302 HW #3

2.6

4. First divide both sides by $(2xy+2)$. We now have $M(x,y) = y$ and $N(x,y) = x$. Since $M_y = N_x = 0$, the resulting equation is *exact*. Integrating $M$ with respect to $x$, while holding $y$ constant, results in $\psi(x,y) = xy + h(y)$. Differentiating with respect to $y$, $\psi_y = x + h'(y)$. Setting $\psi_y = N$, we find that $h'(y) = 0$, and hence $h(y) = 0$ is acceptable. Therefore the solution is defined *implicitly* as $xy = c$. Note that if $xy + 1 = 0$, the equation is trivially satisfied.

5. Writing the equation in the form $M(x,y)dx + N(x,y)dy = 0$ gives $M(x,y) = ax + by$ and $N(x,y) = bx + cy$. Now $M_y = b = N_x$ and the equation is exact. Integrating $M(x,y)$ with respect to $x$ yields $\psi(x,y) = (a/2)x^2 + bxy + h(y)$. Differentiating $\psi$ with respect to $y$ ($x$ constant) and setting $\psi_y(x,y) = N(x,y)$ we find that $h'(y) = cy$ and thus $h(y) = (c/2)y^2$. Hence the solution is given by $(a/2)x^2 + bxy + (c/2)y^2 = k$.

6. Write the given equation as $(ax - by)dx + (bx - cy)dy$. Now $M(x,y) = ax - by$ and $N(x,y) = bx - cy$. Since $M_y \neq N_x$, the differential equation is *not* exact.

7. $M_y(x,y) = e^x\cos y - 2\sin x = N_x(x,y)$ and thus the D.E. is exact. Integrating $M(x,y)$ with respect to $x$ gives $\psi(x,y) = e^x\sin y + 2y\cos x + h(y)$. Finding $\psi_y(x,y)$ from this and setting that equal to $N(x,y)$ yields $h'(y) = 0$ and thus $h(y)$ is a constant. Hence an implicit solution of the D.E. is $e^x\sin y + 2y\cos x = c$. The solution $y = 0$ is also valid since it satisfies the D.E. for all $x$.

8. $M(x,y) = e^x \sin y + 3y$ and $N(x,y) = -3x + e^x \sin y$. Note that $M_y \neq N_x$, and hence the differential equation is *not* exact.

9. If you try to find $\psi(x,y)$ by integrating $M(x,y)$ with respect to $x$ you must integrate by parts. Instead find $\psi(x,y)$ by integrating $N(x,y)$ with respect to $y$ to obtain $\psi(x,y) = e^{xy}\cos 2x - 3y + g(x)$. Now find $g(x)$ by differentiating $\psi(x,y)$ with respect to $x$ and set that equal to $M(x,y)$, which yields $g'(x) = 2x$ or $g(x) = x^2$.

25. The equation is not exact so we must attempt to find an integrating factor. Since $\frac{1}{N}(M_y - N_x) = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} = 3$ is a function of x alone there is an integrating factor depending only on x, as shown in Eq.(26). Then $d\mu/dx = 3\mu$, and the integrating factor is $\mu(x) = e^{3x}$. Hence the equation can be solved as in Example 4.

26. An integrating factor can be found which is a function of x only, yielding $\mu(x) = e^{-x}$. Alternatively, you might recognize that $y' - y = e^{2x} - 1$ is a linear first order equation which can be solved as in Section 2.1.

27. Using the results of Problem 23, it can be shown that $\mu(y) = y$ is an integrating factor. Thus multiplying the D.E. by y gives $ydx + (x - y\sin y)dy = 0$, which can be identified as an exact equation. Alternatively, one can rewrite the last equation as $(ydx + xdy) - y\sin y\, dy = 0$. The first term is $d(xy)$ and the last can be integrated by parts. Thus we have $xy + y\cos y - \sin y = c$.

29. Multiplying by $\sin y$ we obtain $e^x \sin y\, dx + e^x \cos y\, dy + 2y\, dy = 0$, and the first two terms are just $d(e^x \sin y)$. Thus, $e^x \sin y + y^2 = c$.

28. The equation is not exact, since $N_x - M_y = 2y - 1$. However, $(N_x - M_y)/M = = (2y-1)/y$ is a function of $y$ alone. Hence there exists $\mu = \mu(y)$, which is a solution of the differential equation $\mu' = (2 - 1/y)\mu$. The latter equation is *separable*, with $d\mu/\mu = 2 - 1/y$. One solution is $\mu(y) = exp(2y - \ln y) = e^{2y}/y$. Now rewrite the given ODE as $e^{2y}dx + (2x\,e^{2y} - 1/y)dy = 0$. This equation is *exact*, and it is easy to see that $\psi(x,y) = x\,e^{2y} - \ln y$. Therefore the solution of the given equation is defined implicitly by $x\,e^{2y} - \ln y = c$.

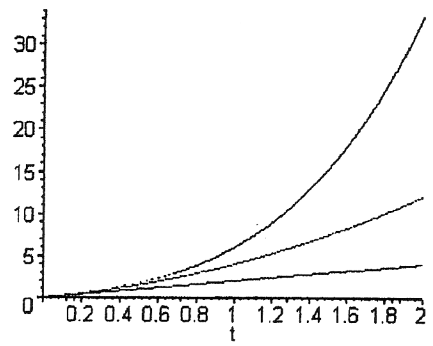
28

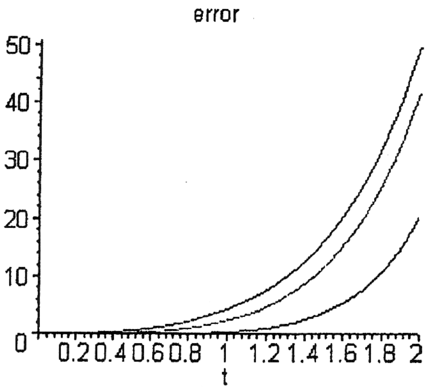
3. The approximating functions are defined recursively by $\phi_{n+1}(t) = \int_0^t 2[\phi_n(s) + 1]ds$. Setting $\phi_0(t) = 0$, $\phi_1(t) = 2t$. Continuing, $\phi_2(t) = 2t^2 + 2t$, $\phi_3(t) = \frac{4}{3}t^3 + 2t^2 + 2t$, $\phi_4(t) = \frac{2}{3}t^4 + \frac{4}{3}t^3 + 2t^2 + 2t, \cdots$. Given convergence, set

$$\begin{aligned}\phi(t) &= \phi_1(t) + \sum_{k=1}^{\infty}[\phi_{k+1}(t) - \phi_k(t)] \\ &= 2t + \sum_{k=2}^{\infty}\frac{a_k}{k!}t^k.\end{aligned}$$

Comparing coefficients, $a_3/3! = 4/3$, $a_4/4! = 2/3$, $\cdots$. It follows that $a_3 = 8$, $a_4 = 16$,
and so on. We find that in general, that $a_k = 2^k$. Hence

$$\begin{aligned}\phi(t) &= \sum_{k=1}^{\infty} \frac{2^k}{k\,!} t^k \\ &= e^{2t} - 1\,.\end{aligned}$$



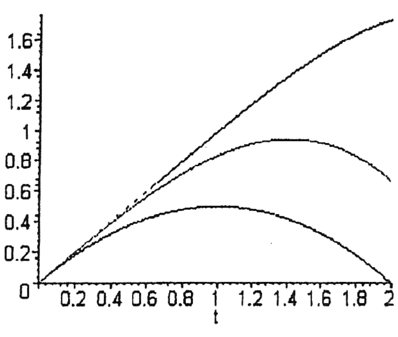

6. The approximating functions are defined recursively by

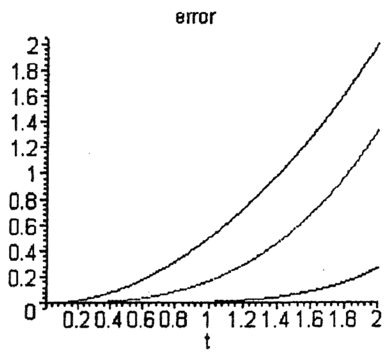
$$\phi_{n+1}(t) = \int_0^t [\phi_n(s) + 1 - s] ds\,.$$

Setting $\phi_0(t) = 0$, $\phi_1(t) = t - t^2/2$, $\phi_2(t) = t - t^3/6$, $\phi_3(t) = t - t^4/24$, $\phi_4(t) =$
$= t - t^5/120$, $\cdots$. Given convergence, set

$$\begin{aligned}\phi(t) &= \phi_1(t) + \sum_{k=1}^{\infty} [\phi_{k+1}(t) - \phi_k(t)] \\ &= t - t^2/2 + \left[t^2/2 - t^3/6\right] + \left[t^3/6 - t^4/24\right] + \cdots \\ &= t + 0 + 0 + \cdots\,.\end{aligned}$$

Note that the terms can be rearranged, as long as the series converges *uniformly*.



18. An algebraic equation with roots $-2$ and $-1/2$ is $2r^2+5r+2=0$. This is the characteristic equation for the ODE $2y''+5y'+2y=0$.

22. The characteristic equation is $4r^2-1=0$, with roots $r=\pm 1/2$. Hence the general solution is $y=c_1e^{-t/2}+c_2e^{t/2}$, with derivative $y'=-c_1e^{-t/2}/2+c_2e^{t/2}/2$. Invoking the initial conditions, we require that $c_1+c_2=2$ and $-c_1+c_2=\beta$. The specific solution is $y(t)=(1-\beta)e^{-t/2}+(1+\beta)e^{t/2}$. Based on the form of the solution, it is evident that as $t\to\infty$, $y(t)\to 0$ as long as $\beta=-1$.

3.2.

2. $W(\cos t,\sin t)=\begin{vmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{vmatrix}=\cos^2 t+\sin^2 t=1.$

3.

$$W\left(e^{-2t},te^{-2t}\right)=\begin{vmatrix}e^{-2t} & te^{-2t}\\ -2e^{-2t} & (1-2t)e^{-2t}\end{vmatrix}=e^{-4t}.$$

7. Write the equation as $y''+(3/t)y'=1$. $p(t)=3/t$ is continuous for all $t>0$. Since $t_0>0$, the IVP has a unique solution for all $t>0$.

21. From Section 3.1, $e^t$ and $e^{-2t}$ are two solutions, and since $W(e^t,e^{-2t})\neq 0$ they form a fundamental set of solutions. To find the fundamental set specified by Theorem 3.2.5, let $y(t)=c_1e^t+c_2e^{-2t}$, where $c_1$ and $c_2$ satisfy
$c_1+c_2=1$ and $c_1-2c_2=0$ for $y_1$. Solving, we find
$y_1=\frac{2}{3}e^t+\frac{1}{3}e^{-2t}$. Likewise, $c_1$ and $c_2$ satisfy
$c_1+c_2=0$ and $c_1-2c_2=1$ for $y_2$, so that
$y_2=\frac{1}{3}e^t-\frac{1}{3}e^{-2t}$.

22. The general solution is $y=c_1e^{-3t}+c_2e^{-t}$. $W(e^{-3t},e^{-t})=2e^{-4t}$, and hence the exponentials form a *fundamental set* of solutions. On the other hand, the *fundamental solutions* must also satisfy the conditions $y_1(1)=1$, $y_1'(1)=0$; $y_2(1)=0$, $y_2'(1)=1$. For $y_1$, the initial conditions require $c_1+c_2=e$, $-3c_1-c_2=0$. The coefficients are $c_1=-e^3/2$, $c_2=3e/2$. For the solution, $y_2$, the initial conditions require $c_1+c_2=0$, $-3c_1-c_2=e$. The coefficients are $c_1=-e^3/2$, $c_2=e/2$. Hence the fundamental solutions are $\left\{y_1=-\frac{1}{2}e^{-3(t-1)}+\frac{3}{2}e^{-(t-1)},\ y_2=-\frac{1}{2}e^{-3(t-1)}+\frac{1}{2}e^{-(t-1)}\right\}$.