1. p. 248 #8 \( \text{Let } R = \mathbb{Z} \times \mathbb{Z} \)

a) \( \text{No: } (1, 0) \cdot (1, 1) = (1, 0) \notin I \)

b) \( \text{Yes: } (c, d) \cdot (2a, 2b) = (2ac, 2bd) \in I \)

\[ (2a, 2b) + (2c, 2d) = (2(ac), 2(md)) \in I \]

c) \( \text{Yes: } (c, d)(2a, 0) = (2ac, 0) \)

\[ (2a, 0) + (2b, 0) = (2(a+c), 0) \]

d) \( \text{No: } (1, 0) \cdot (1, -1) = (1, 0) \notin I \)

p. 248 #9 \( \text{Let } R = \mathbb{Z}[x] \)

c) \( \text{Yes: if } f(x) \in \mathbb{Z}[x], q(x) = 3k \text{, then } p(x) \cdot q(x) = 3b \text{, and } p(x) + q(x) = 3b \text{, then } f(x) \in I \)

b) \( \text{No: } f(x) = 1 + 2x^2, g(x) = x^2 \text{. Then } f \in I \text{ but } f + g = x^2 + 3x^4 \text{ is not } I \)

d) \( \text{Yes: clearly closed } \ast + \ast \text{. If } f(x) = a_0 + a_1 x + \ldots + a_n x^n, g(x) = b_0 + b_1 x + \ldots + b_n x^n, \text{ then } f(x) + g(x) \in I \text{ if } f(x) \in I \text{ and } g(x) \in I \)

e) \( \text{No: } f(x) = x^2 \in I, g(x) \cdot x \in R : f(x) = x^3 \notin I \)

p. 256 #9 \( I \text{ is an ideal since it is the kernel of the ring homomorphism } \phi : R \rightarrow R \times R \text{ def by } \)

\[ f(x) \mapsto (f(1/2), f(1/3)) \]

I not prime since \((x - 1/2)(x - 1/3) \in I \text{, but } (x - 1/2) \notin I, (x - 1/3) \notin I \)

p. 257 #14 \( \text{Let } R \text{ be comm, } f(x) \in R[x]\) whose degree \( n \geq 1 \).

c) \( \text{If } f(x) = x^n + b_{n-1} x^{n-1} + \ldots + b_0, \text{ then } \overline{x}^n = -\overline{b_{n-1} x^{n-1} + \ldots + b_0} \)

- for powers of \( x \) greater than or equal to \( x^n \), replace with \( \overline{f(x)} \) to get \( \overline{f(x)} = \{a_n + a_{n-1} x + \ldots + a_0 x^n \} / a_i \in R \)

b) \( \text{Let } p(x) \neq q(x) \in R[x], \text{ both of deg } < n \)

\[ \overline{p(x)} \neq \overline{q(x)} \text{ since } \overline{p(x)} = \overline{q(x)} \text{ mod } f(x), \overline{p(x)} - \overline{q(x)} \equiv 0 \text{ mod } \overline{f(x)} \]

ie \( f(x) | p(x) - q(x) \). But \( \deg(p(x) - q(x)) < \deg f(x) \rightarrow \)
3. \( \mathbb{R} \) = set maps from \( \{0,1\} \) to \( \mathbb{R} \)

\[ \mathbb{R}^{0,1} = \{ a+br' | a, b \in \{0,1\}, r, r' \in \mathbb{R} \} \]

The zero divisors of \( \mathbb{R} \) are elements of the form \( (0, r) \) and \( (r', 0) \) for \( r, r' \in \mathbb{R} \)

All prime ideals are of the form \( \mathbb{R} \times \mathbb{P} \) and \( \mathbb{P} \times \mathbb{R} \), so since \( \mathbb{R} \) is a field, this is just \( \mathbb{R} \times \{0,1\}, \{0,1\} \times \mathbb{R} \)

These prime ideals are maximal since \( \mathbb{R}^{0,1} / \{0,1\} \mathbb{R} \cong \mathbb{R}^{0,1} / (0,0) \mathbb{R} \cong \{0,1\} \)

Clearly \( \mathbb{R}^{0,1} = \{ a+br' | a, b \in \{0,1\}, r, r' \in \mathbb{R} \} \) as \( \mathbb{R} \) vector spaces

The idempotent elements are \((0,0), (0,1), (1,0), (1,1)\)

4. Let \( A \) be a commutative ring with a unit, let \( I, J \) be two maximal ideals. Then map is surjective by Chinese Remainder Theorem, with kernel \( IJ = \langle n J \rangle \)

5. Let \( A \) be commutative ring without proper non-trivial ideals. If \( 1 \notin A \), then \( A \) is a field by Pop 9, p 253

Now assume \( 1 \in A \). If \( 3d, a \in A \) s.t. \( da \neq 0 \), then \( Ra \) is an ideal, so \( Ra = A \)

: \( 3 \in A, a \in A \) s.t. \( ba = a \) when \( a, b \neq 0 \).

Then \( ta \in A \), \( r = r' a \) for some \( r' \in R \)

: \( br = br' a = r' ba = ra = r \) \( : b = 1 \) or \( a = 0 \)

: if \( 1 \notin A \), the product of any two elements is zero

Now all ideals are just subgroups, so \( R \) must contain no proper non-trivial subgroups, i.e. \( R = (\mathbb{Z}_p, +, x) \), where mult is 0.

4. Let \( (p) \) be the prime ideal in \( R \). Then \( R/(p) \) is a finite integral domain, which by Cor 3 p 228 is a field, so by Pop 12 p 234, \( (p) \) is maximal
7. This is prop. 4.5 p. 717
\[ R = \{ \frac{f(x)}{g(x)} \mid g(x) \neq 0 \} \] is a local ring since this ring has unique max ideal \( \{ f(x) \in \mathbb{A} \mid f(a) = 0 \} \).

8. Let \( A \) be a local ring. Let \( x \in A \). Since \( x + (1-x) = 1 \) is invertible by 7.1, either \( x \) or \( 1-x \) is invertible.

Let \( e = e^2 \) be idempotent.

Then \( (1-e)^2 = 1-e \) so \( (1-e) \) idempotent as well.

- Either \( e \) or \( 1-e \) is invertible.

  - If \( e \) invertible, \( 3e^{-1} A : e^2 = e \Rightarrow e = 1 \)

  - If \( 1-e \) invertible, \( (1-e)^2 = (1-e) \Rightarrow (1-e) = 1 \Rightarrow e = 0 \).

  \[ e = 0 \text{ or } 1. \]