

1. Evaluate the following:

(a)

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x \, dx}{\sqrt{1-x^2}} = x \arcsin x + \sqrt{1-x^2} + C.$$

(b)

$$\begin{aligned} \int \frac{e^{2x} \, dx}{e^{2x} - 1} &= \frac{1}{2} \int \frac{du}{u}, & u &= e^{2x} - 1 \\ &= \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |e^{2x} - 1| + C. \end{aligned}$$

(c)

$$\begin{aligned} \frac{d}{dx} \int_{\arcsin x}^{\pi/2} \ln(\sin t) \, dt &= -\ln(\sin(\arcsin x)) \cdot \frac{d(\arcsin x)}{dx} = \\ &= -\frac{\ln x}{\sqrt{1-x^2}}. \end{aligned}$$

2. The volume  $V$  (in cubic inches) and pressure  $p$  (in pounds per square inch) of the air inside a balloon satisfy the equation  $pV = 1000$ . At what rate is the volume of the balloon changing if the pressure is currently  $100 \text{ lb/in}^2$  and is dropping at the rate of  $2 \text{ lb/in}^2$  per second?

Since  $pV = 1000$ , differentiating with respect to the time  $t$  we get  $p \frac{dV}{dt} + \frac{dp}{dt} V = \frac{d(1000)}{dt} = 0$ . If  $\frac{dp}{dt} = -2$  and  $p = 100$  then  $V = 1000/p = 1000/100 = 10 \text{ in}^3$  and

$$\frac{dV}{dt} = -\frac{V}{p} \frac{dp}{dt} = -\frac{10}{100}(-2) = 0.2,$$

so the volume of the balloon is increasing at the rate of  $0.2 \text{ in}^3/\text{s}$ .

3. Solve the initial value problem:

$$\frac{dy}{dx} = \frac{\sqrt{x}}{1+x}, \quad y = 2 \text{ when } x = 1.$$

$$\begin{aligned}
y &= \int \frac{\sqrt{x} dx}{1+x} = \int \frac{u \cdot 2u du}{1+u^2}, \quad u = \sqrt{x} \\
&= 2 \int \left(1 - \frac{1}{1+u^2}\right) du = 2u - 2 \arctan u + C \\
&= 2\sqrt{x} - 2 \arctan \sqrt{x} + C.
\end{aligned}$$

Since  $2 = y|_{x=1} = 2\sqrt{1} - 2 \arctan \sqrt{1} + C = 2 - 2(\frac{\pi}{4}) + C$  we find  $C = \frac{\pi}{2}$  so  $y = 2\sqrt{x} - 2 \arctan \sqrt{x} + \frac{\pi}{2}$  is the solution to the IVP.

4. (a) Find the first (linear) and second (quadratic) Taylor polynomials for the function  $f(x) = \sqrt{x}$  centered about the point 1.

$f(1) = \sqrt{1} = 1$ ,  $f'(1) = (1/2)x^{-1/2}|_{x=1} = 1/2$  and  $f''(1) = -(1/4)x^{-3/2}|_{x=1} = -1/4$  so the first and second Taylor polynomials are

$$\begin{aligned}
p_1(x) &= 1 + \frac{1}{2}(x-1); \\
p_2(x) &= 1 + \frac{1}{2}(x-1) + \frac{-1/4}{2!}(x-1)^2 = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2.
\end{aligned}$$

- (b) Use part (a) to find the first- and second-order approximations to the exact value of  $\sqrt{101}$ . [Hint:  $\sqrt{101} = 10\sqrt{1.01}$ .]

The first approximation is

$$\sqrt{101} = 10f(1.01) \approx 10p_1(1.01) = 10 \left(1 + \frac{1}{2}(0.01)\right) = 10.05.$$

The second approximation is

$$\begin{aligned}
\sqrt{101} &= 10f(1.01) \approx 10p_2(1.01) = 10 \left(1 + \frac{1}{2}(0.01) - \frac{1}{8}(0.01)^2\right) \\
&= 10.049875.
\end{aligned}$$

(This last value is the exact value of  $\sqrt{101}$  to all digits shown!)

5. Decide if the improper integral

$$\int_1^{\infty} \frac{x dx}{x^4 + 1} \tag{1}$$

converges or diverges. Justify your answer.

The integrand is a nonnegative continuous function when  $x \geq 1$ . Hence the integral will either converge to some finite positive number or diverge to  $+\infty$ . However, it is clear that for  $x \geq 1$  (in fact for any  $x > 0$ ):

$$\frac{x}{x^4 + 1} \leq \frac{x}{x^4} = \frac{1}{x^3}$$

so we can use the comparison theorem for improper integrals and conclude that

$$\begin{aligned} \int_1^\infty \frac{x \, dx}{x^4 + 1} &\leq \int_1^\infty \frac{dx}{x^3} = \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^3} \\ &= \lim_{b \rightarrow \infty} \left( -\frac{1}{2x^2} \Big|_1^b \right) = \lim_{b \rightarrow \infty} \left( \frac{1}{2} - \frac{1}{2b^2} \right) = \frac{1}{2}. \end{aligned}$$

Hence the integral (1) itself must be convergent.

Alternatively, the change of variable  $y = x^2$  brings (1) into the form (note that  $y = 0$  when  $x = 0$  and  $y \rightarrow \infty$  when  $x \rightarrow \infty$  as well, so the integration limits are unchanged):

$$\begin{aligned} \frac{1}{2} \int_0^\infty \frac{dy}{y^2 + 1} &= \frac{1}{2} \lim_{b \rightarrow \infty} \int_0^b \frac{dy}{y^2 + 1} = \frac{1}{2} \lim_{b \rightarrow \infty} (\arctan y|_0^b) \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} (\arctan b - \arctan 0) = \frac{1}{2} \left( \frac{\pi}{2} - 0 \right) = \frac{\pi}{4}. \end{aligned}$$

(Needless to say,  $\frac{\pi}{8} < \frac{1}{2}$ .)

6. A rectangle with sides parallel to the coordinate axes has one vertex at the origin, one on the positive  $x$ -axis, one on the positive  $y$ -axis, and its fourth vertex on the line  $y = 100 - 2x$  (see the figure). What is the maximum possible area of such rectangle? What are its dimensions?

The area of such a rectangle is  $A = xy$  where  $y = 100 - 2x$ , hence  $A = x(100 - 2x)$ , a continuous function for  $0 \leq x \leq 50$  (note the minimum width  $x$  of such a rectangle is 0 and the maximum possible width is 50.) Also,  $A = 0$  for  $x = 0$  or 50 so the maximum of the area function  $A$  will not be attained at the endpoints. Now,  $\frac{dA}{dx} = 100 - 4x$  is well-defined throughout the entire closed interval  $[0, 50]$  and  $\frac{dA}{dx} = 0$

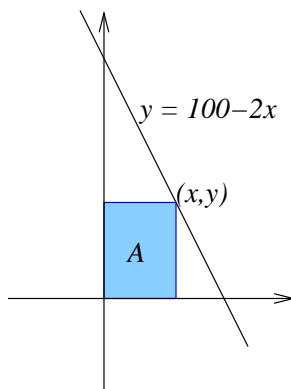


Figure 1: The rectangle inscribed under  $y = 100 - 2x$ .

only when  $x = 25$ . Also, when  $x = 25$ ,  $y = 100 - 2(25) = 50$ . Hence, the rectangle of maximal area has dimensions  $x = 25$ ,  $y = 50$  and has area  $25 \cdot 50 = 1250$  square units.

7. A spherical container of radius 10 feet is partly filled with water (see figure). The depth of the water remaining in the container is 5 feet. Would you intuitively expect that the container is exactly one-quarter full, or less than one-quarter full, or more? Find the exact volume of water inside (for reference, the capacity of the full container is  $\frac{4000}{3}\pi$  cubic feet.)

[Hint: Find the volume of the solid of revolution obtained by revolving the shaded circular wedge of the semi-circle  $y = \sqrt{100 - x^2}$  around the  $x$ -axis.]

The volume we seek is:

$$\begin{aligned} \int_{-10}^{-5} \pi(\sqrt{100 - x^2})^2 dx &= \pi \int_{-10}^{-5} (100 - x^2) dx = 100\pi x - \frac{\pi}{3}x^3 \Big|_{-10}^{-5} \\ &= \frac{625}{3}\pi, \end{aligned}$$

which is less than a quarter of  $\frac{4000}{3}\pi$ . This is to be expected since the tank is wider near its equator and narrower near its endpoints (“poles”).

8. The graph of  $(x^2 + y^2 + y)^2 = x^2 + y^2$  is a cardioid (“heart-shaped”). Find the equation of the tangent line to the cardioid passing through the point  $(1,0)$ . (Hint: use implicit differentiation.)

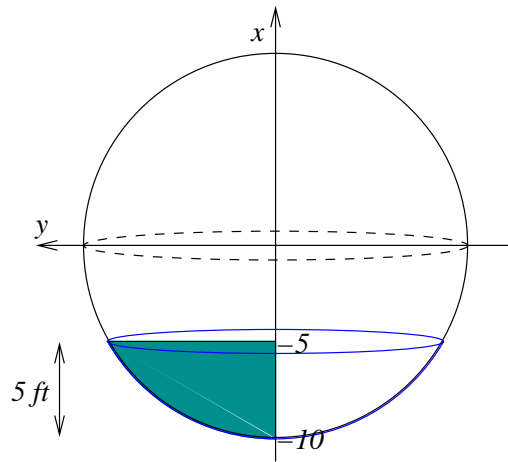


Figure 2: Cross-section of the water container.

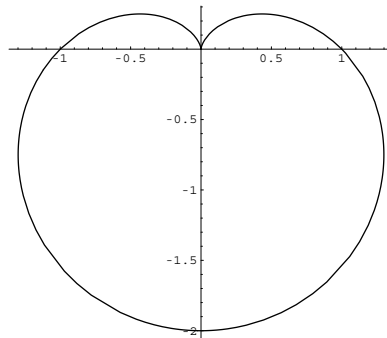


Figure 3: The cardioid  $(x^2 + y^2 + y)^2 = x^2 + y^2$ .

We differentiate the equation implicitly with respect to  $x$  and obtain:

$$2(x^2 + y^2 + y) \left( 2x + 2y \frac{dy}{dx} + \frac{dy}{dx} \right) = 2x + 2y \frac{dy}{dx}.$$

Now plug in  $x = 1, y = 0$  to obtain:

$$2 \left( 2 + \frac{dy}{dx} \right) = 2.$$

Hence  $\frac{dy}{dx} = -1$  is the slope of the tangent line at the point  $(1, 0)$ , and the tangent line itself has equation  $y - 0 = -1(x - 1)$  or  $y = 1 - x$ .