H(q_n) \to 0$.

Which is contradiction, since the domain is bounded, the mean curvature of the graph next to the boundary should converge to the mean curvature of the cylinder

$\partial\Omega \times \mathbb{R}$, which is not zero. ■

Corollary 5.1.3 *If Σ is a complete translating graph over a domain $\Omega \subset \mathbb{R}^2$, then next to the boundary of Ω, Σ is asymptotic to a plane. So a translating graph over $\mathbb{R}^2$ can only be between 2 parallel planes or in one side of a plane or an entire graph.*

Proof. From the proof of Theorem 5.1.2, next to the boundary, Σ converges to a minimal surface. Since Σ is complete, it can only converge to a vertical plane. ■

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