CALCULUS 106 – FIRST MIDTERM EXAM SOLUTIONS

1 . *Solution.* (i)

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} &= \lim_{x\to 0}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\right)\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)\\ &= \frac{1+x-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}\\ &= \frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}\\ &= 1\ .\end{aligned}$$

(ii)

$$\begin{aligned}\lim_{w\to 0^+}\frac{1-\cos\sqrt{w}}{\sin(2w)} &= \lim_{w\to 0^+}\left(\frac{1-\cos\sqrt{w}}{\sin(2w)}\right)\left(\frac{1+\cos\sqrt{w}}{1+\cos\sqrt{w}}\right)\\ &= \lim_{w\to 0^+}\frac{1-\cos^2\sqrt{w}}{\sin(2w)(1+\cos\sqrt{w})}\\ &= \lim_{w\to 0^+}\frac{\sin^2\sqrt{w}}{\sin(2w)(1+\cos\sqrt{w})}\\ &= \lim_{w\to 0^+}\left(\frac{\sin^2\sqrt{w}}{(\sqrt{w})^2}\right)\left(\frac{1}{2}\right)\left(\frac{2w}{\sin(2w)}\right)\left(\frac{1}{(1+\cos\sqrt{w})}\right)\\ &= \frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}\ .\end{aligned}$$

(iii)

$$\lim_{x\to\infty}\frac{2e^x+3x^2e^{-x}}{3e^x+4x^4e^{-x}} = \frac{\lim_{x\to\infty}2+3x^2e^{-2x}}{\lim_{x\to\infty}3+4x^4e^{-2x}} = \frac{2}{3}\ .$$

2 . *Solution.* Differentiate implicitly:

$$2yy'(2-x)+y^2(-1)+y+xy' = 3x^2\ .$$

At $x=1$, $y=1$,

$$2y'-1+1+y'=3$$

or $y'=1$. Hence the line has slope 1 and passes through the point $(1,1)$. It's equation is: $y=x$.

3 . Yes! First, $f(x)=4x+e^{-x}$ is a continuous function. Next, $f(0)=1$, and $f(-1)=-4+e<0$. Since $f(0)>0$ and $f(-1)<0$, by the Intermediate Value Theorem, $f(x)$ must have a zero in between -1 and 0.

4 . *Solution.* (i)

$$\frac{d}{dx}\frac{x}{x+1} = \frac{1\cdot(x+1)-x\cdot 1}{(x+1)^2} = \frac{1}{(x+1)^2}\ .$$

(ii)

$$\frac{d}{x}\cot x = \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\csc^2 x\ .$$

(iii)

$$\frac{d}{dx}x^{\ln x} = x^{\ln x}\frac{d}{dx}\ln(x^{\ln x} = x^{\ln x}\frac{d}{dx}(\ln x)^2 = 2x^{\ln x}\frac{\ln x}{x} = 2x^{\ln x - 1}\ln x\ .$$

5 . *Solution.* The first step is to observe that when $x = \ln 2$,

$$y = e^{\ln 2} - e^{-\ln 2} + 1/2 = 2 - 1/2 + 1/2 = 2\ .$$

Now the formula for the derivative of the inverse is:

$$(f^{-1})'(2) = \frac{1}{f'(\ln 2)}\ .$$

$f'(x) = e^x + e^{-x}$, so $f'(\ln 2) = 2 + 1/2 = 5/2$. Hence, $(f^{-1})'(2) = 2/5$.

6 . Let y be the distance from the car to the officer, and x the distance from the car to the intersection. Then by the Pythagorean theorem, $y^2 = (.3)^2 + x^2 = .09 + x^2$. In particular, when $y = .5$, $x = .4$. Moreover, differentiating implicitly with respect to time, we have $yy' = xx'$. So when $y = .5$ and $y' = -60$, we have $-.5(60) = .4x'$, or in other words, $x' = -75$. So you're driving at 75 mph, and the officer should give you a ticket!