THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences
MIDTERM EXAM - FALL SESSION 2006
110.401 - ADVANCED ALGEBRA I.

Examiner: Professor C. Consani
Duration: 50 MINUTES (11am-11:50am), October 25, 2006.

No calculators allowed. **Total Marks = 100**

SOLUTIONS

1. [20 marks] Show that the elements

$$1,\ (1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)$$

form a subgroup V_4 of A_4 ($A_4 =$ the alternating group of degree 4).

Define a group isomorphism

$$\varphi : V_4 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

i.e.: define explicitly the map φ and show that φ is a group isomorphism.

Sol. The set $V_4 := \{1,\ (1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)\}$ is a subset of A_4 since $\epsilon(\sigma) = 1$, $\forall \sigma \in V_4$, where ϵ denotes the sign-homomorphism $\epsilon : S_4 \to \{\pm 1\}$ (any element in V_4, besides the identity, is a product of 2 transpositions).
We use The Subgroup Criterion: let σ and σ_1 be two elements in $V_4 \subset A_4$, then a direct computations shows that $\sigma \cdot \sigma_1^{-1} \in A_4$, in fact first of all we notice that

$$(1\ 2)(3\ 4) \cdot (1\ 2)(3\ 4) = 1, \quad (1\ 3)(2\ 4) \cdot (1\ 3)(2\ 4) = 1,$$
$$(1\ 2)(3\ 4) \cdot (1\ 3)(2\ 4) = (1\ 4)(2\ 3), \quad (1\ 4)(2\ 3) \cdot (1\ 4)(2\ 3) = 1$$

where $\cdot$ is the group operation in A_4 (i.e. composition of permutations). Therefore, every element of V_4, besides the identity, has order 2. This implies that $\forall\ \sigma \in V_4$, $\sigma^{-1} = \sigma$. Then we conclude by verifying directly that $\forall\ \sigma, \sigma_1 \in V_4$:

$$\sigma \cdot \sigma_1^{-1} = \sigma \cdot \sigma_1 \in V_4.$$

Call $a = (1\ 2)(3\ 4)$, $b = (1\ 3)(2\ 4)$, $c = (1\ 4)(2\ 3)$. Then we have

$$a^2 = b^2 = c^2 = 1,\ ab = c,\ ba = (ab)^{-1} = ab,\ ac = ca = b,\ bc = cb = a.$$

This shows also that V_4 is a commutative subgroup of order 4 of A_4 and that a and b are 2 generators of V_4, i.e. $V_4 =< a,\ b\ : a^2 = 1 = b^2 >$.

We define

$$\varphi : V_4 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

by first describing this map on the generators of V_4 and then we extend its definition to the whole of V_4, compatibly with the group structures (i.e. $\varphi(c) = \varphi(ab) = \varphi(a) + \varphi(b)$, $\varphi(1) = \varphi(a^2) = \varphi(a) + \varphi(a)$)

$$\varphi(a) := (1 + 2\mathbb{Z}, 2\mathbb{Z}),\ \varphi(b) := (2\mathbb{Z}, 1 + 2\mathbb{Z}).$$

By construction, φ is a group homomorphism. Moreover, φ is surjective since the generators of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are reached by elements (in fact generators) of V_4. Finally it is immediate to check that φ is also injective.

2. [20 marks] Let Q_8 be the quaternion group and let V_4 be the Klein group (i.e. the group of order 4 isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and considered in the previous question).

Show that the center of Q_8, $Z(Q_8)$, is the kernel of a group homomorphism

$$\varphi : Q_8 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

Deduce the description of V_4 as a quotient of Q_8.

Sol. It is very easy to verify that $Z(Q_8) = \{\pm 1\}$: $ij = -ji$, $jk = -kj$ therefore $\pm i, \pm j, \pm k \notin Z(Q_8) := \{g \in Q_8 \ : \ gg' = g'g, \ \forall g' \in Q_8\}$.

We define the following group homomorphism

$$\varphi : Q_8 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$
$$\varphi(i) = (1 + 2\mathbb{Z}, 2\mathbb{Z}),\ \varphi(j) = (2\mathbb{Z}, 1 + 2\mathbb{Z}),\ \varphi(k) = (1 + 2\mathbb{Z}, 1 + 2\mathbb{Z}).$$

We have: $\varphi(-1) = \varphi(i^2) = \varphi(i)+\varphi(i) = (2\mathbb{Z}, 2\mathbb{Z}) = \varphi(1) = \varphi(-i^2) = \varphi(-i)+\varphi(i) = \varphi(k) + \varphi(j) + \varphi(i)$. Also, $\varphi(-i) = \varphi(i)$, $\varphi(-j) = \varphi(j)$ and $\varphi(-k) = \varphi(k)$. This shows that $Ker(\varphi) = Z(Q_8)$.
Moreover, the homomorphism is clearly surjective, hence the First Isomorphism Theorem implies that φ induces the isomorphism: $Q_8/Z(Q_8) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \simeq V_4$.

3. [30 marks] Is the map

$$\varphi : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}, \qquad \varphi(m + 6\mathbb{Z}) = m + 8\mathbb{Z}$$

a group homomorphism? If yes, show in details your proof; if not, explain why is so and define a suitable group homomorphism ψ between these 2 groups.

How many different group homomorphisms do there exist connecting these two groups? Define them explicitly.

Sol. The map φ defined by $\varphi(m + 6\mathbb{Z}) = m + 8\mathbb{Z}$ is not a group homomorphism. In fact, $1 \in 1 + 6\mathbb{Z}$ and its period (in $\mathbb{Z}/6\mathbb{Z}$) is $|1| = 6$, also $1 \in 1 + 8\mathbb{Z}$ and its period (in $\mathbb{Z}/8\mathbb{Z}$) is $o(1) = 8$. However 8 does not divide 6. From such definition we would get for example that $\varphi(6\mathbb{Z}) = \varphi(6(1+6\mathbb{Z})) = 6(1+8\mathbb{Z}) \neq 8\mathbb{Z}$, that is the identity of the first group is not sent to the identity of the second group.
In order to define an appropriate group homomorphism a chosen generator of the first group (whose order is 6) should be sent to an element of the second group, whose order (divides 8) must also divide the order of the chosen generator of the first group. For example, we could choose $1 + 6\mathbb{Z}$ as a generator of $\mathbb{Z}/6\mathbb{Z}$ and define

$$\psi : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}, \qquad \psi(1 + 6\mathbb{Z}) = 4 + 8\mathbb{Z}$$

That is $\psi(m + 6\mathbb{Z}) = 4m + 8\mathbb{Z}$.
Using the condition on the divisibility of the orders as explained above, we conclude that $\psi(m+6\mathbb{Z}) = 8\mathbb{Z}$ (i.e. the trivial homomophism) and $\psi(m+6\mathbb{Z}) = 4m+8\mathbb{Z}$ are the only possible homomorphisms.

4. [30 marks] Let G be a finite group and let $H < G$ be a subgroup of G, with $|G : H| = 2$.

Show that H contains all the elements of G of odd order (=period).

Sol. $|G : H| = 2$ implies that H is normal in G (cfr. Dummit and Foote for a proof). Hence G/H is a well-defined quotient group. Let $x \in G$ have odd order n. Then $(xH)^n = x^n H = 1H = H$, and so the order of xH divides n. In particular, the order of xH is odd. But G/H has order $|G : H| = 2$, and since the nontrivial element of G/H has order 2 we must have $xH = H$, which shows that $x \in H$.