THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences
FINAL EXAM - SPRING SESSION 2005
110.201 - LINEAR ALGEBRA.

Examiner: Professor C. Consani
Duration: 3 HOURS (9am-12noon), May 12, 2005.

No calculators allowed. **Total Marks = 100**

Student Name: ______________________________

TA Name & Session: _________________________

1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
Total	

1. [10 marks] Consider the matrix $A = \begin{bmatrix} 2 & 1 & 0 & 4 \\ 2 & 1 & 1 & 2 \\ 4 & 2 & 3 & 2 \end{bmatrix}$.

1a. [2 marks] Compute the reduced row-echelon form of A.

1b. [2 marks] Determine the rank of A.

1c. [2 marks] Determine a basis of the column space of A.

1d. [2 marks] Determine a basis of the nullspace of A.

1e. [2 marks] For what value(s) of $r \in \mathbb{R}$ is the following system solvable

$$A\underline{x} = \begin{bmatrix} 2 \\ 3 \\ r \end{bmatrix}?$$

Sol.

[1a.]

$$A = \begin{bmatrix} 2 & 1 & 0 & 4 \\ 2 & 1 & 1 & 2 \\ 4 & 2 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 4 & 2 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & -6 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \mathbf{1} & 1/2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} \mathbf{1} & 1/2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

[1b.] It follows from the description of rref(A) that rk$(A) = 2$.

[1c.] A basis of the column space of A is given by the vectors $\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ corresponding to the pivot columns in A.

[1d.] There are two non-trivial relations among the columns of A. Let denote by $\underline{v}_1, \ldots, \underline{v}_4$ the first,...,fourth column of A. We have

$$\underline{v}_1 - 2\underline{v}_2 = \underline{0}, \qquad 2\underline{v}_1 - 2\underline{v}_3 - \underline{v}_4 = \underline{0}.$$

Hence, a basis of the nullspace of A is given by the vectors $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -2 \\ -1 \end{bmatrix}$.

[1e.] Consider the complete matrix B associated to the system

$$B = \left[\begin{array}{cccc|c} 2 & 1 & 0 & 4 & 2 \\ 2 & 1 & 1 & 2 & 3 \\ 4 & 2 & 3 & 2 & r \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & -6 & r-4 \end{array}\right] \to \left[\begin{array}{cccc|c} \mathbf{2} & 1 & 0 & 4 & 2 \\ 0 & 0 & \mathbf{1} & -2 & 1 \\ 0 & 0 & 0 & 0 & r-7 \end{array}\right]$$

The system is solvable if and only if $r = 7$.

2. [15 marks] Consider the matrix $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$.

2a. [5 marks] Give a factorization $A = QR$, where R is an upper-triangular matrix and Q is a matrix with orthonormal columns.

2b. [5 marks] Find the least square solution to the system

$$A\underline{x} = \underline{b}, \qquad \text{for} \quad \underline{b} = \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix}.$$

2c. [5 marks] The projection matrix $P = A(A^T A)^{-1} A^T$ projects all vectors onto the column space of A. Find a vector $\underline{q}$, not in the column space of A such that

$$P\underline{q} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}.$$

Sol.

[2a.] Perform Gram-Schmidt process on the (linearly independent) columns of A and get two orthonormal vectors fitting the columns of

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \end{bmatrix}.$$

Then, obtain R via $R = Q^T A$, i.e. $R = \begin{bmatrix} \sqrt{3} & 2\sqrt{3} \\ 0 & \sqrt{6} \end{bmatrix}$.

[2b.] The system $(A^T A)\underline{x} = A^T \underline{b}$ is

$$\begin{bmatrix} 3 & 6 \\ 6 & 18 \end{bmatrix} \underline{x} = \begin{bmatrix} 18 \\ 30 \end{bmatrix}$$

with (unique) solution $\begin{bmatrix} 8 \\ -1 \end{bmatrix}$. This is the least square solution to the system $A\underline{x} = \underline{b}$.

[2c.] The vector $\underline{p} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}$ is in the column space of A. Moreover, the vector $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ spans the nullspace of P. Hence there are infinitely many choices for the vector $\underline{q}$ and they are

$$\underline{q} = \underline{p} + t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad t \in \mathbb{R}.$$

3. [15 marks]

3a. [4 marks] Give a 3×3-matrix A with the following properties:

i. $A^T = A^{-1}$.
ii. $\det(A) = 1$. (A is not allowed to be a diagonal matrix)

3b. [4 marks] Give a 3×3-matrix with the following properties:

i. $A^T = A$.
ii. $A^2 = A$.
iii. $\text{rk}(A) = 1$. (A is not allowed to be a diagonal matrix)

3c. [4 marks] Suppose A is a 5×3-matrix with orthonormal columns. Evaluate the following determinants:

i $\det(A^T A)$
ii $\det(AA^T)$
iii $\det(A(A^T A)^{-1} A^T)$.

3d. [3 marks] Which value(s) of $\alpha \in \mathbb{R}$ give $\det(A) = 0$, if

$$A = \begin{bmatrix} \alpha & 2 & 3 \\ -\alpha & \alpha & 0 \\ 3 & 2 & 5 \end{bmatrix}?$$

Sol.

[3a.] We consider, for example an orthogonal matrix A, with $\det(A) = 1$. Permutation matrices such as $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ have this property.

[3b.] We may consider a projection matrix of rank 1: for example (cfr. question 2c.) $A = \underline{v}(\underline{v}^T\underline{v})^{-1}\underline{v}^T$, where $\underline{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Namely the matrix $A = \frac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

[3c.]

$$\det(A^T A) = \det(I) = 1$$
$$\det(AA^T) = 0$$
$$\det(A(A^T A)^{-1} A^T) = 0$$

AA^T must have dependent columns and determinant zero because $A(A^T\underline{x}) = \underline{0}$ for any non-zero vector $\underline{x}$ in the nullspace of A^T. The 3×5-matrix A^T has 3 linearly independent (orthonormal!) rows and a non-trivial nullspace of dimension $5 - 3 = 2$. Notice that $\det(A(A^T A)^{-1} A^T) = \det(AA^T) = 0$ as $A^T A = I$.

[3d.] $\det(A) = 5\alpha(\alpha - 1) = 0$. Therefore, $\alpha = 0$ or $\alpha = 1$.

4. [15 marks] Suppose the following information is known about a matrix A:

i. $A \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}$

ii. $A \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 6 \end{bmatrix}$

iii. A is symmetric.

The following questions refer to any matrix A with the above properties

4a. [3 marks] Is $\text{Ker}(A) = \{\underline{0}\}$? Explain your answer.

4b. [3 marks] Is A invertible? Why?

4c. [3 marks] Does A have linearly independent eigenvectors? Explain.

4d. [6 marks] Give a specific example of a matrix A satisfying the above three properties and whose eigenvalues add up to zero.

Sol.

[4a.] A has linearly dependent columns: for example, it follows from conditions i. and ii. that $A(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}) = \underline{0}$. This implies that $\text{Ker}(A) \neq \{\underline{0}\}$.

[4b.] A is not invertible as $\text{Ker}(A) \neq \{\underline{0}\}$.

[4c.] Yes, the eigenvectors of a symmetric matrix are linearly independent (and can be chosen to be orthonormal).

[4d.] For example $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -3 & -6 & a_{33} \end{bmatrix}$, with $a_{33} = -5$. In fact: $A \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ gives 2 times the first column of A and $A \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ gives -1 times the second column of A. Then, from the symmetry condition iii. we get $a_{13} = a_{31}$ and $a_{23} = a_{32}$. For a_{33} we impose $\sum \lambda_k = \text{trace}(A) = 1 + 4 + a_{33} = 0$. From this relation we deduce $a_{33} = -5$.

5. [10 marks] Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

5a. [2 marks] Find the eigenvalues of A.

5b. [3 marks] Give a factorization $A = QDQ^T$ where Q has orthonormal columns and D is a diagonal matrix.

5c. [4 marks] As $t \to \infty$, what is the limit of $\underline{u}(t)$ for

$$\frac{d\underline{u}(t)}{dt} = -A\underline{u}(t)$$

given the initial condition $\underline{u}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$?

5d. [1 marks] Is A a positive definite matrix? Why? Give the quadratic form $q(x, y)$ associated to A.

Sol.

[5a.] Solve $\det(A - \lambda I) = 0$. We get the equation $\lambda(\lambda - 5) = 0$. Hence $\lambda_1 = 5$ and $\lambda_2 = 0$.

[5b.] $A = QDQ^T$ is an "eigenvalue-eigenvector" factorization of a symmetric matrix. D is a diagonal matrix containing the eigenvalues of A and Q is a 2×2-matrix whose orthonormal columns are eigenvectors of A. For example

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

[5c.] Note that the eigenvalues and eigenvectors of $-A$ need to be used

$$\underline{u}(t) = c_1 e^{-\lambda_1 t} \underline{x}_1 + c_2 e^{\lambda_2 t} \underline{x}_2 = c_1 e^{-5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

The initial condition determines the values of c_1 and c_2: $c_1 = 1$ and $c_2 = 1$. Hence, as $t \to \infty$, $\underline{u}(t) = e^{-5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \to \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

[5d.] A is not a positive definite matrix as $\lambda_2 = 0$. The quadratic form (singular) associated to A is $q(x, y) = x^2 + 4xy + 4y^2$.

6. [10 marks]

6a. [3 marks] If possible, find an invertible matrix M such that

$$M^{-1}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}M = \begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 2 & 2\end{bmatrix}.$$

If it is not possible, state why M cannot exist.

6b. [3 marks] For what real values of c (if any) is

$$A = \begin{bmatrix}-1 & c & 2\\ c & -4 & -3\\ 2 & -3 & 4\end{bmatrix}$$

a symmetric positive definite matrix?

6c. [4 marks] Let $A = \begin{bmatrix}3 & 4\\ 4 & 3\end{bmatrix}$. Is the quadratic form $q(x, y)$ associated to A positive definite? Find its principal axes.

<u>Sol.</u>

[6a.] Not possible. The condition means that $B = \begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 2 & 2\end{bmatrix}$ is similar to $A = \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$. Similar matrices have equal traces and rank. But $\text{trace}(A) = 3 \neq \text{trace}(B) = 5$ and $\text{rk}(A) = 1 \neq \text{rk}(B) = 2$.

[6b.] Not possible. For symmetric positive-definite matrices all upper-left determinants are greater than zero. Note that the 1 by 1 upper-left determinant is -1.

[6c.] $\det(A) < 0$, hence A (or equivalently its associated quadratic form $q(x, y) = 3x^2 + 8xy + 3y^2$) is not positive definite. The eigenvalues of A are: $\lambda_1 = -1$ and $\lambda_2 = 7$. The principal axes are the eigenspaces of A, namely $E_1 = \text{span}\{\begin{bmatrix}-1\\ 1\end{bmatrix}\}$ and $E_2 = \text{span}\{\begin{bmatrix}1\\ 1\end{bmatrix}\}$.

7. [15 marks]

7a. [6 marks] Let $A_1 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Is A_1 diagonalizable? Why? Is A_1 invertible? Why? Determine the spectral decomposition of A_1 into projection matrices.

7b. [3 marks] Let $A_2 = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$. Is A_2 invertible? Why? Is A_2 diagonalizable? Why? Determine (if exists) a matrix S and a diagonal matrix D such that $S^{-1}A_2S = D$.

7c. [6 marks] Describe the linear transformation $T_{A_2} : \mathbb{R}^2 \to \mathbb{R}^2$ associated to A_2. Does such A_2 have a decomposition into projection matrices? If yes, give it.

Sol.

[7a.] A_1 is symmetric, hence A_1 is diagonalizable. A_1 is invertible as $\det(A_1) = \prod \lambda_i = 2 \cdot 1 \cdot (-1) = -2 \neq 0$. The spectral decomposition of A_1 is given by

$$A_1 = \sum_{i=1}^{3} \lambda_i \underline{x}_i \underline{x}_i^T$$

where $\underline{x}_i$ are eigenvectors associated to the eigenvalues λ_i. We can choose $\underline{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as eigenvector associated to $\lambda_1 = 2$, $\underline{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ as eigenvector associated to $\lambda_2 = 1$ and $\underline{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ as eigenvector associated to $\lambda_3 = -1$. It follows that the spectral decomposition of A_1 is

$$A_1 = 2P_1 + P_2 - P_3 = 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

[7b.] $\det(A_2) = 0$ hence A_2 is not invertible. The eigenvalues of A_2 are $\mu_1 = 0$ and $\mu_2 = -4$. They are distinct, hence A_2 is diagonalizable. The columns of the matrix S are made by 2 eigenvectors of A i.e. $S = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$, whereas $D = \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}$ is the eigenvalues matrix.

[7c.] The linear transformation $T_{A_2} : \mathbb{R}^2 \to \mathbb{R}^2$ is determined by a projection P_2 onto the line spanned by $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$$A_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underline{0}, \qquad A_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} = -4 \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

It follows that

$$A_2 = -4 \begin{bmatrix} 3/4 & -3/4 \\ -1/4 & 1/4 \end{bmatrix} = -4P_2$$

is the required decomposition, where P_2 is a projection matrix.

8. [10 marks]

8a. [3 marks] Find the lengths and the inner product $\underline{x} \cdot \underline{y}$ of the following complex vectors

$$\underline{x} = \begin{bmatrix} 2-4i \\ 4i \end{bmatrix}, \qquad \underline{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad (i^2 = -1).$$

8b. [3 marks] Let $A = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$. Let $\underline{x}_1$, $\underline{x}_2$ be two (linearly independent) eigenvectors of A. Compute $\underline{x}_1 \cdot \underline{x}_2$ and show that $\det(A) \in \mathbb{R}$.

8c. [4 marks] Prove that for any complex vector $\underline{x}$

$$\underline{x}^H A \underline{x} \in \mathbb{R}. \qquad (H = \text{Hermitian})$$

Sol.

[8a.] $\text{length}(\underline{x}) = (\underline{x}^H \underline{x})^{1/2} = (\begin{bmatrix} 2+4i & -4i \end{bmatrix} \cdot \begin{bmatrix} 2-4i \\ 4i \end{bmatrix})^{1/2} = 6$; $\text{length}(\underline{y}) = (\underline{y}^T \underline{y})^{1/2} = \sqrt{20}$ and $\underline{x} \cdot \underline{y} := \underline{x}^H \underline{y} = 4(1-2i)$.

[8b.] Notice that $A = A^H$, furthermore let λ_i be the 2 eigenvalues of A: $0 = \lambda_1 \neq \lambda_2 = 3$, then $\underline{x}_1 \cdot \underline{x}_2 = 0$. Also, one knows that every eigenvalue of a Hermitian matrix is real and so will be its determinant ($\det(A) = \lambda_1 \lambda_2 = 0$).

[8c.] We have $(\underline{x}^H A \underline{x})^H = \underline{x}^H A \underline{x}$, as $A = A^H$. It follows that $\underline{x}^H A \underline{x} \in \mathbb{R}$.