CALCULUS 106 – SECOND MIDTERM EXAM SOLUTIONS

1 . First compute the first and second derivatives. We find:

$$f'(x) = \frac{2x(x+2) - (x^2-3)}{(x+2)^2} = \frac{(x+1)(x+3)}{(x+2)^2} \ ,$$

and

$$f''(x) = \frac{2}{x+2} - \frac{2(x^2+4x+3)}{(x+2)^3} = \frac{2}{(x+2)^3} \ .$$

Hence, the critical points are $x = -1$ and $x = -3$. There are no inflection points. Notice, however, that f is concave up for $x > -2$ and concave down for $x < -2$. By the second derivative test, $(-1, -2)$ is a local minimum, and $(-3, -6)$ is a local maximum. The line $x = -2$ is a vertical asymptote, with

$$\lim_{x \to -2^+} f(x) = +\infty \ , \quad \lim_{x \to -2^-} f(x) = -\infty \ .$$

To find the oblique asymptote, notice that $x^2 - 3 = (x+2)(x-2) + 1$, or

$$\frac{x^2-3}{x+2} = x - 2 + \frac{1}{x+2} \ .$$

So $y = x - 2$ is an oblique asymptote. For the graph, see Figure 1.

2 . From Figure 2, we see that the area of the rectangle is a function of x, the location of the lower right corner. Moreover, $A(x) = x(3 - x^2)$. So the maximum occurs at $A'(x) = 3 - 3x^2 = 0$, or $x = 1$. Then the height is $y = 3 - x^2 = 2$, so the maxmimal area is 2.

3 . (a) Let $L = \lim_{x\to\infty} (1 + \sin(1/x))^x$. Then

$$\begin{aligned}
\ln L &= \lim_{x\to\infty} \ln\left(1 + \sin(1/x)\right)^x \\
&= \lim_{x\to\infty} x \ln\left(1 + \sin(1/x)\right) \\
\text{(substitute } y = 1/x\text{)} \qquad &= \lim_{y\to 0^+} \frac{\ln(1+\sin y)}{y} \\
\text{(L'Hôpital's rule)} \qquad &= \lim_{y\to 0^+} \frac{1}{(1+\sin y)} \cos y \\
&= 1 \ .
\end{aligned}$$

Hence, the limit $L = e^1 = e$.

(b) Just use L'Hôpital's rule:

$$\lim_{x \to 0} \frac{e^x - 1}{\ln(1+x)} = \lim_{x \to 0} \frac{e^x}{\frac{1}{1+x}} = 1 \ .$$

4 . The general antiderivative is $T(t) = -(1/\pi)\cos(\pi t) + C$, where C is any constant. From the initial condition:

$$T(0) = 3 = -\frac{1}{\pi} + C \quad \Longrightarrow \quad C = 3 + \frac{1}{\pi} \ .$$

So the solution is:

$$T(t) = -\frac{1}{\pi}\cos(\pi t) + 3 + \frac{1}{\pi} \ .$$

5 . (a) Notice that $\frac{d}{dx} 3^x = 3^x \ln 3$, or

$$\frac{d}{dx}\left(\frac{3^x}{\ln 3}\right) = 3^x \ .$$

Therefore,

$$\int_0^1 3^x \, dx = \left.\frac{3^x}{\ln 3}\right]_0^1 = \frac{2}{\ln 3} \ .$$

(b) By the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_2^{\ln x} e^{-u} du = e^{-\ln x}\frac{d}{dx}\ln x = \frac{1}{x^2} \ .$$

6 .

$$\begin{aligned}
\int_{-a}^{a}(1-|x|)dx &= \int_0^a (1-x)dx + \int_{-a}^{0}(1+x)dx \\
&= \left.\left(x - \frac{x^2}{2}\right)\right]_0^a + \left.\left(x + \frac{x^2}{2}\right)\right]_{-a}^0 \\
&= a - \frac{a^2}{2} - \left(-a + \frac{a^2}{2}\right) \\
&= a(2-a)
\end{aligned}$$

So we should choose $a = 2$.

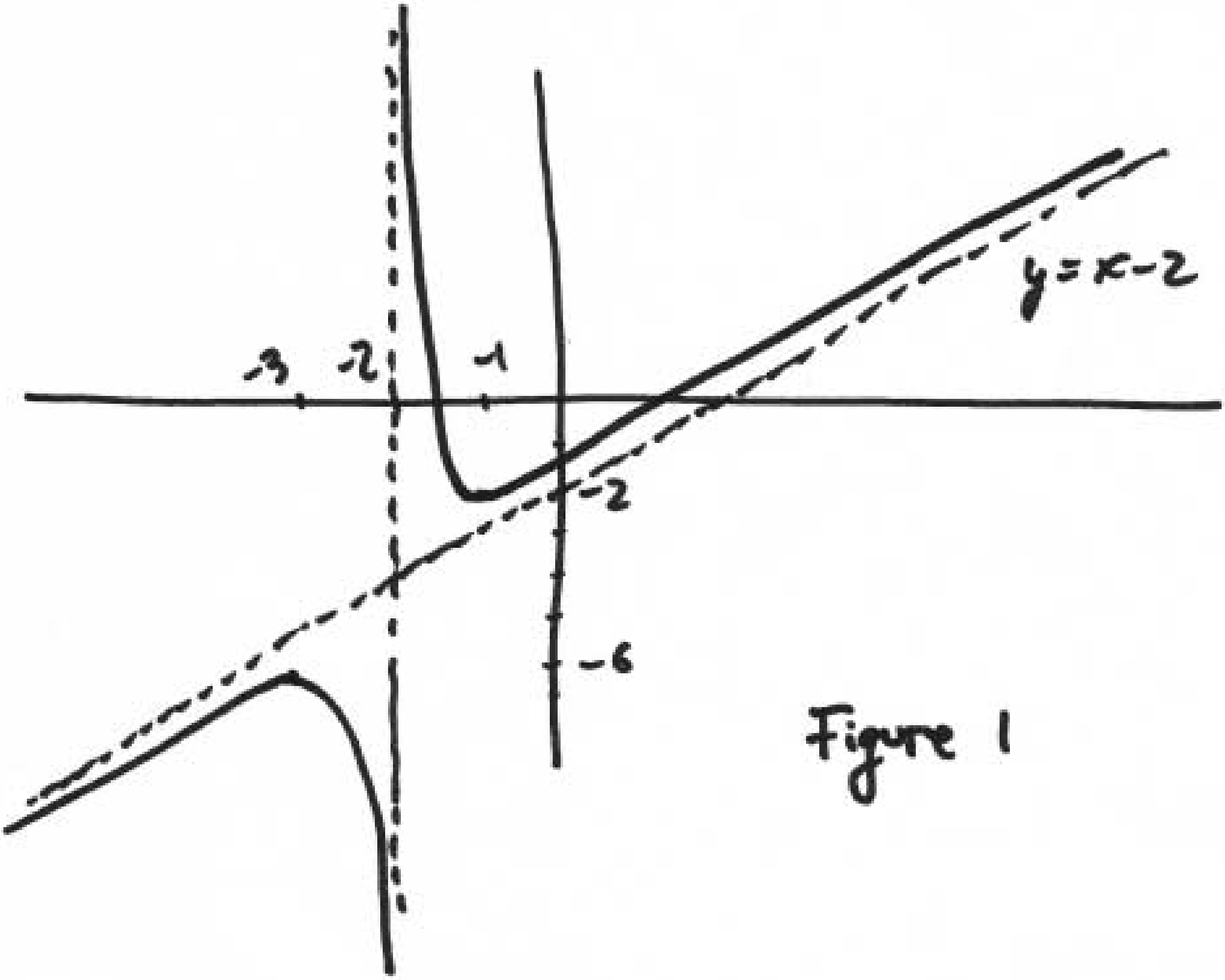

Figure 1

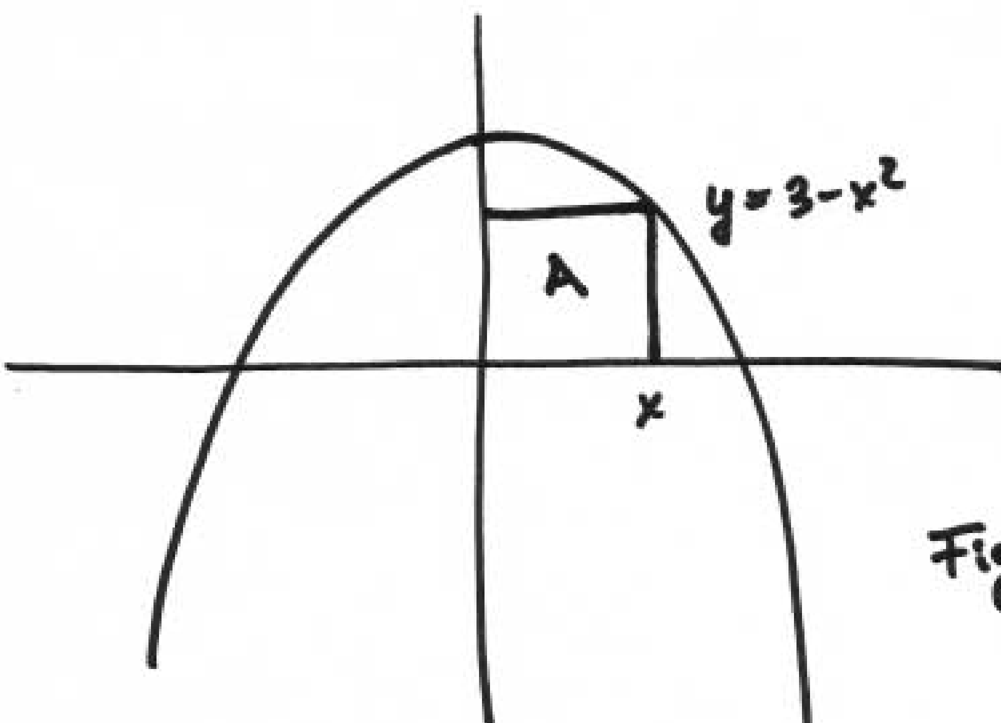

Figure 2