THE ASYMPTOTIC DISTRIBUTION OF NODAL SETS ON SPHERES

by

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Abstract

A nodal set is the zero locus of an eigenfunction of the Laplacian. We show that for the standard unit sphere in Euclidean space of dimension at least three, certain measures supported on nodal sets of normalized eigenfunctions, chosen uniformly and at random, converge to the usual Lebesgue measure on the sphere as the corresponding eigenvalues increase without bound; moreover, the exact nature of this convergence is discussed.

We also show that the supremum norms of random eigenfunctions chosen from the Nth eigenspace grow only on the order of the square root of the logarithm of N. Finally, we compute the scaling limit of the two point correlation functions for the measures mentioned above.

To perform the above computations, we must develop the relationship between the eigenspace projection map, defined by duality with the evaluation map, and the value distribution of eigenfunctions. We also develop a method of describing the gradient of an eigenfunction geometrically in terms of the differential of the eigenspace projection map.

READERS: Steve Zelditch (Advisor) and Bernard Shiffman.

To Mary Allison,

without whom this work

would have been lifeless;

and to my Muse,

for staying with me.

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Chapter 1

Introduction

The study of spectral theory is concerned with the relationship between the geometry of a manifold and the eigenvalues and eigenfunctions of its Laplacian. The standard unit sphere in Euclidean space has long held a privileged place in the theory, due in part to the relative ease with which one can compute both its eigenvalues and eigenfunctions. Yet the sphere still contains a rich structure that lies just beneath the surface of what is known, and invites us to ask subtle questions about the eigenfunctions of any compact real manifold.

We begin our discussion by recalling that the Laplacian is an elliptic linear operator. By the Hodge Theorem [20], the space of square integrable functions on the sphere admits a direct sum decomposition into the (complex) eigenspaces of the Laplacian. On the spheres of dimension at least 2, these eigenspaces have increasing dimension as we allow the corresponding eigenvalues to increase, a property which is not true for arbitrary manifolds.

For quite some time, people have been concerned with studying the zero sets, called *nodal sets*, of these eigenfunctions [5]. Let's consider for a moment the simplest sphere, namely the unit circle S^1 in the plane. The Laplacian on S^1 takes the form

$$\Delta = -\frac{\partial^2}{\partial\theta^2},$$

and its real eigenfunctions are of the form

$$a \cos N\theta + b \sin N\theta,$$

for $N = 0, 1, \ldots,$ and a, b real. One can easily check that each such non-zero eigenfunction with $N \geq 1$ vanishes at exactly $2N$ distinct, equally spaced points on the circle. So what can we say about nodal sets of higher-dimensional spheres? Yau suggested in problem #74 of [30] that the hypersurface Hausdorff measure of nodal sets on compact manifolds grows like the square root of the associated eigenvalue, which is known in the literature as Yau's Conjecture. Donnelly and Fefferman have proven Yau's Conjecture in the case of compact real-analytic manifolds [8]—a class which includes the spheres. Also of interest, Berard computed in [2] the average hypersurface measure of nodal sets on rank-one symmetric spaces, thereby verifying an analogue of Yau's conjecture *"en moyenne"*.

It's not immediately clear what generalizations of the "equal spacing" of nodal points on S^1 can be made to higher dimensional spheres. In an attempt to provide an answer, we make the following observation: suppose one picks a sequence of non-zero eigenfunctions $f_1, f_2, \ldots,$ on S^1, with

$$\Delta f_N = N^2 f_N.$$

Let $\theta_1^N, \ldots, \theta_{2N}^N$ be the zeros of f_N. By the definition of the Riemann integral, we have for any continuous φ on S^1,

$$\frac{1}{2N} \sum_{j=1}^{2N} \varphi(\theta_j^N) \to \int_{S^1} \varphi(\theta)\, d\theta,$$

as $N \to \infty$. In words, we say that unit mass measures placed on the nodal sets of the f_N converge weakly to the Lebesgue measure $d\theta$. This is the idea of "equal spacing" we'll try to generalize in this work.

To get another view of how such a notion of equal spacing arises, we mention that in [23], Shiffman and Zelditch computed the average volume of zero sets of holomorphic sections of high powers $L^{\otimes N}, N \gg 1$, of a positive line bundle L over a compact complex manifold. They showed that sequences of natural measures supported on the zeros of random sections converge weakly to the curvature form of the line bundle in the limit $N \to \infty$. Zelditch then asked the question: does such a "uniformity in the limit" theorem hold for measures placed on nodal sets of the sphere as well? He conjectured that it did, motivated by the following analogy: on complex projective space, holomorphic sections of powers of the hyperplane bundle can be identified with homogeneous (holomorphic) complex polynomials. On the sphere, eigenfunctions can be identified with homogeneous harmonic polynomials. In essence, studying nodal sets is the real analogue of studying zero sets of such holomorphic sections.

As a final note, one could also approach this work from the study of random polynomials. The interested reader is invited to peruse the references found in [9] for a list of works devoted to the study of zeros of random polynomials in one variable. As mentioned above, the present work is as an attempt to understand the zeros of a particular type of random multi-variable polynomial—the homogeneous harmonic polynomials.

Chapter 2

Statement of Results

Let S^m, $m \geq 2$, denote the unit sphere in $\mathbb{R}^{m+1}$, equipped with the standard round metric. Denote by Δ the Laplacian on smooth functions. It is well known that the eigenvalue problem

$$\Delta f = \lambda f \tag{2.1}$$

has non-trivial solutions for λ given by

$$\lambda_N = N(N+m-1), \tag{2.2}$$

with N a non-negative integer. We are interested in studying the distribution of zero sets of random real eigenfunctions as $\lambda_N \to \infty$. To explain our results, we set some notation.

Denote by E_N the eigenspace of all real solutions of (2.1) with $\lambda = \lambda_N$ given as in (2.2). Classically, E_N is the vector space of all homogeneous, harmonic, real polynomials of degree N in $\mathbb{R}^{m+1}$, restricted to S^m. E_N is an inner product space under the usual L^2-inner product

$$\langle f, g\rangle_{L^2} = \int_{S^m} f(x)g(x)\, d\sigma_m(x).$$

Here σ_m is the usual Lebesgue measure on S^m. Let $d_N \stackrel{\text{def}}{=} \dim E_N$. It is known [12] that

$$d_N = \frac{2N+m-1}{N+m-1}\binom{N+m-1}{m-1} = \frac{2}{(m-1)!}N^{m-1}(1+o(1)).$$

Here $o(1)$ represents an error term which decays to 0 as $N \to \infty$.

Let SE_N be the unit sphere in E_N, consisting of all eigenfunctions with L^2-norm 1. By identifying SE_N with S^{d_N-1}, it inherits the Lebesgue measure μ_N, which we normalize to be a probability measure:

$$\mu_N(SE_N) = 1.$$

The term 'random' will always be with respect to μ_N. Let $SE_\infty = \prod_{N=0}^{\infty} SE_N$, $\mu_\infty = \prod_{N=0}^{\infty} \mu_N$.

Our technique for studying the zero sets of eigenfunctions is to associate to each one a measure. We then try to understand the weak limit of these measures as $\lambda_N \to \infty$. There are two choices of natural measures which will interest us in this work. The first is defined as follows: it is known [5] that for any $f \in E_N$, its *nodal set* $\mathrm{Zero}(f) \stackrel{\mathrm{def}}{=} f^{-1}(0) \subset S^m$ is an $(m-1)$-dimensional submanifold apart from a closed set of smaller Hausdorff dimension. More precisely, there exists a closed subset Ξ_f of $\mathrm{Zero}(f)$, with Hausdorff dimension less than $m-1$, such that $\mathrm{Zero}(f) \setminus \Xi_f$ is an embedded submanifold of S^m. As such, $\mathrm{Zero}(f)$ inherits the Riemannian hypersurface measure Z_f^N, considered as a measure on S^m.

The second measure is given for those $f \in E_N$ with no singular points by

$$\delta_f^N = f^*\delta_0,$$

[19] where δ_0 is the unit measure concentrated at the origin in $\mathbb{R}$. We will refer to δ_f^N as the Léray nodal measure. From the Coarea Formula (3.2.12 of [10]), we can derive the following relationship:

$$d\delta_f^N = \frac{1}{|\nabla f|} dZ_f^N.$$

We note here that Z_f^N is homogeneous of degree 0, whereas δ_f^N is homogeneous of degree -1. Moreover, while Z_f^N does depend on the metric chosen for S^m, it does

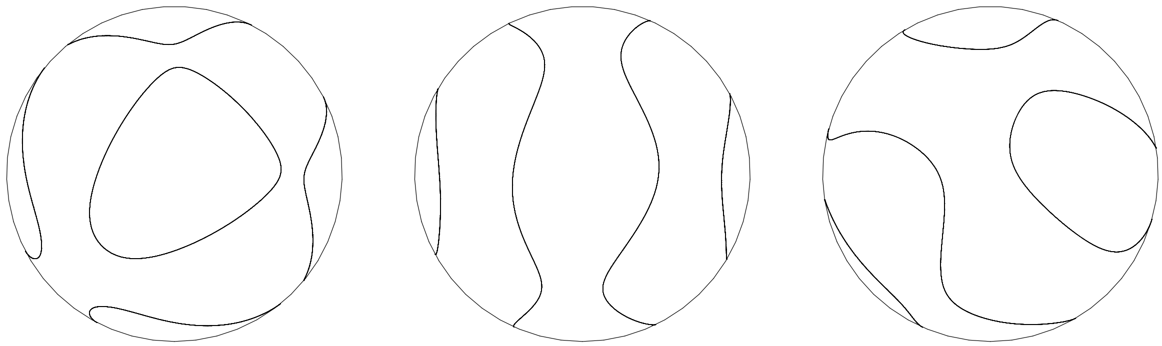

Figure 2.1: Some Nodal Sets With $N = 4, \lambda_N = 20$

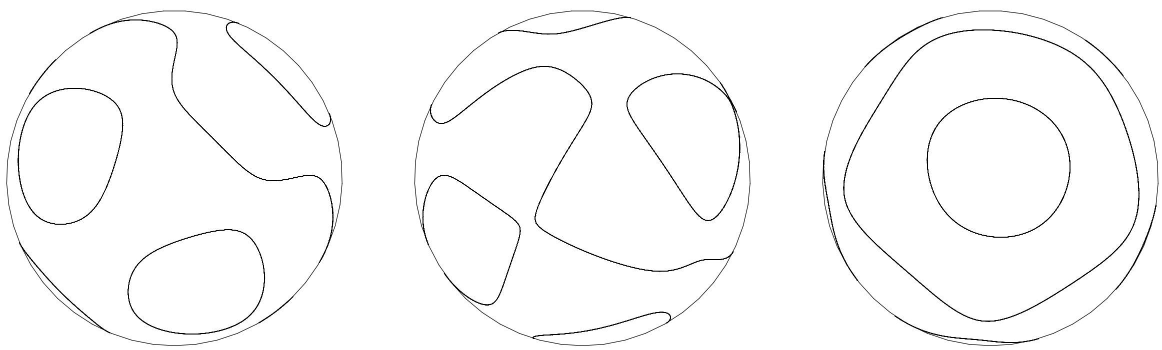

Figure 2.2: Some Nodal Sets With $N = 5, \lambda_N = 30$

not depend on the the function f defining Zero(f). On the other hand, δ_f^N depends heavily on f, but not on the metric chosen for S^m. For some examples of nodal sets, see Figures 2.1, 2.2, 2.3, and 2.4.

Our impetus for taking this approach comes from the work of Shiffman and Zelditch [23], where they are able to show that zero sets of random sequences of sections of high powers of a positive line bundle over a compact complex manifold become uniformly distributed. The interested reader is invited to read [3, 4], and note that Lemmas 6.1 and 7.2 of the present work can be obtained as corollaries of Theorem 2.2 of [4].

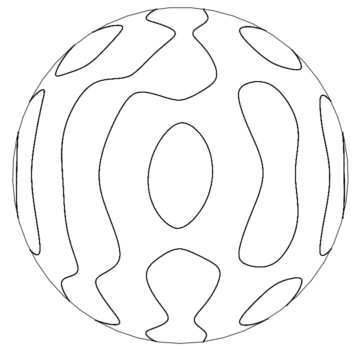

Figure 2.3: A Nodal Set With $N = 10, \lambda_N = 110$

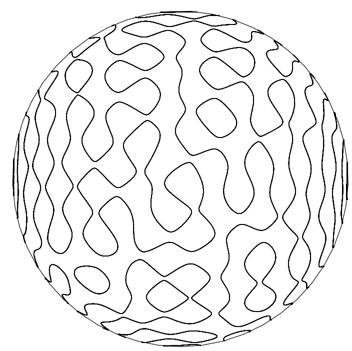

Figure 2.4: A Nodal Set With $N = 21, \lambda_N = 462$

Given a probability space (Ω, ω), we will denote the expected value and variance of a random variable X, respectively, by

$$\begin{aligned} E_\omega X &= \int_\Omega X(x)\, d\omega(x), \\ V_\omega X &= \int_\Omega (X(x) - E_\omega X)^2\, d\omega(x) \\ &= \int_\Omega X^2(x)\, d\omega(x) - (E_\omega X)^2, \end{aligned}$$

provided the integrals are finite. We will also have occasion to use the notation

$$(d\mu, \varphi) = \int_{S^m} \varphi(x)\, d\mu(x)$$

for a measure μ and continuous function φ.

We will begin by showing the following proposition:

Proposition 2.1. *The expected values of Z_f^N and δ_f^N as distributions are proportional to the volume measure σ_m. Explicitly, for all continuous functions φ on S^m, we have*

$$E_{\mu_N}(Z_f^N, \varphi) = \frac{|S^{m-1}|}{|S^m|}\sqrt{\frac{\lambda_N}{m}}\int_{S^m} \varphi(x)\, d\sigma_m(x), \tag{2.3}$$

$$E_{\mu_N}(\delta_f^N, \varphi) = \frac{|S^{d_N-2}|}{|S^{d_N-1}|}\sqrt{\frac{|S^m|}{d_N}}\int_{S^m} \varphi(x)\, d\sigma_m(x). \tag{2.4}$$

Equality (2.3) was shown in [2] with $\varphi \equiv 1$ using integral geometry. For reference, we recall the volume formula for spheres:

$$|S^m| = \frac{2\pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}\right)},$$

where Γ signifies, as usual, the Euler gamma function. Thus, while the constant on the right-hand side of (2.3) grows like N, the constant on the right-hand side of (2.4) is

$$\sqrt{\frac{|S^m|}{2\pi}} + o(1).$$

This proposition allows us to define normalized nodal measures $\widetilde{Z}_f^N$ and $\tilde{\delta}_f^N$ by requiring that

$$E_{\mu_N}(\widetilde{Z}_f^N, \varphi) = E_{\mu_N}(\tilde{\delta}_f^N, \varphi) = \int_{S^m} \varphi(x)\, d\sigma_m(x).$$

We now come to our main theorems. Recall that we say a sequence of measures $\{\omega_N\}$ defined on a compact measure space Ω converges weakly to a measure ω on Ω as $N \to \infty$ if given any continuous function φ on Ω,

$$\lim_{N\to\infty} \int_\Omega \varphi\, d\omega_N = \int_\Omega \varphi\, d\omega.$$

Theorem 2.2. *1. As $N \to \infty$, we have*

$$\widetilde{Z}_f^N \to \sigma_m$$

almost surely in the sense of Cesàro.

By this we mean that for μ_∞-almost all sequences of random normalized eigenfunctions $\{f_N\}_{N\geq 0}$, with $f_N \in SE_N$,

$$\frac{1}{M}\sum_{N=1}^{M} \widetilde{Z}_{f_N}^N \to \sigma_m$$

weakly as $M \to \infty$.

2. *As $N \to \infty$, we have*

$$\widetilde{Z}_f^N \to \sigma_m$$

weakly in probability.

By this we mean that for any continuous function φ on S^m, and any $\epsilon > 0$,

$$\mu_N \left\{ f \in SE_N : \left| \int_{\mathrm{Zero}(f)} \varphi \, d\widetilde{Z}_f^N - \int_{S^m} \varphi \, d\sigma_m \right| > \epsilon \right\} \to 0$$

as $N \to \infty$.

3. *Suppose $m \geq 6$. Then as $N \to \infty$, we have*

$$\widetilde{Z}_f^N \to \sigma_m$$

almost surely in the sense of weak convergence.

By this we mean that for μ_∞-almost all sequences of random normalized eigenfunctions $\{f_N\}_{N\geq 0}$, with $f_N \in SE_N$,

$$\widetilde{Z}_{f_N}^N \to \sigma_m$$

weakly as $N \to \infty$.

Theorem 2.3. *Parts 1 and 2 of Theorem 2.2 hold with $\widetilde{Z}_f^N$ replaced by $\tilde{\delta}_f^N$. Part 3 holds with $\widetilde{Z}_f^N$ replaced by $\tilde{\delta}_f^N$ for $m \geq 4$.*

Theorems 2.2 and 2.3 tell us that the measures Z_f^N and δ_f^N become evenly distributed with respect to the volume measure in the limit $N \to \infty$. What changes among the parts of the theorems is the method of convergence. We mention here that even though the proofs presented in this work are not strong enough to prove a.s. weak convergence for all m, we conjecture that it does hold.

We will prove Theorem 2.2 in section 8.1, then explain the necessary modifications to prove Theorem 2.3 in section 8.2. Before we do that, we will first describe the relationship between the values of eigenfunctions and their eigenspace in chapter 3. As a corollary of this relationship, we will derive L^∞ bounds for sequences of random eigenfunctions in chapter 4. In chapter 5 we will discuss some smoothness properties of nodal sets, and in chapters 6 and 7 we will compute the expected value and variance of sequences of random nodal measures. We will then examine the scaling limit of the two point correlation functions in chapter 9.

Chapter 3

Eigenspace Geometry

For all $N > 0$, we now define the eigenspace map $\Phi_N : S^m \to E_N$. Consider the following diagram:

$$\begin{tikzcd} & E_N^* \\ S^m \arrow[ur, "\mathrm{ev}"] \arrow[r, "\Phi_N"'] & E_N \arrow[u, "\iota"', "\cong"] \end{tikzcd}$$

Here ι is the duality isomorphism defined by

$$\iota(f)(g) = \langle f, g \rangle_{L^2},$$

and ev is the evaluation mapping given by

$$\mathrm{ev}(x)(f) = f(x).$$

We define Φ_N so as to make the above diagram commute. Thus

$$\Phi_N(x) = \left(\iota^{-1} \circ \mathrm{ev}\right)(x),$$

or

$$\langle \Phi_N(x), f \rangle_{L^2} = f(x). \tag{3.1}$$

If we choose an orthonormal basis $\{f_1, \ldots, f_{d_N}\}$ of E_N, then Φ_N is given in this basis by

$$\Phi_N(x) = (f_1(x), \ldots, f_{d_N}(x)), \tag{3.2}$$

or

$$\big(\Phi_N(x)\big)(y) = \sum_{j=1}^{d_N} f_j(y) f_j(x).$$

The eigenspace map has several useful properties that allow us to use the inner product structure of E_N to study nodal sets. In particular [9], one can check from the definitions that for all $f \in E_N$,

$$\Phi_N(\mathrm{Zero}(f)) = \Phi_N(S^m) \cap f^{\perp},$$

where $f^{\perp} = \{g \in E_N : \langle f, g \rangle_{L^2} = 0\}$.

We will also need to understand the differential—or equivalently the gradient—of an eigenfunction in terms of the eigenspace map. Fix $N > 0$. For notational convenience, we will temporarily drop the N dependence of Φ. Fix $x \in S^m$. Consider the following (non-commutative) diagram of linear maps between vector spaces:

$$\begin{tikzcd}
E_N \arrow[r, "d_x"] & T_x^* S^m \\
T_{\Phi(x)} E_N \arrow[u, "\alpha_x", "\cong"'] & T_x S^m \arrow[l, "d\Phi_x"] \arrow[u, "\cong", "\iota_{S^m,x}"']
\end{tikzcd}$$

where

$$\begin{aligned}
d_x(f) &= df_x, & f &\in E_N \\
\alpha_x(u) &= \exp_{\Phi(x)}(u) - \Phi(x), & u &\in T_{\Phi(x)} E_N, \\
\iota_{S^m,x}(v)(w) &= \langle v, w \rangle_{T_x S^m}, & v, w &\in T_x S^m.
\end{aligned}$$

We condense the above diagram to obtain the following one:

$$\begin{tikzcd}
E_N \arrow[r, bend left, "d_x"] & T_x^* S^m \arrow[l, bend left, "J_x"]
\end{tikzcd}$$

where

$$J_x = \alpha_x \circ d\Phi_x \circ \iota_{S^m,x}^{-1}. \tag{3.3}$$

It is useful to view d_x and J_x in coordinate form. To this end, fix an orthonormal basis $\{f_1, \ldots, f_{d_N}\}$ of E_N. Write $f \in E_N$ as

$$f = \sum_{j=1}^{d_N} a_j f_j. \tag{3.4}$$

Choose a coordinate neighborhood $U \subset S^m$ of x with coordinate functions $x_1, \ldots, x_m$. With these choices, d_x is given by

$$d_x(f) = \sum_{j=1}^{d_N} a_j df_j|_x,$$

and J_x is given by

$$J_x(dx_j) = \sum_{l=1}^{d_N} \sum_{h=1}^{m} g^{jh} \frac{\partial f_l}{\partial x_h}\bigg|_x f_l,$$

where $g^{jh}(x) = \langle dx_j, dx_h \rangle_{T_x^* S^m}$.

We are now able to construct the operator we are really interested in. Define $Q_x^N : E_N \to E_N$ by

$$Q_x^N = J_x \circ d_x.$$

Lemma 3.1. *For $f, g \in E_N$, we have*

$$\langle Q_x^N f, g \rangle_{L^2} = \langle df_x, dg_x \rangle_{T_x^* S^m}. \tag{3.5}$$

Moreover, by (3.5) we see that Q_x^N is a symmetric operator.

Proof. Recalling the decomposition (3.4) of f, we have

$$\begin{aligned}
\langle Q_x^N f_j, f_k \rangle_{L^2} &= \langle J_x df_j, f_k \rangle_{L^2} \\
&= \sum_{p=1}^{m} \frac{\partial f_j}{\partial x_p} \langle J_x dx_p, f_k \rangle_{L^2} \\
&= \sum_{l=1}^{d_N} \sum_{p,h=1}^{m} g^{ph} \frac{\partial f_j}{\partial x_p} \frac{\partial f_l}{\partial x_h} \langle f_l, f_k \rangle_{L^2} \\
&= \sum_{p,h=1}^{m} g^{ph} \frac{\partial f_j}{\partial x_p} \frac{\partial f_k}{\partial x_h} \\
&= \langle df_j, df_k \rangle_{T_x^* S^m}.
\end{aligned}$$

The lemma now follows by linearity. $\square$

We now want to use the symmetries of S^m to deduce information about Φ_N and Q_x^N. Recall that S^m can be identified with the homogeneous space $SO(m+1)/SO(m)$ [29], and as such $SO(m+1)$ acts transitively on S^m by isometries. This action induces an action on continuous functions by

$$g \cdot f(x) = f(g^{-1} \cdot x),$$

for $g \in SO(m+1)$, $f \in C(S^m)$, $x \in S^m$. It may be instructive to note that Φ_N is an equivariant map [7]. By this we mean the following: if we denote by v the image under Φ_N of the identity $e \in SO(m+1)$, we have

$$\Phi_N(g) = g \cdot v.$$

Recall also that σ_m is bi-invariant under the $SO(m+1)$ action. It follows that $\langle\, , \rangle_{L^2}$ is also $SO(m+1)$-invariant. Therefore, for an ordered orthonormal basis $\bar{f} = (f_1, \ldots, f_{d_N})$ of E_N, $g \cdot \bar{f}$ is also an orthonormal basis. Hence there is an orthogonal matrix $O_g = (o_{kl})$ so that $g \cdot \bar{f} = O_g \bar{f}$. We have then

$$\begin{aligned} g \cdot \sum_{j=1}^{d_N} f_j^2(x) &= \sum_{j=1}^{d_N} f_j^2(g^{-1} \cdot x) \\ &= \sum_{j=1}^{d_N} (g \cdot f_j(x))^2 \\ &= \sum_{j,k,l=1}^{d_N} o_{kl} o_{kl} f_j^2(x) \\ &= \sum_{j=1}^{d_N} f_j^2(x). \end{aligned}$$

Therefore by (3.2), $\|\Phi_N(x)\|_{L^2}$ is independent of x. In particular, Φ_N maps into a sphere. Moreover, we can compute

$$\begin{aligned} \sum_{j=1}^{d_N} f_j^2(x) &= \frac{1}{|S^m|} \sum_{j=1}^{d_N} \int_{S^m} f_j^2(x)\, d\sigma_m(x) \\ &= \frac{d_N}{|S^m|}, \end{aligned}$$

and hence

$$\|\Phi_N(x)\|_{L^2} = \sqrt{\frac{d_N}{|S^m|}}. \tag{3.6}$$

Consider now the symmetric tensor $\sum_{j=1}^{d_N} df_j \otimes df_j$. As above, we have for any $g \in SO(m+1)$,

$$\begin{aligned} g^* \sum_{j=1}^{d_N} df_j \otimes df_j &= \sum_{j=1}^{d_N} g^* df_j \otimes g^* df_j \\ &= \sum_{j=1}^{d_N} d(g \cdot f_j) \otimes d(g \cdot f_j) \\ &= \sum_{j,k,l=1}^{d_N} o_{kl} o_{kl} df_j \otimes df_j \\ &= \sum_{j=1}^{d_N} df_j \otimes df_j. \end{aligned}$$

Since the round metric on S^m is the unique $SO(m+1)$-invariant metric, up to multiplicative constant, it follows that

$$\sum_{j=1}^{d_N} df_j \otimes df_j = c_N \sigma_m \tag{3.7}$$

for some constant $c_N \geq 0$. To compute c_N, we can use Takahashi's Theorem [26] which states that given an isometric immersion $h : M^m \to \mathbb{R}^{m+k}$ of a compact manifold M satisfying

$$\Delta h = \lambda h$$

componentwise, we can conclude that h is a minimal immersion into a sphere of radius $\sqrt{\frac{m}{\lambda}}$. It follows then that

$$c_N = \frac{\lambda_N d_N}{m|S^m|}. \tag{3.8}$$

Now let $\mathcal{R}(T)$ and $\mathcal{N}(T)$ denote the range and null space, respectively, of a linear operator T.

Lemma 3.2. *For all $x \in S^m$, $\frac{1}{c_N}Q_x^N$ is a projection operator on E_N. Moreover,* $\operatorname{rank} Q_x^N = m$*, and $\mathcal{R}(Q_x^N)$ can be identified with the tangent space to $\Phi_N(S^m)$ at $\Phi_N(x)$. Also, $\Phi_N(x) \in \mathcal{N}(Q_x^N)$.*

Proof. We have $\left(\frac{1}{c_N}Q_x^N\right)^2 = \frac{1}{c_N^2}\left(J_x \circ d_x \circ J_x \circ d_x\right)$. If we can show that

$$Q'_x \stackrel{\text{def}}{=} d_x \circ J_x = c_N I,$$

where I is the identity operator on $T_x^* S^m$, we would have $\left(\frac{1}{c_N}Q_x^N\right)^2 = \frac{1}{c_N}Q_x^N$ as desired.

On U, (3.7) is equivalent to

$$\sum_{l=1}^{d_N} \frac{\partial f_l}{\partial x_j}\frac{\partial f_l}{\partial x_k} = c_N g_{jk},$$

where $g_{jk} = \langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\rangle_{T_x S^m}$. One checks from the definitions that

$$\begin{aligned} Q'_x(dx_j) &= \sum_{l=1}^{d_N}\sum_{h,k=1}^{m} g^{jh}\frac{\partial f_l}{\partial x_h}\frac{\partial f_l}{\partial x_k}dx_k \\ &= c_N \sum_{h,k=1}^{m} g^{jh} g_{hk} dx_k \\ &= c_N dx_j. \end{aligned}$$

Hence $Q'_x = c_N I$, as desired. Note in particular then that d_x, J_x, and Q_x^N all have rank m. That $\mathcal{R}(Q_x^N)$ is the tangent space to $\Phi_N(S^m)$ at $\Phi_N(x)$ follows from (3.3).

Since Φ_N maps into a sphere, we have

$$\sum_{j=1}^{d_N} \frac{\partial f_j(x)}{\partial x_k} f_j(x) \equiv 0$$

for $k = 1, \ldots, m$. We compute

$$\begin{aligned} Q_x^N \Phi_N(x) &= J_x\left(\sum_{j=1}^{d_N} f_j df_j\right) \\ &= \sum_{j=1}^{d_N}\sum_{k=1}^{m} f_j \frac{\partial f_j}{\partial x_k} J_x(dx_k) \\ &= 0, \end{aligned}$$

which proves that $\Phi_N(x) \in \mathcal{N}(Q_x^N)$. $\square$

From the preceding two lemmas we obtain geometrically the following non-trivial sharp pointwise bound.

Corollary 3.3. *For all $f \in SE_N$, we have*

$$|\nabla f(x)| \leq \sqrt{\frac{\lambda_N d_N}{m|S^m|}}.$$

Before leaving this chapter, we note that the eigenspace maps Φ_N have been studied extensively as examples of minimal immersions of spheres into spheres [6, 7, 15, 26, 28]. Also, the L^p mapping properties of the Φ_N in the case $m = 2$ have been considered in [24].

Chapter 4

L^∞ Norms

We will now consider an application of the previous chapter. From (3.1), (3.6), and the Cauchy-Schwarz inequality, it follows that for all $f \in SE_N$ and $x \in S^m$,

$$|f(x)| \leq \|f\|_{L^2} \|\Phi_N(x)\|_{L^2} = \sqrt{\frac{d_N}{|S^m|}}. \tag{4.1}$$

From the upper bound

$$d_N = O\left(N^{m-1}\right),$$

we obtain the estimate

$$\|f\|_\infty \leq \sqrt{\frac{d_N}{|S^m|}} = O\left(N^{\frac{m-1}{2}}\right).$$

Moreover, this inequality is sharp; in particular, from (3.1) we see that equality occurs in (4.1) at $f(x)$ for f given by

$$f = \frac{1}{\|\Phi_N(x)\|_{L^2}} \Phi_N(x).$$

When we bound the L^∞ norms of *random* eigenfunctions, however, we can do much better.

Theorem 4.1. *For μ_∞-almost all sequences $\{f_N\}_{N\geq 0}$, with $f_N \in SE_N$, we have the following estimate as $N \to \infty$:*

$$\|f_N\|_\infty = O\left(\sqrt{\log N}\right). \tag{4.2}$$

The proof follows the ideas of Nonnenmacher and Voros [18]. For similar theorems, see [27].

Proof. So as to avoid confusion, $d(x,y)$ will denote the distance between x and y as measured on S^m, while $\tilde{d}(f,g)$ will denote the distance between f and g measured on SE_N. Also, for notational convenience, let

$$|\Phi_N(x)| \stackrel{\text{def}}{=} \|\Phi_N(x)\|_{L^2}$$

for the remainder of the proof.

We know that for $f \in SE_N$, $x \in S^m$,

$$|f(x)| = |\langle f, \Phi_N(x)\rangle_{L^2}| = \sqrt{\frac{d_N}{|S^m|}} \left|\cos \tilde{d}\left(f, \frac{\Phi_N(x)}{|\Phi_N(x)|}\right)\right|. \tag{4.3}$$

Moreover,

$$\tilde{d}\left(\frac{\Phi_N(x)}{|\Phi_N(x)|}, \frac{\Phi_N(y)}{|\Phi_N(y)|}\right) \leq \sqrt{\frac{\lambda_N}{m}} d(x,y).$$

Fix $p > 1 + \frac{1}{2}(m-1)$. Lemma 2.6 of [17] tells us that it is possible to pick at most $M = c_1 N^{(m+1)p}$ points $\{x_j\} \subset S^m$ in such a way that given any $x \in S^m$, there exists a j such that $d(x, x_j) < c_2 N^{-p}$. For $f \in SE_N$, $r > 0$, we denote by $B(f,r)$ the open ball of radius r around f in SE_N. For a fixed $0 < r < \frac{\pi}{2}$, we have

$$\begin{aligned}
\mu_N \left\{ \bigcup_{j=1}^{M} B\left(\frac{\Phi_N(x_j)}{|\Phi_N(x_j)|}, r\right)\right\} &\leq M \frac{|S^{d_N-2}|}{|S^{d_N-1}|} \int_{\cos r}^{1} \left(1-t^2\right)^{\frac{d_N-3}{2}}\, dt \\
&\leq M \frac{|S^{d_N-2}|}{|S^{d_N-1}|} (\sin r)^{d_N-1} \\
&\leq c_1 N^{m(p+\frac{1}{2})+p-\frac{1}{2}} \left(1-\cos^2 r\right)^{\frac{d_N-1}{2}}.
\end{aligned}$$

Working in reverse, we ask: If

$$q = \mu_N \left\{ \bigcup_{j=1}^{M} B\left(\frac{\Phi_N(x_j)}{|\Phi_N(x_j)|}, r\right)\right\},$$

what information do we gain about r in terms of q? We have, with $s = m(p+\frac{1}{2})+p-\frac{1}{2}$,

$$r \geq \cos^{-1}\sqrt{\frac{2s\log N + 2|\log q| + c_3}{d_N - 1}} \stackrel{\text{def}}{=} g(p,N,q),$$

with $c_3 = 2 \log c_1$.

Now if $f \notin \bigcup_{j=1}^{M} B\left(\frac{\Phi_N(x_j)}{|\Phi_N(x_j)|}, r\right)$, we have

$$\begin{aligned} \tilde{d}\left(f, \frac{\Phi_N(x)}{|\Phi_N(x)|}\right) &\geq \tilde{d}\left(f, \frac{\Phi_N(x_j)}{|\Phi_N(x_j)|}\right) - \tilde{d}\left(\frac{\Phi_N(x)}{|\Phi_N(x)|}, \frac{\Phi_N(x_j)}{|\Phi_N(x_j)|}\right) \\ &\geq g(p, N, q) - c_2\sqrt{\frac{\lambda_N}{m}} N^{-p}. \end{aligned} \tag{4.4}$$

Choosing r so that $q \leq N^{-2}$, we have by (4.3), (4.4), and the addition formula for cosines the following pointwise bound for these f:

$$\begin{aligned} |f(x)| &\leq \sqrt{\frac{d_N}{|S^m|}} \left(\cos g(p, N, q) + \sin\left(c_4 N^{-p+1}\right)\right) \\ &\leq c_5 \sqrt{d_N} \cos g(p, N, q) \\ &= O\left(\sqrt{\log N}\right), \end{aligned}$$

on a set of measure at least $1 - N^{-2}$. In particular, (4.2) holds for a set of μ_∞-positive measure. By the "zero-one" law [14], the theorem follows. $\square$

Chapter 5

Nodal Sets

We say that a function f has a *singular point* at x if and only if $f(x) = |\nabla f(x)| = 0$. We will show that most eigenfunctions do not admit singular points, and thereby conclude that their nodal sets are embedded submanifolds.

Lemma 5.1. *The set of all $f \in SE_N$ which have at least one singular point has Hausdorff dimension at most $d_N - 2$.*

By the Implicit Function Theorem, the zero set of a smooth function with no singular points is an embedded submanifold. Hence we have the following useful corollary:

Corollary 5.2. *For μ_N-almost all $f \in SE_N$, $\mathrm{Zero}(f) \subset S^m$ is an embedded submanifold of dimension $m-1$.*

Proof (of Lemma 5.1). We will denote elements of T^*S^m by (x, v), where $x \in S^m$, $v \in T^*_x S^m$. Define

$$\Psi : S^m \times E_N \to T^*S^m \times \mathbb{R}$$

by

$$\begin{aligned}\Psi(x, f) &= \big((x, df_x), f(x)\big)\\ &= \big((x, d_x(f)), \langle \Phi_N(x), f\rangle_{L^2}\big).\end{aligned}$$

Clearly Ψ is C^∞.

As before, choose an orthonormal basis $\{f_1, \ldots, f_{d_N}\}$ of E_N, with corresponding coordinate functions $a_1, \ldots, a_{d_N}$. We identify E_N with $\mathbb{R}^{d_N}$ by $f_j \leftrightarrow e_j$, where $\{e_j\}$ is the standard basis of $\mathbb{R}^{d_N}$. Choose a coordinate neighborhood $U \subset S^m$ with coordinate mapping $\varphi(x) = (x_1, \ldots, x_m)$. Let $V = \varphi(U) \subset \mathbb{R}^m$. We identify $T^*_U S^m$ with $V \times \mathbb{R}^m$ by $dx_j \leftrightarrow e_j$, $\{e_j\}$ the standard basis of $\mathbb{R}^m$. Write

$$df_j = \sum_{k=1}^{m} c_{jk} dx_k.$$

In this setup, we can write Ψ as

$$\Psi : V \times \mathbb{R}^{d_N} \to V \times \mathbb{R}^m \times \mathbb{R}$$

where

$$\Psi(x_1, \ldots, x_m; a_1, \ldots, a_{d_N}) = \left(x_1, \ldots, x_m; \sum_{j=1}^{d_N} a_j c_{j1}, \ldots, \sum_{j=1}^{d_N} a_j c_{jm}; \sum_{j=1}^{d_N} a_j f_j(x) \right).$$

The Jacobian matrix of Ψ is given by

$$\begin{pmatrix} I_{m\times m} & 0 \\ * & A_{m\times d_N} \\ * & B_{1\times d_N} \end{pmatrix}_{(2m+1)\times(m+d_N)}$$

where $B_{1j} = f_j$, and $A_{jk} = c_{jk}$. From the fact that $\operatorname{rank} d_x = m$, it follows that $\operatorname{rank} A = m$. Moreover, from the fact that $\Phi_N(x) \in \mathcal{N}(Q^n_x)$, we conclude that this Jacobian has rank $2m+1$, i.e., the differential of Ψ is surjective.

Now define $K \subset S^m \times E_N$ by $K = \Psi^{-1}((S^m, 0), 0)$. Denote by $\pi : S^m \times E_N \to E_N$ projection onto the second component. Then $\pi(K) \subset E_N$ is the set of all $f \in E_N$ with at least one singular point. By the preceding paragraph and the Implicit Function Theorem, K is a $(d_N - 1)$-dimensional embedded submanifold. In particular, it has Hausdorff dimension $d_N - 1$. Being linear, π cannot increase Hausdorff dimension. Hence $\pi(K)$ has Hausdorff dimension at most $d_N - 1$. Observing that $cf \in \pi(K)$ for all $f \in \pi(K)$, $c \in \mathbb{R}$, it follows that the Hausdorff dimension of $\pi(K)|_{SE_N}$ is at most $d_N - 2$, and the lemma is proved. $\square$

For $a, b > 0$, denote by $H_a^b \subset SE_N$ the set

$$H_a^b = \{f \in SE_N : |f(x)| \le a \Rightarrow |\nabla f(x)| > b\}. \tag{5.1}$$

The following technical lemma will be used extensively in what follows.

Lemma 5.3. *The following statements hold:*

1. $H_{a'}^{b'} \subset H_a^b$ *if* $a \le a'$, $b \le b'$,
2. H_a^b *is open in* SE_N,
3. *If* $f \in SE_N$ *has no singular points, then* $f \in H_a^b$ *for some* $a, b > 0$,
4. $\mu_N\left(SE_N \setminus \cup_{a>0} \cup_{b>0} H_a^b\right) = 0$, *and*
5. *For* $f \in H_a^b$, *and* $0 \le \epsilon < a$, $f^{-1}(-\epsilon, \epsilon)$ *is contained in a tubular neighborhood around* $\mathrm{Zero}(f)$ *of radius* $\frac{\epsilon}{b}$.

Proof. Statement 1 follows from (5.1). Define

$$\nu : SE_N \times S^m \to \mathbb{R} \times \mathbb{R}$$

by

$$\nu(f, x) = (|f(x)|, |\nabla f(x)|).$$

Clearly ν is continuous. Let $\pi : SE_N \times S^m \to SE_N$ denote projection onto the first factor. Then $SE_N \setminus H_a^b = \pi(\nu^{-1}([0, a] \times [0, b]))$, which is closed. This proves 2.

Now pick $f \in SE_N$ with no singular points. We note that both $\mathrm{Zero}(f)$ and $\mathrm{Crit}(f) \overset{\text{def}}{=} \{x \in S^m : |\nabla f(x)| = 0\}$ are closed in S^m. Since S^m is a normal topological space, we can find two disjoint open sets A and B in S^m such that $\mathrm{Zero}(f) \subset A$ and $\mathrm{Crit}(f) \subset B$. On $S^m \setminus A$, $|f(x)| > a$ for some $a > 0$. Therefore $f^{-1}[-a, a] \subset A$. On

$S^m \setminus B$, $|\nabla f(x)| > b$ for some $b > 0$. Since $A \subset (S^m \setminus B)$, we obtain 3. From 3 and Lemma 5.1 we now get 4.

It remains to prove 5. Pick a point $x_0 \in f^{-1}(-\epsilon, \epsilon)$, with $f(x_0) > 0$. Let γ be a curve $\gamma : [0, M] \to S^m$ with $\gamma(0) = x_0$, $f(\gamma(M)) = 0$, and

$$\gamma'(t) = -\nabla f(\gamma(t)) \tag{5.2}$$

for all $t \in [0, M]$. Such a curve exists since $|\nabla f(x)| > b$ on $f^{-1}(-\epsilon, \epsilon)$. Then we have

$$\epsilon > f(x_0) = -\int_\gamma df \ \geq bd(x_0, \gamma(M)),$$

where d denotes distance. Therefore, $f^{-1}[0, \epsilon)$ is contained in a tubular neighborhood of radius $\frac{\epsilon}{b}$ around $\mathrm{Zero}(f)$. The same proof works for those x with $f(x) < 0$; just replace $-\nabla f$ with ∇f in (5.2). $\square$

Chapter 6

Expected Value

We now want to prove the first equality in Proposition 2.1. To do so, we will need the following lemma. Heuristically, it states that in order to compute the expected value of Z_f^N, it suffices to average the norm of the gradient of all eigenfunctions which vanish at a point.

Lemma 6.1. *Fix $N > 1$. Let $\mathcal{P} = \{\Phi_N(x)\}^{\perp} \cap SE_N$. For all continuous functions φ on S^m, we have*

$$E_{\mu_N}(Z_f^N, \varphi) = \int_{S^m} \varphi(x) K^N(x)\, d\sigma_m(x), \tag{6.1}$$

where

$$K^N(x) = \frac{1}{\|\Phi_N(x)\|_{L^2}} \int_{\mathcal{P}} |\nabla f(x)|\, d\mu_N|_{\mathcal{P}}(f).$$

Here $\mu_N|_{\mathcal{P}}$ denotes the natural Lebesgue measure on $\mathcal{P}$ (which is also a sphere), normalized so that

$$\mu|_{\mathcal{P}}(\mathcal{P}) = \frac{|S^{d_N-2}|}{|S^{d_N-1}|}.$$

Proof. Define

$$I(f, \epsilon; x) = \begin{cases} 1 & \text{if } |f(x)| < \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

By the Coarea Formula, we have, for those f with no singular points,

$$(Z_f^N, \varphi) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{S^m} I(f, \epsilon; x) |\nabla f(x)| \varphi(x) \, dx.$$

For those continuous ψ on SE_N with support lying in H_a^b for some a and b, we claim that the following equalities hold:

$$\int_{SE_N} \psi(f)(Z_f^N, \varphi) \, d\mu_N(f)$$

$$= \int_{SE_N} \psi(f) \left\{ \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{S^m} I(f, \epsilon; x) |\nabla f(x)| \varphi(x) \, dx \right\} \, d\mu_N(f)$$

$$(6.2) \qquad = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{SE_N} \psi(f) \int_{S^m} I(f, \epsilon; x) |\nabla f(x)| \varphi(x) \, dx \, d\mu_N(f)$$

$$(6.3) \qquad = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{S^m} \varphi(x) \int_{SE_N} \psi(f) I(f, \epsilon; x) |\nabla f(x)| \, d\mu_N(f) \, dx$$

$$(6.4) \qquad = \int_{S^m} \varphi(x) K_\psi^N(x) \, dx,$$

where

$$K_\psi^N(x) = \frac{1}{\|\Phi_N(x)\|_{L^2}} \int_{\mathcal{P}} \psi(f) |\nabla f(x)| \, d\mu_N|_{\mathcal{P}}(f).$$

To begin the proofs of these, first note that equality (6.3) holds by Fubini's Theorem. Next, fix $x \in S^m$. We note that $I(f, \epsilon; x) = 1$ if and only if

$$|\langle \Phi_N(x), f \rangle_{L^2}| = |f(x)| < \epsilon.$$

We compute:

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{SE_N} \psi(f) I(f, \epsilon; x) |\nabla f(x)| \, d\mu_N(f) = K_\psi^N(x)$$

uniformly in x (cf. Lemma 1.3.2 of [12]). This proves equality (6.4).

We now verify (6.2). Given $f \in E_N$, denote by $T_f^\eta \subset S^m$ the tubular neighborhood around Zero(f) of radius η. Then for $f \in H_a^b$,

$$\text{supp } I(f, \epsilon; x) \subset T_f^{\frac{\epsilon}{b}},$$

by Lemma 5.3.5, so long as $\epsilon < a$. Let $h = -\operatorname{div}\left(\frac{\nabla f(x)}{|\nabla f(x)|}\right)$ denote the mean curvature along Zero(f). The following upper bound on the size of a tubular domain is borrowed from Theorem 8.4 in [11]:

$$\sigma_m(T_f^\eta) \le \int_0^\eta \int_{\mathrm{Zero}(f)} \sum_{\pm} \max\left(\left(1 \pm \frac{th}{m-1}\right)^{m-1}, 0\right) \, dZ_f^N(x)\, dt.$$

Also, since f is an eigenfunction, we have

$$\begin{aligned}
|h| &= \left|\left\langle \nabla f(x), \nabla\left(\frac{1}{|\nabla f(x)|}\right)\right\rangle\right| \\
&\le \frac{1}{|\nabla f(x)|}|\nabla|\nabla f(x)|| \\
&= \frac{1}{2|\nabla f(x)|^2}|\nabla|\nabla f(x)|^2| \\
&\le \frac{C}{b^2},
\end{aligned}$$

for some constant C depending only on N. We recall also that the volume of Zero(f) is uniformly bounded for all $f \in E_N$ [8], so

$$\frac{1}{2\epsilon}\sigma_m(T_f^{\frac{\epsilon}{b}})$$

is uniformly bounded for all $f \in H_a^b$, with $\epsilon < a$. This justifies equality (6.2) for all ψ whose support lies in H_a^b.

Continuing, we now take the supremum over all ψ with $0 \le \psi \le 1$, and supp $\psi \subset H_a^b$ to obtain

$$\int_{H_a^b} (Z_f^N, \varphi)\, d\mu_N(f) = \int_{S^m} \varphi(x) \left\{ \frac{1}{\|\Phi_N(x)\|_{L^2}} \int_{\mathcal{P}\cap H_a^b} |\nabla f(x)|\, d\mu_N|_{\mathcal{P}}(f) \right\} \, d\sigma_m(x).$$

Since this equality holds for all $a, b > 0$, we obtain

$$E(Z_f^N, \varphi) = \int_{S^m} \varphi(x) \left\{ \frac{1}{\|\Phi_N(x)\|_{L^2}} \int_{\mathcal{P}\cap H} |\nabla f(x)|\, d\mu_N|_{\mathcal{P}}(f) \right\} \, d\sigma_m(x),$$

where $H = \cup_{a>0} \cup_{b>0} H_a^b$. It remains to prove that for all $x \in S^m$,

$$\int_{\mathcal{P}\cap H} |\nabla f(x)|\, d\mu_N|_{\mathcal{P}}(f) = \int_{\mathcal{P}} |\nabla f(x)|\, d\mu_N|_{\mathcal{P}}(f).$$

Fix $x_0 \in S^m$. Let $H_0 = \{f \in SE_N : |\nabla f(x_0)| = f(x_0) = 0\}$. Then

$$\int_{(\mathcal{P}\cap H)\cup H_0} |\nabla f(x_0)|\, d\mu_N|_{\mathcal{P}}(f) = \int_{\mathcal{P}\cap H} |\nabla f(x_0)|\, d\mu_N|_{\mathcal{P}}(f).$$

We will be done if we can show that $\mathcal{P} \setminus (H \cup H_0)$ has dimension at most $d_N - 3$.

To this end, define $\widetilde{\Psi} : S^m \setminus \{\pm x_0\} \times E_N \to T^*S^m \times \mathbb{R} \times \mathbb{R}$ by

$$\widetilde{\Psi}(y, f) = \big((y, df_y), f(y), f(x_0)\big).$$

We know that $\Phi_N(x)$ is not parallel to $\Phi_N(y)$ so long as $y \neq \pm x$ (Facts B.2.2 and B.2.5 from Appendix B). If we can show that $\Phi_N(y) \notin \mathcal{R}(Q_x^N)$, we will know that $d\widetilde{\Psi}$ has rank $2m + 2$ everywhere. Suppose to the contrary that $\Phi_N(y) \in \mathcal{R}(Q_x^N)$. Then $f(x_0) = 0$ for all $f \in E_N$ with $|\nabla f(y)| = 0$. We can show by example that this is not possible. Denote by $K_y \cong SO(m)$ the isotropy subgroup of rotations which fix y. We first show that if y has the property stated above, then all eigenfunctions exhibiting a critical point at y must vanish not only at x_0, but also on $K_y \cdot x_0$. To see this, we note that if an eigenfunction f has a critical point at y but does not vanish at a point $x_1 \in K_y \cdot x_0$, then $k \cdot f$ also has a critical point at y and does not vanish at x_0, where $k \cdot x_1 = x_0$, $k \in K_y$, contradicting our assumption. Of course for $N > 1$, $\Phi_N(y)$ can be rotated by an element not in K_y to have a critical point at y (B.2.5), and thus it will not satisfy the above requirement, supplying the necessary contradiction.

It follows from the Implicit Function Theorem then that $K = \widetilde{\Psi}^{-1}((S^m, 0), 0, 0)$ is a $(d_N - 2)$-dimensional embedded submanifold of $S^m \setminus \{\pm x_0\} \times E_N$. Therefore $\pi(K)$ has Hausdorff dimension at most $d_N - 2$. Observing that $cf \in \pi(K)$ for all $f \in \pi(K)$ and $c \in \mathbb{R}$, it follows that $\pi(K)|_{SE_N}$ has Hausdorff dimension at most $d_N - 3$. We finish by noting that

$$\pi(K)|_{SE_N} = \mathcal{P} \setminus (H \cup H_0).$$

$\square$

Remark 6.2. It might be instructive to note here that $K^N(x)$ is independent of x. In particular, for every $g \in SO(m + 1)$,

$$E_{\mu_N}(Z_f^N, g \cdot \varphi) = E_{\mu_N}(Z_f^N, \varphi).$$

We recall that the sphere S^m has a unique (up to multiplicative constant) $SO(m+1)$-invariant measure. As such, $E_{\mu_N} Z_f^N$ must be a constant multiple of the usual Lebesgue measure σ_m. From (6.1), we see therefore that $K^N(x)$ must be constant.

We are now in a position to prove the first equality of Proposition 2.1.

Proof (of (2.3)*).* Fix $x \in S^m$. We will compute $K^N(x)$. Choose an orthonormal basis $\{f_1, \ldots, f_{d_N}\}$ of E_N and write $f \in E_N$ as

$$f = \sum_{j=1}^{d_N} a_j f_j.$$

Without loss of generality we may assume that $\Phi_N(x) = \|\Phi_N(x)\|_{L^2} f_{d_N}$. We may assume also that $\{f_1, \ldots, f_m\}$ span the range of $\frac{1}{c_N} Q_x^N$. For notational convenience, let $\mathbf{a} = (a_1, \ldots, a_m)$, and $d\mathbf{a} = da_1 \ldots da_m$. We slice integrate, e.g., Theorem A.5 of [1], with $a_1, \ldots, a_m$ all constant to obtain

$$\begin{aligned} \int_{\mathcal{P}} |\nabla f(x)| \, d\mu_N|_{\mathcal{P}}(f) &= \sqrt{c_N} \frac{|S^{d_N-m-2}|}{|S^{d_N-1}|} \int_{B^m} |\mathbf{a}| \left(1 - |\mathbf{a}|^2\right)^{\frac{d_N-m-3}{2}} \, d\mathbf{a} \\ &= \sqrt{c_N} \frac{|S^{d_N-m-2}||S^{m-1}|}{|S^{d_N-1}|} \int_0^1 r^m (1-r^2)^{\frac{d_N-m-3}{2}} \, dr \\ &= \sqrt{c_N} \frac{|S^{m-1}|}{|S^m|}. \end{aligned} \tag{6.5}$$

Equality (2.3) now follows from (6.5), (3.6), and (3.8). $\square$

Chapter 7

Variance

In view of the first equality in Proposition 2.1, we now define normalized nodal measures $\widetilde{Z}_f^N$ for $f \in E_N$ by

$$\widetilde{Z}_f^N = \frac{|S^m|}{|S^{m-1}|}\sqrt{\frac{m}{\lambda_N}}Z_f^N.$$

In particular,

$$E_{\mu_N}(\widetilde{Z}_f^N, \varphi) = \int_{S^m} \varphi(x)\, d\sigma_m(x).$$

The main result which will occupy the next two chapters is the following:

Proposition 7.1. *For $m \geq 2$ fixed, we have the following estimate as $N \to \infty$:*

$$V_{\mu_N}(\widetilde{Z}_f^N, \varphi) = O\left(\frac{\|\varphi\|_{L^\infty}^2}{N^{\frac{(m-1)^2}{3m+1}}}\right).$$

Before we introduce the next lemma, which will serve as the cornerstone for proving the above proposition, we need to make a definition. For $N \geq 1$, let

$$\Upsilon^N(x,y) = \left(1 - \frac{\langle \Phi_N(x), \Phi_N(y)\rangle_{L^2}^2}{\|\Phi_N(x)\|_{L^2}^2 \|\Phi_N(y)\|_{L^2}^2}\right)^{-\frac{1}{2}}.$$

The Funk-Hecke Theorem [12] states that

$$\langle \Phi_N(x), \Phi_N(y)\rangle_{L^2} = \frac{d_N}{|S^m|}P_N^{m+1}(\cos d(x,y)),$$

where d denotes distance as measured on S^m, and P_N^{m+1} is the Nth Legendre polynomial of order $m+1$. See Appendix B for a review of the definition and properties of Legendre polynomials. We can write Υ^N more concretely then as

$$\Upsilon^N(x,y) = \frac{1}{\sqrt{1-\left(P_N^{m+1}(\cos d(x,y))\right)^2}}. \tag{7.1}$$

The next lemma tells us that in order to compute the second moment of $\widetilde{Z}_f^N$, we need to average the product of the norms of the gradient of an eigenfunction evaluated at two points x and y over all eigenfunctions which vanish at both x and y.

Lemma 7.2. *Let $\mathcal{Q} = \{\Phi_N(x)\}^\perp \cap \{\Phi_N(y)\}^\perp \cap SE_N$. For all continuous functions φ on S^m, we have*

$$\int_{SE_N} (\widetilde{Z}_f^N, \varphi)^2 \, d\mu_N(f) = \int_{S^m}\int_{S^m} \varphi(x)\varphi(y) K^N(x,y) \, dx \, dy,$$

where

$$K^N(x,y) = \frac{\Upsilon^N(x,y)}{K^N(x)K^N(y)\|\Phi_N(x)\|_{L^2}\|\Phi_N(y)\|_{L^2}} \int_{\mathcal{Q}} |\nabla f(x)||\nabla f(y)| \, d\mu_N|_{\mathcal{Q}}(f). \tag{7.2}$$

Proof. For those continuous ψ on SE_N with support lying in H_a^b for some a and b, we claim that the following equalities hold:

$$\begin{aligned}
&\int_{SE_N} \psi(f)(Z_f^N,\varphi)^2 \, d\mu_N(f) \\
&\qquad = \int_{SE_N} \psi(f) \left\{ \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{S^m} I(f,\epsilon;x)|\nabla f(x)|\varphi(x)\, dx \right\}^2 d\mu_N(f) \\
&\qquad = \lim_{\epsilon\to 0} \int_{SE_N} \psi(f) \left\{ \frac{1}{2\epsilon} \int_{S^m} I(f,\epsilon;x)|\nabla f(x)|\varphi(x)\, dx \right\}^2 d\mu_N(f) &(7.3)\\
&\qquad = \lim_{\epsilon\to 0} \frac{1}{4\epsilon^2} \int_{S^m}\int_{S^m} \varphi(x)\varphi(y) \int_{SE_N} \psi(f) I(f,\epsilon;x) I(f,\epsilon;y) &(7.4)\\
&\qquad\qquad \times |\nabla f(x)||\nabla f(y)| \, d\mu_N(f)\, dx\, dy \\
&\qquad = \int_{S^m}\int_{S^m} \varphi(x)\varphi(y) K_\psi^N(x,y)\, dx\, dy, &(7.5)
\end{aligned}$$

where

$$K_\psi^N(x,y) = \frac{\Upsilon^N(x,y)}{\|\Phi_N(x)\|_{L^2}\|\Phi_N(y)\|_{L^2}} \int_{\mathcal{Q}} \psi(f)|\nabla f(x)||\nabla f(y)| \, d\mu_N|_{\mathcal{Q}}(f).$$

Again, equality (7.4) holds by Fubini's Theorem. Equality (7.3) holds just as (6.2) for all ψ whose support lies in some H_a^b. Lemma A.1, together with Lemma B.4, (A.1), and the Dominated Convergence Theorem, gives us

$$\lim_{\epsilon \to 0} \frac{1}{4\epsilon^2} \int_{SE_N} \psi(f) I(f, \epsilon; x) I(f, \epsilon; y) |\nabla f(x)| |\nabla f(y)| \, d\mu_N(f) = K_\psi^N(x, y),$$

and justifies equality (7.5). We now finish just as in Lemma 6.1. $\square$

Chapter 8

Proofs of the Main Theorems

8.1 The Riemannian Hypersurface Measure

We begin by proving Proposition 7.1 in small steps. Fix $0 < p < 1 - \frac{4}{m+3}$, to be determined later. The upper limit on p, which is a technical necessity for now, will automatically be satisfied in the end. We define $\Lambda_N^p \subset S^m \times S^m$ by

$$\Lambda_N^p = \left\{ (x, y) : d(x, y) \geq \frac{1}{N^p} \right\}.$$

Let $\overline{\Lambda_N^p} = S^m \times S^m \setminus \Lambda_N^p$. In other words, the bar over Λ_N^p will signify not closure, but rather complement. We have

$$V_{\mu_N}(\widetilde{Z}_f^N, \varphi) = \left(\int_{\Lambda_N^p} + \int_{\overline{\Lambda_N^p}} \right) \varphi(x)\varphi(y)(K^N(x, y) - 1)\, dx\, dy.$$

We concentrate first on the integral over $\overline{\Lambda_N^p}$.

Lemma 8.1. *For $m \geq 2$ fixed, we have the following estimate as $N \to \infty$:*

$$\int_{\overline{\Lambda_N^p}} \varphi(x)\varphi(y)(K^N(x, y) - 1)\, dx\, dy = O\left(\frac{\|\varphi\|_{L^\infty}^2}{N^{p(m-1)}} \right)$$

Before we begin the proof, we need two lemmas.

Lemma 8.2. *For all $x \in S^m$,*

$$\int_{\mathcal{Q}} |\nabla f(x)|^2 \, d\mu_N|_{\mathcal{Q}}(f) \leq c_N \frac{m}{2\pi}.$$

Proof. Choose an orthonormal basis $\{f_1, f_2, g_1, \ldots, g_{d_N-2}\}$ of E_N such that

1. $\Phi_N(x) = \|\Phi_N(x)\|_{L^2} f_1$,
2. $\Phi_N(y) \in \mathrm{Span}\{f_1, f_2\}$, and
3. $\mathcal{R}(Q_x^N) \subset \mathrm{Span}\{f_2, g_1, \ldots, g_m\}$.

Write $f \in E_N$ as $f = a_1 f_1 + a_2 f_2 + \sum_{j=1}^{d_N-2} b_j g_j$. Then

$$\begin{aligned}
\int_{\mathcal{Q}} |\nabla f(x)|^2 \, d\mu_N|_{\mathcal{Q}}(f) &\leq c_N \frac{|S^{d_N-m-3}|}{|S^{d_N-1}|} \int_{B^m} |\bar{b}|^2 \left(1 - |\bar{b}|^2\right)^{\frac{d_N-m-4}{2}} \, db_m \ldots db_1 \\
&= c_N \frac{|S^{d_N-m-3}||S^{m-1}|}{|S^{d_N-1}|} \int_0^1 r^{m+1} \left(1 - r^2\right)^{\frac{d_N-m-4}{2}} \, dr \\
&= \frac{c_N}{2} \frac{|S^{d_N-m-3}||S^{m-1}|}{|S^{d_N-1}|} \int_0^1 (1-r)^{\frac{d_N-m-2}{2}-1} r^{\frac{m+2}{2}-1} \, dr \\
&= c_N \frac{m}{2\pi}.
\end{aligned}$$

$\square$

Lemma 8.3. *For some C independent of both $N \geq 1$ and $u \in [0, \frac{\pi}{2}]$,*

$$\frac{\sin u}{\sqrt{1 - (P_N^M(\cos u))^2}} \leq C. \tag{8.1}$$

Note that C may depend on M.

Proof. One can show easily that (8.1) is equivalent to

$$(P_N^M)^2(\cos u) \leq 1 - \frac{1}{C^2} \sin^2 u,$$

or equivalently, for all $0 \leq t \leq 1$,

$$(P_N^M)^2(t) \leq 1 - \frac{1}{C^2}(1 - t^2).$$

We recall the integral representation of Legendre polynomials found in [12]:

$$P_N^M(t) = \frac{|S^{M-3}|}{|S^{M-1}|} \int_0^\pi \left\{ t + i\sqrt{1-t^2} \cos\varphi \right\}^N (\sin\varphi)^{M-3} \, d\varphi.$$

We note that

$$\begin{aligned}|t + i\sqrt{1-t^2}\cos\varphi| &= \sqrt{1-(1-t^2)\sin^2\varphi}\\ &\le 1.\end{aligned}$$

Hence for $N \geq 2$,

$$\begin{aligned}(P_N^M)^2(t) &\le |P_N^M(t)|\\ &\le 1-(1-t^2)\frac{|S^{M-3}|}{|S^{M-1}|}\int_0^\pi \sin^{M-1}\varphi\, d\varphi.\end{aligned}$$

□

Proof (of Lemma 8.1). By the Cauchy-Schwarz inequality and Lemma 8.2, we have

$$\int_{\mathcal{Q}} |\nabla f(x)||\nabla f(y)|\, d\mu_N|_{\mathcal{Q}}(f) \le c_N \frac{m}{2\pi}.$$

From this inequality, together with (7.2) and (3.6), we have

$$\int_{\Lambda_N^p} K^N(x,y)\, dx\, dy \le c_N \frac{m}{2\pi}\frac{|S^m|}{d_N}\int_{\Lambda_N^p}\frac{\Upsilon^N(x,y)}{K^N(x)K^N(y)}\, dx\, dy. \tag{8.2}$$

From (2.3), (6.1), and Remark 6.2, we have

$$K^N(x) = K^N(y) = \frac{|S^{m-1}|}{|S^m|}\sqrt{\frac{\lambda_N}{m}}. \tag{8.3}$$

Recalling the definition of c_N given by (3.8), we conclude from (8.2) and (8.3) that

$$\int_{\Lambda_N^p} K^N(x,y)\, dx\, dy \le \frac{m}{2\pi}\frac{|S^m|^2}{|S^{m-1}|^2}\int_{\Lambda_N^p}\Upsilon^N(x,y)\, dx\, dy.$$

By the Triangle inequality, (7.1), Lemma 8.3, and direct computation, we conclude

$$\begin{aligned}\left|\int_{\Lambda_N^p}(K^N(x,y)-1)\, dx\, dy\right| &\le \frac{m}{2\pi}\frac{|S^m|^2}{|S^{m-1}|^2}\int_{\Lambda_N^p}\Upsilon^N(x,y)\, dx\, dy + \int_{\Lambda_N^p} dx\, dy\\ &\le C\frac{m}{\pi}\frac{|S^m|^3}{|S^{m-1}|}\frac{1}{N^{p(m-1)}} + 2|S^m||S^{m-1}|\frac{1}{N^{mp}}\\ &= O\left(\frac{1}{N^{p(m-1)}}\right).\end{aligned}$$

The constant C in the second inequality is the same C that appears in Lemma 8.3. □

We can now concentrate on the integral over Λ_N^p.

Lemma 8.4. *Uniformly for $(x, y) \in \Lambda_N^p$,*

$$K^N(x, y) = 1 + O\left(N^{(p-1)(\frac{m+3}{2})+2}\right).$$

Again, we need a couple of lemmas.

Lemma 8.5. *For N sufficiently large, $\mathcal{R}(Q_x) \cap \mathcal{R}(Q_y) = 0$ for all $(x, y) \in \Lambda_N^p$.*

Proof. Fix $x, y \in S^m$, $x \neq y$. Define $B_{xy} : T_x^* S^m \to T_y^* S^m$ by

$$B_{xy} = d_y \circ J_x.$$

We know that $\mathcal{R}(Q_x) \cap \mathcal{R}(Q_y) = 0$ if and only if $\|B_{xy}\| < c_N$. We want to show that this happens for all $(x, y) \in \Lambda_N^p$ for N sufficiently large.

Let $U \subset S^m$ be a coordinate neighborhood. Let $\{X_1, \ldots, X_m\}$ be an orthonormal frame on U with corresponding orthonormal coframe $\{\omega_1, \ldots, \omega_m\}$. Let $\{f_1, \ldots, f_{d_N}\}$ be an orthonormal basis of E_N. Then by direct calculation we obtain

$$\langle B_{xy}\omega_j, \omega_k \rangle = \sum_{h=1}^{d_N} X_j f_h(x) X_k f_h(y).$$

Now

$$\begin{aligned}
&\frac{1}{c_N} \sum_{h=1}^{d_N} X_j f_h(x) X_k f_h(y) \\
&\qquad = \frac{m}{\lambda_N} X_j(x) X_k(y) P_N^{m+1}(\cos d(x, y)) \\
&\qquad = \frac{m}{\lambda_N} X_j(x) \left(-(P_N^{m+1})'(\cos d(x, y)) \sin d(x, y) X_k(y) d(x, y)\right) \\
&\qquad = \frac{m}{\lambda_N} \Big\{ (P_N^{m+1})''(\cos d(x, y)) \sin^2 d(x, y) X_j(x) d(x, y) X_k(y) d(x, y) \\
&\qquad\quad - (P_N^{m+1})'(\cos d(x, y)) \cos d(x, y) X_j(x) d(x, y) X_k(y) d(x, y) \\
&\qquad\quad - (P_N^{m+1})'(\cos d(x, y)) \sin d(x, y) X_j(x) X_k(y) d(x, y) \Big\} \\
&\qquad = O\left(N^{(p-1)(\frac{m+1}{2})} + N^{(p-1)(\frac{m+3}{2})+2}\right) \\
&\qquad = O\left(\frac{1}{N^{(1-p)(\frac{m+3}{2})-2}}\right),
\end{aligned}$$

which decays since we required that

$$p < 1 - \frac{4}{m+3}.$$

$\square$

Lemma 8.6. *Uniformly for* $(x, y) \in \Lambda_N^p$,

$$\int_{\mathcal{Q}} |\nabla f(x)||\nabla f(y)| \, d\mu_N|_{\mathcal{Q}}(f) = c_N \left(\frac{|S^{m-1}|^2}{|S^m|^2} + O\left(N^{(p-1)(\frac{m+3}{2})+2} \right) \right).$$

Proof. Define $\tau : E_N \to \mathbb{R} \times \mathbb{R} \times T_x^* S^m \times T_y^* S^m$ by

$$\tau(f) = (f(x), f(y), df_x, df_y).$$

By Lemma 8.5, τ is a surjective linear map for all sufficiently large N. Denote by $(s, t, \mathbf{u}, \mathbf{v})$ the variables in

$$\mathbb{R} \times \mathbb{R} \times T_x^* S^m \times T_y^* S^m.$$

Choose coordinates as in Lemma 8.5, and write $\nabla_x = (X_1(x), \ldots, X_m(x))$. Using the fact that the image of a Gaussian measure under a surjective linear map is again Gaussian [16], we have

$$\begin{aligned}
&\int_{\mathcal{Q}} |\nabla f(x)||\nabla f(y)| \, d\mu_N|_{\mathcal{Q}}(f) \\
&\qquad = \frac{1}{(2\pi)^{\frac{d_N}{2}}} \int_{\mathbb{R}^{d_N - 2}} |\nabla f(x)||\nabla f(y)| e^{-\frac{1}{2}\|f\|^2} \, df \\
&\qquad = \frac{\|\Phi_N(x)\|_{L^2} \|\Phi_N(y)\|_{L^2}}{\Upsilon^N(x, y)} \frac{1}{(2\pi)^{\frac{d_N}{2}}} \\
&\qquad \times \lim_{\epsilon \to 0} \frac{1}{4\epsilon^2} \int_{E_N} I(f, \epsilon; x) I(f, \epsilon; y) |\nabla f(x)||\nabla f(y)| e^{-\frac{1}{2}\|f\|^2} \, df \qquad (8.4) \\
&\qquad = \frac{\|\Phi_N(x)\|_{L^2} \|\Phi_N(y)\|_{L^2}}{\Upsilon^N(x, y)} \frac{1}{(2\pi)^{m+1} \sqrt{|\Lambda_N|}} \\
&\qquad \times \lim_{\epsilon \to 0} \frac{1}{4\epsilon^2} \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} \iint \|\mathbf{u}\| \|\mathbf{v}\| \\
&\qquad\qquad e^{-\frac{1}{2}(s,t,\mathbf{u},\mathbf{v}) \Lambda_N^{-1} (s,t,\mathbf{u},\mathbf{v})^t} \, d\mathbf{u} \, d\mathbf{v} \, ds \, dt,
\end{aligned}$$

where Λ_N is the $(2m+2)\times(2m+2)$ block matrix given by

$$\Lambda_N = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix},$$

$$A = \frac{d_N}{|S^m|}\begin{pmatrix} 1 & P_N^{m+1}(\cos d(x,y)) \\ P_N^{m+1}(\cos d(x,y)) & 1 \end{pmatrix}_{2\times 2},$$

$$B = \frac{d_N}{|S^m|}\begin{pmatrix} 0 & \nabla_y P_N^{m+1}(\cos d(x,y)) \\ \nabla_x P_N^{m+1}(\cos d(x,y)) & 0 \end{pmatrix}_{2\times(2m)},$$

and

$$C = \frac{d_N}{|S^m|}\begin{pmatrix} \frac{\lambda_N}{m} I & X_j(x)X_k(y)P_N^{m+1}(\cos d(x,y)) \\ X_j(y)X_k(x)P_N^{m+1}(\cos d(x,y)) & \frac{\lambda_N}{m} I \end{pmatrix}_{(2m)\times(2m)}.$$

Here j is a row index and k is a column index, both ranging from 1 to m.

Taking the limit in (8.4), we obtain

$$\int_{\mathcal{Q}} |\nabla f(x)||\nabla f(y)|\, d\mu_N|_{\mathcal{Q}}(f) = \frac{d_N}{|S^m|}\frac{1}{\Upsilon^N(x,y)}\frac{1}{(2\pi)^{m+1}\sqrt{|\Lambda_N|}} \times \iint \|\mathbf{u}\|\|\mathbf{v}\| e^{-\frac{1}{2}(0,0,\mathbf{u},\mathbf{v})\Lambda_N^{-1}(0,0,\mathbf{u},\mathbf{v})^t}\, d\mathbf{u}\, d\mathbf{v}.$$

Here, and in what follows, we write $|\Lambda_N|$ for the determinant of a matrix—in this case Λ_N. We compute that

$$(0,0,\mathbf{u},\mathbf{v})\Lambda_N^{-1}(0,0,\mathbf{u},\mathbf{v})^t = (\mathbf{u},\mathbf{v})F_N^{-1}(\mathbf{u},\mathbf{v})^t,$$

where

$$F_N = C - B^t A^{-1} B. \tag{8.5}$$

We note that by Jacobi's Theorem [16], $|\Lambda_N| = |A||F_N|$. Also,

$$\sqrt{|A|} = \frac{d_N}{|S^m|}\frac{1}{\Upsilon^N(x,y)}.$$

Hence,

$$(8.6)\quad \int_{\mathcal{Q}} |\nabla f(x)||\nabla f(y)|\, d\mu_N|_{\mathcal{Q}}(f) = \frac{1}{(2\pi)^{m+1}\sqrt{|F_N|}} \iint \|\mathbf{u}\|\|\mathbf{v}\| e^{-\frac{1}{2}(\mathbf{u},\mathbf{v})F_N^{-1}(\mathbf{u},\mathbf{v})^t}\, d\mathbf{u}\, d\mathbf{v}.$$

We now state the estimates we will need. So as to not destroy the continuity of the current argument, we postpone the proofs. To facilitate notation, we will say that a matrix if $O(g(N))$ if each of its entries is $O(g(N))$.

Lemma 8.7. *Partition F_N and F_N^{-1} into $m \times m$ block matrices as*

$$F_N = \begin{pmatrix} Q & R \\ R^t & S \end{pmatrix}, \qquad F_N^{-1} = \begin{pmatrix} T & V \\ V^t & W \end{pmatrix}.$$

The following estimates hold uniformly for all $(x, y) \in \Lambda_N^p$ as $N \to \infty$:

1. $\frac{1}{\sqrt{|F_N|}} = c_N^{-m}\left(1 + O\left(N^{(p-1)(\frac{m+3}{2})+2}\right)\right)$,
2. $Q^{-1} = c_N^{-1}\left(I + O\left(N^{(p-1)(m+1)+2}\right)\right)$,
3. $W = c_N^{-1}\left(I + O\left(N^{(p-1)(\frac{m+3}{2})+2}\right)\right)$*; moreover, the same estimate holds for* W^{-1} *with* c_N^{-1} *replaced by* c_N*, and*
4. $V = c_N^{-1} O\left(N^{(p-1)(\frac{m+3}{2})+2}\right)$.

Continuing, we note that A and C are symmetric matrices. Hence F_N, and thus W, is also symmetric. We have therefore

$$(\mathbf{u}, \mathbf{v})F_N^{-1}(\mathbf{u}, \mathbf{v})^t = (\mathbf{v} + \mathbf{u}VW^{-1})W(\mathbf{v} + \mathbf{u}VW^{-1})^t + \mathbf{u}Q^{-1}\mathbf{u}^t.$$

In particular,

$$(8.7)\quad \iint \|\mathbf{u}\|\|\mathbf{v}\| e^{-\frac{1}{2}(\mathbf{u},\mathbf{v})F_N^{-1}(\mathbf{u},\mathbf{v})^t}\, d\mathbf{u}\, d\mathbf{v} = \int \|\mathbf{u}\| \left\{ \int \|\mathbf{v} - \mathbf{u}VW^{-1}\| e^{-\frac{1}{2}\mathbf{v}W\mathbf{v}^t}\, d\mathbf{v} \right\} e^{-\frac{1}{2}\mathbf{u}Q^{-1}\mathbf{u}^t}\, d\mathbf{u}.$$

Now

$$
\begin{aligned}
\big|\|\mathbf{v}-\mathbf{u}VW^{-1}\|-\|\mathbf{v}\|\big| &\le \frac{2\|\mathbf{u}VW^{-1}\|\|\mathbf{v}\|+\|\mathbf{u}VW^{-1}\|^2}{\|\mathbf{v}-\mathbf{u}VW^{-1}\|+\|\mathbf{v}\|} \\
&\le 3\|\mathbf{u}VW^{-1}\|.
\end{aligned}
\tag{8.8}
$$

From (8.7), (8.8), Lemma 8.7.3, and Lemma 8.7.4, we have

$$
\begin{aligned}
&\iint \|\mathbf{u}\|\|\mathbf{v}\|e^{-\frac{1}{2}(\mathbf{u},\mathbf{v})F_N^{-1}(\mathbf{u},\mathbf{v})^t}\,d\mathbf{u}\,d\mathbf{v} = \\
&\qquad\left\{\int \|\mathbf{u}\|e^{-\frac{1}{2}\mathbf{u}Q^{-1}\mathbf{u}^t}\,d\mathbf{u}\right\}\left\{\int \|\mathbf{v}\|e^{-\frac{1}{2}\mathbf{v}W\mathbf{v}^t}\,d\mathbf{v}\right\} + c_N^{m+1}O\left(N^{(p-1)(\frac{m+3}{2})+2}\right).
\end{aligned}
\tag{8.9}
$$

We now calculate

$$
\begin{aligned}
&\left|\int \|\mathbf{u}\|e^{-\frac{1}{2}\mathbf{u}Q^{-1}\mathbf{u}^t}\,d\mathbf{u} - \int \|\mathbf{u}\|e^{-\frac{1}{2}c_N^{-1}\|\mathbf{u}\|^2}\,d\mathbf{u}\right| \\
&\qquad\le \int \|\mathbf{u}\|e^{-\frac{1}{2}c_N^{-1}\|\mathbf{u}\|^2}\left|e^{-\frac{1}{2}c_N^{-1}\mathbf{u}O(N^q)\mathbf{u}^t}-1\right|\,d\mathbf{u} \\
&\qquad\le c_N^{-1}O(N^q)\int \|\mathbf{u}\|^3 e^{-\frac{1}{2}c_N^{-1}(1+O(N^q))\|\mathbf{u}\|^2}\,d\mathbf{u} \\
&\qquad= c_N^{\frac{m+1}{2}}O(N^q),
\end{aligned}
\tag{8.10}
$$

where $q=(p-1)(m+1)+2$. The first inequality follows from Lemma 8.7.2. The second inequality follows from the easy estimate

$$
|e^t-1| \le |t|e^{|t|},
$$

which is valid for all real t, and the final equality follows from direct computation.

Similarly, from Lemma 8.7.3 we can show

$$
\int \|\mathbf{v}\|e^{-\frac{1}{2}\mathbf{v}W\mathbf{v}^t}\,d\mathbf{v} = \int \|\mathbf{v}\|e^{-\frac{1}{2}c_N^{-1}\|\mathbf{v}\|^2}\,d\mathbf{v} + c_N^{\frac{m+1}{2}}O(N^r),
\tag{8.11}
$$

where $r=(p-1)(\frac{m+3}{2})+2$. Combining (8.9),(8.10), and (8.11), we have

$$
\begin{aligned}
&\iint \|\mathbf{u}\|\|\mathbf{v}\|e^{-\frac{1}{2}(\mathbf{u},\mathbf{v})F_N^{-1}(\mathbf{u},\mathbf{v})^t}\,d\mathbf{u}\,d\mathbf{v} = \\
&\qquad c_N^{m+1}\left(|S^{m-1}|^2 2^{m-1}\left(\Gamma(\tfrac{m+1}{2})\right)^2 + O\left(N^{(p-1)(\frac{m+3}{2})+2}\right)\right).
\end{aligned}
\tag{8.12}
$$

Lemma 8.6 now follows from (8.6), Lemma 8.7.1, and (8.12). □

For completeness, we now prove our estimates.

Proof (of Lemma 8.7). We first introduce some notation which will be used only in this proof. Let

$$
\begin{aligned}
P_N &\stackrel{\text{def}}{=} P_N^{m+1}(\cos d(x,y)),\\
X_j P_N &\stackrel{\text{def}}{=} X_j(x) P_N^{m+1}(\cos d(x,y)),\\
Y_j P_N &\stackrel{\text{def}}{=} X_j(y) P_N^{m+1}(\cos d(x,y)),\\
X_j Y_k P_N &\stackrel{\text{def}}{=} X_j(x) X_k(y) P_N^{m+1}(\cos d(x,y)),\\
\alpha &\stackrel{\text{def}}{=} \frac{1}{1-\left(P_N^{m+1}(\cos d(x,y))\right)^2}.
\end{aligned}
$$

We compute from (8.5) that

$$
c_N^{-1} Q = I - \frac{m\alpha}{\lambda_N}\big((X_j P_N)(X_k P_N)\big). \tag{8.13}
$$

Here, and for the rest of the proof, j denotes a row index and k denotes a column index, both running from 1 to m.

For $(x,y) \in \Lambda_N^p$, we have $d(x,y) \geq N^{-p}$. From Facts B.2.3 and B.2.4 in Appendix B, we compute

$$
\begin{aligned}
|X_j P_N| &= \left|\left(P_N^{m+1}\right)'(\cos d(x,y)) \sin d(x,y) X_j(x) d(x,y)\right|\\
&\leq \frac{\lambda_N}{m}\left|P_{N-1}^{m+3}(\cos d(x,y))\right|\\
&= O\left(N^{(p-1)(\frac{m+1}{2})+2}\right).
\end{aligned} \tag{8.14}
$$

Also, we have

$$
\begin{aligned}
\alpha &= 1 + \left(P_N^{m+1}(\cos d(x,y))\right)^2 \frac{1}{1-\left(P_N^{m+1}(\cos d(x,y))\right)^2}\\
&= 1 + O\left(N^{(p-1)(m-1)}\right),
\end{aligned} \tag{8.15}
$$

where the error estimate follows from Fact B.2.4 and the observation that the fraction in the last equality is uniformly bounded.

Combining (8.13), (8.14), and (8.15), we conclude that

$$Q = c_N \left(I + O\left(N^{(p-1)(m+1)+2}\right)\right). \tag{8.16}$$

The same estimate holds for the sub-block S via the same argument. The needed estimate for Q^{-1} follows from (8.16) by the classical adjoint formula for inverses (cf. Theorem 5.4 of [13]).

Also from (8.5), we have

$$\begin{aligned} c_N^{-1} R &= \left(\frac{m}{\lambda_N} X_j Y_k P_N + \frac{m\alpha}{\lambda_N} P_N (X_j P_N)(Y_k P_N)\right) \\ &= O\left(N^{(p-1)(\frac{m+3}{2})+2}\right) + O\left(N^{(p-1)(\frac{3m+1}{2})+2}\right) \\ &= O\left(N^{(p-1)(\frac{m+3}{2})+2}\right) \end{aligned} \tag{8.17}$$

It follows immediately that

$$F_N = c_N \left(I + O\left(N^{(p-1)(\frac{m+3}{2})+2}\right)\right),$$

and

$$F_N^{-1} = c_N^{-1} \left(I + O\left(N^{(p-1)(\frac{m+3}{2})+2}\right)\right).$$

The needed estimate for V and W, and therefore for W^{-1} also, is then immediate. To complete the proof, we observe that the estimate

$$\frac{1}{|F_N|} = |F_N^{-1}| = c_N^{-2m} \left(1 + O\left(N^{(p-1)(\frac{m+3}{2})+2}\right)\right)$$

follows directly from the definition of the determinant. The required estimate for $|F_N|^{-\frac{1}{2}}$ then follows. □

We now complete the proof of the main new lemma of this chapter.

Proof (of Lemma 8.4). We begin by noting that uniformly for $(x, y) \in \Lambda_N^p$, we have

$$\Upsilon^N(x, y) = 1 + O\left(N^{(p-1)(m-1)}\right). \tag{8.18}$$

This follows from (7.1), Fact B.2.4, and the following equality which holds for all real $|t| < 1$:

$$\frac{1}{\sqrt{1-t^2}} = 1 + \frac{t^2}{\sqrt{1-t^2}(1+\sqrt{1-t^2})}.$$

The present lemma now follows from (7.2), (8.18), and Lemma 8.6. $\square$

Combining Lemma 8.1 and Lemma 8.4, and optimizing over p, we obtain Proposition 7.1. Note that the variance is bounded; it's also summable for $m > 5$. We are finally in a position to prove the main theorem of the work.

Proof (of Theorem 2.2). Part 2 follows immediately from Proposition 7.1 and Chebyshev's inequality [14].

We will now prove part 1. Let φ be a continuous function on S^m. Let $\mathcal{B}_\varphi \subset SE_\infty$ be the set of all sequences $\{f_N\}$, with $f_N \in SE_N$, such that

$$\frac{1}{M}\sum_{N=1}^{M}\int_{S^m} \varphi \, d\widetilde{Z}_{f_N}^N \nrightarrow \int_{S^m} \varphi \, d\sigma_m$$

as $M \to \infty$. By Proposition 7.1 and Kolmogorov's Strong Law of Large Numbers [14], $\mathcal{B}_\varphi$ has μ_∞-measure zero.

Let $\mathcal{A}$ be a countable L^∞-dense subset of $C_\mathbb{R}(S^m)$, the set of all *real-valued* continuous function on S^m. Such a subset is constructed in Lemma C.1 found in Appendix C. Define $\mathcal{B}$ to be the following union:

$$\mathcal{B} \stackrel{\text{def}}{=} \bigcup_{\psi\in\mathcal{A}} \mathcal{B}_\psi.$$

Since $\mathcal{A}$ is countable, $\mathcal{B}$ also has μ_∞-measure zero.

Now take any real-valued continuous function φ on S^m. There exists a sequence of elements of $\mathcal{A}$, say $\varphi_1, \varphi_2, \ldots$, such that

$$\|\varphi - \varphi_j\|_{L^\infty} \leq \frac{1}{j}.$$

For any sequence $\{f_N\}$, with $f_N \in SE_N$, which does not lie in $\mathcal{B}$, we have

$$\begin{aligned}
\left|\frac{1}{M}\sum_{N=1}^{M}\int_{S^m}\varphi\, d\widetilde{Z}_{f_N}^N - \int_{S^m}\varphi\, d\sigma_m\right| &\le \left|\frac{1}{M}\sum_{N=1}^{M}\int_{S^m}\varphi\, d\widetilde{Z}_{f_N}^N - \frac{1}{M}\sum_{N=1}^{M}\int_{S^m}\varphi_j\, d\widetilde{Z}_{f_N}^N\right| \\
&\quad + \left|\frac{1}{M}\sum_{N=1}^{M}\int_{S^m}\varphi_j\, d\widetilde{Z}_{f_N}^N - \int_{S^m}\varphi_j\, d\sigma_m\right| \\
&\quad + \left|\int_{S^m}\varphi_j\, d\sigma_m - \int_{S^m}\varphi\, d\sigma_m\right| \\
&\le Cj^{-1} \\
&\quad + \left|\frac{1}{M}\sum_{N=1}^{M}\int_{S^m}\varphi_j\, d\widetilde{Z}_{f_N}^N - \int_{S^m}\varphi_j\, d\sigma_m\right| \\
&\quad + |S^m|j^{-1}.
\end{aligned}$$

Here C is a uniform upper bound on $(\widetilde{Z}_{f_N}^N, 1)$, the existence of which was shown in [8]. We conclude therefore that

$$\limsup_{M\to\infty}\left|\frac{1}{M}\sum_{N=1}^{M}\int_{S^m}\varphi\, d\widetilde{Z}_{f_N}^N - \int_{S^m}\varphi\, d\sigma_m\right| \le (C+|S^m|)\frac{1}{j}.$$

Part 1 now follows for all real-valued φ from the observation that this inequality holds for all positive integers j. We finish by simply considering the real and imaginary parts of an arbitrary continuous φ on S^m.

It remains to prove part 3. Denote an element of SE_∞ by $\mathcal{F} = \{f_N\}_{N\ge 0}$, with $f_N \in SE_N$. Given a real continuous function φ on S^m, let $Y_\varphi^N(\mathcal{F}) = (\widetilde{Z}_{f_N}^N - \sigma_m, \varphi)^2$. Then

$$\sum_{N=0}^{\infty}\int_{SE_\infty} Y_\varphi^N(\mathcal{F})\, d\mu_\infty(\mathcal{F}) = \sum_{N=0}^{\infty} V_{\mu_N}(\widetilde{Z}_{f_N}^N, \varphi) < \infty.$$

In particular, for μ_∞-almost all $\mathcal{F}$,

$$\sum_{N=0}^{\infty} Y_\varphi^N(\mathcal{F}) < \infty,$$

and thus $Y_\varphi^N(\mathcal{F}) \to 0$ as $N \to \infty$. Since this holds for any real φ in a countable L^∞-dense subset of $C_{\mathbb{R}}(S^m)$, we complete the proof as above. $\square$

8.2 The Léray Nodal Measure

Explicitly, we have for any continuous function φ on S^m,

$$\int_{S^m} \varphi \, d\delta_f^N = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{S^m} I(f, \epsilon; x) \, d\sigma_m(x).$$

From this the second equality in Proposition 2.1 follows just as the first. The analogue of Lemma 7.2 is then the following:

Lemma 8.8. *For all continuous functions φ on S^m, we have*

$$\int_{SE_N} (\tilde{\delta}_f^N, \varphi)^2 \, d\mu_N(f) = \int_{S^m} \int_{S^m} \varphi(x)\varphi(y) K_\delta^N(x, y) \, dx \, dy,$$

where

$$K_\delta^N(x, y) = \frac{|S^{d_N-1}||S^{d_N-3}|}{|S^{d_N-2}|^2} \Upsilon^N(x, y). \tag{8.19}$$

We note that

$$\begin{aligned} \frac{|S^{d_N-1}||S^{d_N-3}|}{|S^{d_N-2}|^2} &= 1 + O\left(\frac{1}{d_N}\right) \\ &= 1 + O\left(\frac{1}{N^{m-1}}\right). \end{aligned} \tag{8.20}$$

The following analogue of Proposition 7.1 also holds.

Proposition 8.9. *For m fixed, we have the following estimate as $N \to \infty$:*

$$V_{\mu_N}(\tilde{\delta}_f^N, \varphi) = O\left(\frac{\|\varphi\|_{L^\infty}^2}{N^{\frac{m-1}{2}}}\right). \tag{8.21}$$

From this Theorem 2.3 follows.

Proof. By (8.19), (8.20), and Lemma B.4, we compute

$$V_{\mu_N}(\tilde{\delta}_f^N, 1) = \int_{S^m} \int_{S^m} (\Upsilon^N(x, y) - 1) \, dx \, dy + O\left(\frac{1}{N^{m-1}}\right). \tag{8.22}$$

By (8.18), we have (for any $p > 0$)

$$\int_{\Lambda_N^p} (\Upsilon^N(x, y) - 1) \, dx \, dy = O\left(N^{(p-1)(m-1)}\right). \tag{8.23}$$

Just as in the proof of Lemma B.4, we have by direct computation

$$\int_{\overline{\Lambda_N^p}} (\Upsilon^N(x,y) - 1)\, dx\, dy =$$

$$2|S^m||S^{m-1}| \int_0^{\frac{1}{N^p}} \left\{ \frac{1}{\sqrt{1 - (P_N^{m+1}(\cos u))^2}} - 1 \right\} \sin^{m-1} u\, du.$$

By Lemma 8.3, we have

$$\int_{\overline{\Lambda_N^p}} (\Upsilon^N(x,y) - 1)\, dx\, dy = O\left(\frac{1}{N^{p(m-1)}}\right). \tag{8.24}$$

Choosing $p = \frac{1}{2}$ and combining (8.22), (8.23), and (8.24), we obtain (8.21). $\square$

Chapter 9

Scaling Limits

Also of interest are the local statistics of random nodal sets, e.g., the scaling limits of the two-point correlation functions $K^N(x,y)$ and $K^N_\delta(x,y)$. To define the scaling limits, we note first that both K^N and K^N_δ depend only on the distance $d = d(x,y)$ between x and y. For K^N_δ this follows from (8.19), and for K^N this follows from (7.2) and the fact that spheres are two-point homogeneous spaces. We define the scaling limits then to be

$$K^{\mathrm{sc}}(d) \stackrel{\mathrm{def}}{=} \lim_{N\to\infty} K^N\left(\frac{d}{N}\right), \text{ and } K^{\mathrm{sc}}_\delta(d) \stackrel{\mathrm{def}}{=} \lim_{N\to\infty} K^N_\delta\left(\frac{d}{N}\right).$$

Roughly speaking, $K^N(x,y)$ can be thought of as the joint probability density that a random eigenfunction $f_N \in SE_N$ simultaneously vanishes in small neighborhoods of x and y. We can view the scaling limits then as the limit distribution of these joint probabilities rescaled to have a finite non-zero limit. Such scaling limits have been computed in the analogous complex case in [3, 4].

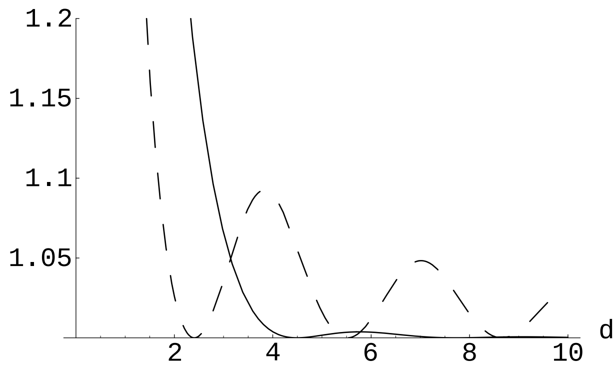

Figure 9.1: The scaling limit $K_\delta^{\mathrm{sc}}(d)$ when $m = 2$ (dashed curve) and $m = 5$ (solid curve)

9.1 The Léray Scaling Limit

Computing the scaling limit of K_δ^N is straightforward. We use the following limit which can be found in [25]:

$$\lim_{N\to\infty} P_N^{m+1}\left(\cos\frac{d}{N}\right) = \Gamma\left(\frac{m}{2}\right)\left(\frac{2}{d}\right)^{\frac{m-2}{2}} J_{\frac{m-2}{2}}(d), \tag{9.1}$$

where J_η is the η-order Bessel function of the first kind. From (8.19), (8.20), (9.1), and (7.1), it follows that

$$K_\delta^{\mathrm{sc}}(d) = \frac{1}{\sqrt{1 - \left(\frac{2}{d}\right)^{m-2}\left[\Gamma\left(\frac{m}{2}\right) J_{\frac{m-2}{2}}(d)\right]^2}}. \tag{9.2}$$

See Figure 9.1, noting that the ordinate scale starts at 1.

9.2 The Riemannian Hypersurface Scaling Limit

In contrast to the previous computation, finding K^{sc} will in fact be quite difficult. We will carry out the calculations only in the case $m = 2$. We begin by recalling from

(7.2) and (8.6) that

$$(9.3)\quad K^N(x,y) = \frac{1}{2\pi^3 c_N}\frac{\Upsilon^N(x,y)}{\sqrt{|F_N|}}$$
$$\iiiint \sqrt{u_1^2+u_2^2}\sqrt{v_1^2+v_2^2}e^{-\frac{1}{2}(u_1,u_2,v_1,v_2)F_N^{-1}(u_1,u_2,v_1,v_2)^t}\,du_1\,du_2\,dv_1\,dv_2,$$

where F_N is defined in (8.5). We can interchange the limit and integral via the Dominated Convergence Theorem. In fact, we will see in a moment that the scaling limit (after suitable normalization) of F_N^{-1} is positive definite, and as such there exists a constant c, which may depend on d, such that the exponential in (9.3) is eventually bounded above by

$$e^{-\frac{1}{2}c\left(u_1^2+u_2^2+v_1^2+v_2^2\right)}.$$

We now choose spherical coordinates (φ,θ) on S^2, with $\cos\varphi = z$ as usual. For convenience, we choose $0<\varphi<\pi$, and $-\pi<\theta<\pi$. Then

$$\left\{\frac{\partial}{\partial\varphi}, \frac{1}{\sin\varphi}\frac{\partial}{\partial\theta}\right\}$$

is an orthonormal frame. We choose x to be the point $(\frac{\pi}{2},0)$, and y to be the point $(\frac{\pi}{2},d)$. Then in the scaling limit,

$$(9.4)\qquad \frac{1}{c_N}F_N \to Y^T Y,$$

where

$$Y = \begin{pmatrix} \alpha(d) & 0 & \gamma(d) & 0 \\ 0 & 1 & 0 & \beta(d) \\ 0 & 0 & \sqrt{\alpha^2(d)-\gamma^2(d)} & 0 \\ 0 & 0 & 0 & \sqrt{1-\beta^2(d)} \end{pmatrix},$$

with

$$\alpha(d) = \sqrt{1-\frac{2J_1^2(d)}{1-J_0^2(d)}},$$
$$\beta(d) = \frac{2J_1(d)}{d},$$
$$\gamma(d) = \frac{\beta(d)+J_0(d)(\alpha^2(d)-1)-J_2(d)}{\alpha(d)}.$$

Note that (9.4) implies that the scaling limit of $\frac{1}{c_N} F_N$ is positive definite.

We now change variables in the integral in $K^{\mathrm{sc}}(d)$ by setting

$$\begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = Y^t \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

and then change again to spherical coordinates in $\mathbb{R}^4$ (which should cause little confusion with the spherical coordinates on S^2) by

$$\begin{aligned} a &= r \sin\varphi \cos\psi, \\ b &= r\cos\varphi, \\ c &= r \sin\varphi \sin\psi \sin\theta, \\ d &= r \sin\varphi \sin\psi \cos\theta, \end{aligned}$$

with $0 \le r \le \infty$, $0 \le \theta \le 2\pi$, and $0 \le \varphi, \psi \le \pi$. Integrating in r we obtain

$$(9.5) \quad K^{\mathrm{sc}}(d) = \frac{4}{\pi^3} \frac{1}{\sqrt{1 - J_0^2(d)}} \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} h(d, \sin\varphi\cos\psi, \cos\varphi, \sin\varphi\sin\psi\sin\theta, \sin\varphi\sin\psi\cos\theta) \sin^2\varphi \sin\psi \, d\varphi \, d\psi \, d\theta,$$

where

$$h(d,s,t,u,v) = \sqrt{t^2 + s^2\alpha^2(d)} \sqrt{\left(t\beta(d) + v\sqrt{1-\beta^2(d)}\right)^2 + \left(u\sqrt{\alpha^2(d) - \gamma^2(d)} + s\gamma(d)\right)^2}.$$

We can then numerically graph $K^{\mathrm{sc}}(d)$ as in Figure 9.2.

For fixed η, the following well-known approximation holds as $d \to \infty$:

$$(9.6) \qquad J_\eta(d) = \sqrt{\frac{2}{\pi d}} \cos\left(d - \frac{\pi}{2}\eta - \frac{\pi}{4}\right) + O\left(\frac{1}{d^{\frac{3}{2}}}\right).$$

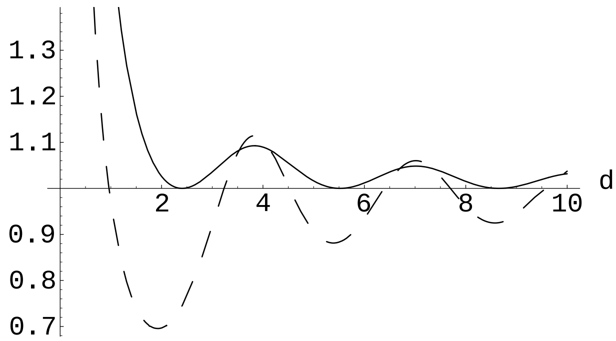

Figure 9.2: The scaling limits $K^{\mathrm{sc}}(d)$ (dashed curve) and $K^{\mathrm{sc}}_\delta(d)$ (solid curve) when $m = 2$

We can conclude from (9.2) that

$$\begin{aligned} K^{\mathrm{sc}}_\delta(d) &\geq 1 + 2^{m-3}\left[\Gamma\left(\frac{m}{2}\right)\right]^2 \left(\frac{1}{d}\right)^{m-2} J^2_{\frac{m-2}{2}}(d) \\ &= 1 + O\left(\frac{1}{d^{m-1}}\right), \end{aligned}$$

as $d \to \infty$. Moreover, by (9.6), the error term cannot be improved. Also, (9.5) exhibits $K^{\mathrm{sc}}(d)$ as a product of the scaling limit of Υ^N, given by

$$\frac{1}{\sqrt{1 - J_0^2(d)}},$$

and an integral. It follows that K^{sc} will also have at best $\frac{1}{d}$ decay. The slow decay of the scaling limit correlation functions tells us that different points on the nodal sets are strongly correlated and are not independent from one another.

Appendix A

A Lemma on Spherical Integration

Lemma A.1. *Let $S^m \subset \mathbb{R}^{m+1}$, $m \geq 2$, be the standard unit sphere with (non-normalized) Lebesgue measure σ_m. Fix $p, q \in S^m$, $p \neq q$. Fix $\epsilon_1, \epsilon_2 > 0$. Let*

$$A = \{x \in S^m : |x \cdot p| < \epsilon_1 \quad \textit{and} \quad |x \cdot q| < \epsilon_2\},$$

where $\cdot$ represents the standard Euclidean inner product in $\mathbb{R}^{m+1}$. Then for all continuous functions φ on S^m, we have

$$\lim_{(\epsilon_1,\epsilon_2)\to(0,0)} \frac{1}{4\epsilon_1\epsilon_2} \int_A \varphi(x)\, d\sigma(x) = \frac{1}{\sqrt{1-(\cos d(p,q))^2}} \int_B \varphi(x)\, d\sigma|_B(x),$$

where

$$B = \{x \in S^m : x \cdot p = x \cdot q = 0\}.$$

In particular,

$$\lim_{(\epsilon_1,\epsilon_2)\to(0,0)} \frac{1}{4\epsilon_1\epsilon_2} \sigma(A) = \frac{|S^{m-2}|}{\sqrt{1-(\cos d(p,q))^2}}.$$

Proof. For simplicity of notation, we will prove the lemma in the case where $\varphi \equiv 1$. The reader should have no difficulty modifying the proof to handle the general case. It is clear also that we may assume $d(p,q) \leq \frac{\pi}{2}$, where d denotes distance as measured on S^m. We write coordinates for $x \in \mathbb{R}^{m+1}$ as $x = (x_1, \ldots, x_{m+1})$, with $p = (1, 0, \ldots, 0)$

and $q = (t, \sqrt{1-t^2}, 0, \ldots, 0)$, where $t = \cos d(p,q)$. We have then

$$x \cdot p = x_1,$$
$$x \cdot q = x_1 t + x_2\sqrt{1-t^2}.$$

For $\epsilon_1, \epsilon_2 > 0$, we have that $x \in S^m$ is an element of A if and only if

$$-\epsilon_1 < x_1 < \epsilon_1, \quad \text{and}$$
$$\frac{-\epsilon_2 - x_1 t}{\sqrt{1-t^2}} < x_2 < \frac{\epsilon_2 - x_1 t}{\sqrt{1-t^2}}.$$

To find $\sigma_m(A)$ we slice integrate in x_1 and x_2 to obtain

$$\sigma_m(A) = |S^{m-2}| \int_{-\epsilon_1}^{\epsilon_1} \int_{\frac{-\epsilon_2 - x_1 t}{\sqrt{1-t^2}}}^{\frac{\epsilon_2 - x_1 t}{\sqrt{1-t^2}}} \left(\sqrt{1 - x_1^2 - x_2^2}\right)^{m-3} \, dx_2 \, dx_1.$$

We now change variables in the inner integral by setting

$$v = x_1 t + x_2\sqrt{1-t^2}.$$

Then

(A.1) $\sigma_m(A) =$

$$|S^{m-2}| \left(1-t^2\right)^{-\frac{m-2}{2}} \int_{-\epsilon_1}^{\epsilon_1} \int_{-\epsilon_2}^{\epsilon_2} \left((1-x_1^2)(1-t^2) - (v - x_1 t)^2\right)^{\frac{m-3}{2}} \, dv \, dx_1.$$

By the mean value theorem for integrals, we have

$$\lim_{(\epsilon_1,\epsilon_2)\to(0,0)} \frac{1}{4\epsilon_1\epsilon_2} \sigma_m(A) = \frac{|S^{m-2}|}{\sqrt{1-t^2}},$$

as desired. $\square$

Appendix B

Legendre Polynomials

Fix an integer $M \geq 3$. Define an inner product $\langle \, , \rangle_M$ on $L^2[-1,1]$ by

$$\langle f, g \rangle_M = \int_{-1}^{1} f(t) g(t) (1-t^2)^{\frac{M-3}{2}} \, dt.$$

Definition B.1. Let $\{P_0^M(t), \ldots, P_N^M(t), \ldots\}$ be the set of real polynomials uniquely determined by the following requirements:

1. degree $P_N^M(t) = N$,
2. $\langle P_N^M(t), P_{N'}^M(t) \rangle_M = 0$ if $N \neq N'$, and
3. $P_N^M(1) = 1$.

We call $P_N^M(t)$ the *Nth Legendre polynomial of order M.*

We collect some facts on Legendre polynomials which can be found in [12] and [25].

Fact B.2. *The following statements hold:*

1. $P_N^M(-t) = (-1)^N P_N^M(t)$,
2. $|P_N^M(t)| < 1$ *for* $-1 < t < 1$,

3. $\frac{d}{dt}P_N^M(t) = \frac{N(N+M-2)}{M-1}P_{N-1}^{M+2}(t)$ *for* $N \geq 1$*,*

4. $P_N^M(\cos\theta) = \theta^{-\frac{M-2}{2}}O\left(N^{-\frac{M-2}{2}}\right)$*,* $0 < \theta \leq \pi$*, where the implied constant is uniform in* θ *but depends on* M*, and*

5. On S^m*,*

$$\langle \Phi_N(x), \Phi_N(y)\rangle_{L^2} = \Phi_N(x)(y) = \frac{d_N}{|S^m|}P_N^{m+1}(\cos d(x,y)),$$

where $d(x,y)$ *denotes the distance between* x *and* y*.*

As a corollary of Corollary 3.3 taking $f(x)$ to be $\Phi_N(y)(x)$ properly normalized, and applying Fact B.2.3, we have the following estimate.

Corollary B.3. *For* $\theta \in (0,\pi)$*, we have*

$$|P_{N-1}^{m+3}(\cos\theta)| < \sqrt{\frac{m}{\lambda_N}\frac{1}{\sin\theta}}.$$

In the case $m = 2$, compare with (18) in III.10 of [22].

Recall from chapter 7 the definition of Υ^N. We need to show for technical reasons that $\Upsilon^N \in L^1(S^m \times S^m)$.

Lemma B.4. *For some* C*, independent of* N*, we have*

$$\int_{S^m}\int_{S^m} \Upsilon^N(x,y)\,dx\,dy \leq C < \infty$$

Proof. Fix $x_0 \in S^m$. We have

$$\begin{aligned}
\int_{S^m}\int_{S^m} \Upsilon^N(x,y)\,dx\,dy &= \int_{S^m}\int_{S^m} \frac{1}{\sqrt{1-\left(P_N^{m+1}(\cos d(x,y))\right)^2}}\,dx\,dy \\
&= |S^m|\int_{S^m} \frac{1}{\sqrt{1-\left(P_N^{m+1}(\cos d(x_0,y))\right)^2}}\,dy \\
&= 2|S^m||S^{m-1}|\int_0^1 \frac{(1-t^2)^{\frac{m-2}{2}}}{\sqrt{1-(P_N^{m+1}(t))^2}}\,dt \\
&= 2|S^m||S^{m-1}|\int_0^{\frac{\pi}{2}} \frac{\sin^{m-1}u}{\sqrt{1-(P_N^{m+1}(\cos u))^2}}\,du, \qquad \text{(B.1)}
\end{aligned}$$

where $t = \cos u$. The integrand in (B.1) is uniformly bounded in N by Lemma 8.3, and the lemma follows. ∎

Appendix C

Separability

Let $C_{\mathbb{R}}(S^m)$ denote the set of all real-valued continuous functions on S^m.

Lemma C.1. *The space $C_{\mathbb{R}}(S^m)$ is L^∞-separable. By this we mean there exists a countable L^∞-dense subset of $C_{\mathbb{R}}(S^m)$.*

Proof. Throughout this proof, m will be fixed. We will construct the needed subset. For each $N \geq 0$, choose a basis $\{f_1^N, \ldots, f_{d_N}^N\}$ of E_N. Let A be the union of all these basis functions, i.e.,

$$A = \bigcup_{N=0}^{\infty} \bigcup_{j=1}^{d_N} \{f_j^N\}.$$

Let $\mathcal{A}_{\mathbb{R}}$ be the algebra of all $\mathbb{R}$-linear combinations of finite products of elements of A. Let $\mathcal{A}_{\mathbb{Q}} \subset \mathcal{A}_{\mathbb{R}}$ be the set of all $\mathbb{Q}$-linear combinations of finite products of elements of A.

Note that $\mathcal{A}_{\mathbb{Q}}$ is a countable set. Moreover, since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can easily see that $\mathcal{A}_{\mathbb{Q}}$ is L^∞-dense in $\mathcal{A}_{\mathbb{R}}$. By the Stone-Weierstrass Theorem [21], $\mathcal{A}_{\mathbb{R}}$ is L^∞-dense in $C_{\mathbb{R}}(S^m)$. Hence $\mathcal{A}_{\mathbb{Q}}$ is a countable L^∞-dense subset of $C_{\mathbb{R}}(S^m)$, as needed. $\square$

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